Proc. Summer School on Diff. Geometry
Dep. de Matemática, Universidade de Coimbra
September 1999, 53-58.

Perturbation Theory of Overintegrable Differential Systems

J. P. Françoise

Université de Paris 6, UFR 920, tour 45-46, 4 place Jussieu, B.P. 172
"Géométrie Différentielle, Systèmes Dynamiques et Applications"
75252 Paris, France

Summary. Several differential systems have their orbits contained in curves globally defined as common level sets of functions. This is so for instance of the maximally superintegrable Hamiltonian systems with m degrees of freedom, where the number of independent first integrals is $2m-1$. The first part of the article displays one example of interest related to mathematical physics. The second part presents a perturbation theory to describe the periodic orbits of the perturbed systems. The main result is the generalization of the algorithm of the successive derivatives of return mappings for 2-dimensional systems to any dimension.

1. Introduction

These last years, the dynamics of plane systems was extensively studied and several new techniques were developed. Some are specific to 2-dimensional systems but mostly often these methods can be appropriately extended to multidimensional systems. Purpose of this article is to generalize the algorithm of the successive derivatives of return mappings ([F1, [F2]) to any dimension. The algorithm was derived some years ago to find the first non-vanishing derivative (relatively to the parameter ϵ) of the return mapping (near the origin) of a plane vector field $X_0 + \epsilon X_1$ of type:

$$X_0 + \epsilon X_1 = x\partial/\partial y - y\partial/\partial x + \epsilon \sum_{i,j/i+j=2}^{d} [a_{i,j}x^i y^j \partial/\partial x + b_{i,j}x^i y^j \partial/\partial y]. \tag{1}$$

The algorithm was then used in the center-focus problem (cf. [FP]), which directly relates to Hopf bifurcations of higher order and to several other problems on limit cycles of plane vector fields. In the recent past, several authors emphasized the need for finding systematic algorithmic methods for studying bifurcation theory of periodic orbits. Such a method is presented here.

1'. The unperturbed dynamical system

Let $f = (f_1, \ldots, f_{n-1}) : R^n \to R^{n-1}$ be a generic submersion (meaning that f is a submersion outside a critical set $f^{-1}(C)$, where C is a set of isolated points). Let $\Omega = dx_1 \wedge dx_2 \wedge \cdots dx_n$ be a volume form on R^n. Consider the vector field X_0 such that:

$$\iota_{X_0} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = df_1 \wedge \cdots \wedge df_{n-1}. \tag{2}$$

The functions $f_i, (i = 1, \ldots, n-1)$ are first integrals of the vector field X_0:

$$df_i \wedge \iota_{X_0} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = (X_0.f_i) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = df_i \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_{n-1} = 0. \tag{3}$$

For c varying in a neighborhood of 0, assume that the curves $f^{-1}(c)$ have a compact connected component γ_c. Let Σ be a small neighborhood of the zero-section of the normal bundle to γ_0. For c small enough, the curves γ_c are closed periodic orbits of X_0 and they cut transversely Σ. Choose c as a coordinate on the transverse section Σ to the flow of X_0. Lastly, assume that there are 1-forms ω_i such that:

$$\iota_{X_0} \omega_i = df_i; i = 1, \ldots, n-1. \tag{4}$$

Depending of the type of regularity of the 1-form ω_i, this condition may be a consequence of the preceding assumptions.

The appropriated extension of the (*)-property first discussed in [F1, [F2] is presented in the following

Definition 1. *Let $f = (f_1, \ldots, f_{n-1}) : R^n \to R^{n-1}$ be a generic submersion. Assume that $f^{-1}(c)$ contains a compact curve γ_c. The application displays the (*)-property if for all 1-form ω such that*

$$\int_{\gamma_c} \omega = 0, \tag{5}$$

for all c; there exist g_i, R such that:

$$\omega = g_1 df_1 + \cdots + g_{n-1} df_{n-1} + dR. \tag{6}$$

It was proved in [F1, [F2] that the function $f_1 : R^2 \to R^1, f_1 : (x_1, x_2) \to (x_1^2 + x_2^2)$ displays the (*)-property. Several generalizations were proposed after but the core of the argument in the computation of the successive derivatives is captured in this notion. The generalization proposed in this article provides a new presentation of the (*)-property which seems interesting as well for the 2-dimensional case. Indeed, the definition of the vector field X_0 given in the preceding introduction yields the

Proposition 2. *Let ω be a 1-form such that $\omega(X_0) = 0$, then there are functions $g_1, \ldots, g_{n-1}$ so that:*

$$\omega = g_1 df_1 + \cdots + g_{n-1} df_{n-1}. \tag{7}$$

Note that the condition $\omega(X_0) = 0$, equivalent to $\omega \wedge df_1 \wedge \cdots \wedge df_{n-1} = 0$, yields $\omega = g_1 df_1 + \cdots + g_{n-1} df_{n-1}$ where the coefficients g_k are obtained as ratio of minors of the Jacobian matrix of the f_j.

This displays an alternative to the (*)-property now presented as follows:

Proposition 3. *A generic submersion $f : R^n \to R^{n-1}$ displays the (*)-property if for any 1-form ω such that* (5)*, i.e.*

$$\int_{\gamma_c} \omega = 0,$$

for all c; then there exists a function R such that:

$$\omega(X_0) = X_0 \cdot R. \tag{8}$$

Such a function R can be (in principle) constructed with the following pattern. Choose R arbitrarily on the transverse section Σ, then extends R to the whole tubular neighborhood of γ_0 saturated by the orbits γ_c by integration of the 1-form ω along the orbits of X_0.

2. Examples of unperturbed systems of interest in mathematical physics

Let $H : V^{2m} \to R$ be a Hamiltonian system defined on a symplectic manifold V^{2m} of dimension $2m$ equipped with a symplectic form ω. Recall that H is said to be integrable in Arnol'd-Liouville sense if H displays m generically independent first integrals (one of these maybe the Hamiltonian itself) which are in involution for the Poisson bracket associated with the symplectic form ω. A vector field X on a manifold V of dimension n defines a flow and a dynamical system. The vector field (not necessarily Hamiltonian) is classically said to be maximally superintegrable if it has $n-1$ generically independent global first integrals $f_1, \ldots, f_{n-1}$. The orbits of X are then contained in the connected components of the common level sets of the functions $f_i, i = 1, \ldots, n-1$. Some Hamiltonian systems are known to be maximally superintegrable and so they display $2m-1$ first integrals. A discussion of the rational Calogero-Moser system with an external quadratic potential is provided now as a featuring example. The system is described by the Hamiltonian:

$$H = (1/2)\sum_{i=1}^{m} y_i^2 + g^2 \sum_{ij} (x_i - x_j)^{-2} + (\lambda^2/2)\sum_{i=1}^{m} y_i^2. \tag{9}$$

Introduce the matrix function:

$$L(x, y)/L_{ij} = y_i \delta_{ij} + g\mathrm{i}(x_i - x_j)^{-1}(1 - \delta_{ij}), \tag{10}$$

and observe that the time evolution of this matrix function $L(x, y)$ along the flow is:

$$\dot{L} = [L, M] - \lambda^2 X. \tag{11}$$

This equation is supplemented with the equation:

$$\dot{X} = [X, M] + L, \tag{12}$$

displayed by the diagonal matrix X:

$$X(x, y)/X_{ij} = x_i \delta_{ij}. \tag{13}$$

The classical approach consists in introducing the matrices:

$$Z = L + \mathrm{i}\lambda X, \tag{14a}$$
$$W = L - \mathrm{i}\lambda X. \tag{14b}$$

These matrices undergo the time evolution:

$$\dot{Z} = \mathrm{i}\lambda Z + [Z, M], \tag{15a}$$
$$\dot{W} = -\mathrm{i}\lambda W + [W, M]. \tag{15b}$$

It was then observed ([F1, [F2]) that the matrix $P = ZW$ defines a Lax matrix for the system:

$$\dot{P} = [P, M]. \tag{16}$$

Here, we note that the functions:

$$F_k = tr(ZP^k), \tag{17a}$$
$$G_k = tr(WP^k), \tag{17b}$$

yield:

$$\dot{F}_k = \mathrm{i}\lambda F_k, \tag{18a}$$
$$\dot{G}_k = \mathrm{i}\lambda G_k. \tag{18b}$$

Appropriated combinations of these functions provide the first integrals of the flow of the Hamiltonian system X_0. All the orbits of X_0 are periodic and thus this is an example of system to which the preceding approach applies.

3. The successive derivatives of the holonomy of the perturbed system

Now perturb X_0 into $X_\epsilon = X_0 + \epsilon X_1$. Let M be a point of Σ close to 0 and let γ_ϵ be the trajectory of X_ϵ passing by the point M. The next first intersection point of γ_ϵ with Σ defines the so-called Poincaré return mapping (or holonomy) of X_ϵ relatively to the transverse section Σ: $c \longmapsto L(c, \epsilon)$. The mapping L is analytic and it displays a Taylor development (in ϵ):

$$L(c, \epsilon) = c + \epsilon L_1(c) + \cdots + \epsilon^k L_k(c) + O(\epsilon)^{k+1}. \tag{19}$$

The expression of the first coefficient $L_1(c)$ is classical and belongs to the lore of bifurcation theory. With the vector field X_ϵ and the 1-forms ω_i (cf. [F1, [F2]), introduce the 1-forms:

$$\iota_{X_\epsilon}\omega_i = \iota_{X_0}\omega_i + \epsilon\iota_{X_1}\omega_i = df_i + \epsilon\iota_{X_1}\omega_i. \tag{20}$$

Recall that the parameter c chosen as coordinates on the transverse section Σ is the restriction of the functions $f = (f_1, \dots, f_{n-1})$ to the section.

Then the i^{th}-component of $L_1(c)$ is equal to:

$$L_{1,i}(c) = \int_{\gamma_0} \iota_{X_1}\omega_i. \tag{21}$$

Assume now that the first derivative $L_1(c)$ vanishes identically and that the submersion f displays the (*)-property then there exist g_{ij} and R_i such that:

$$\iota_{X_1}\omega_i = \sum_j g_{ij}df_j + dR_i. \tag{22}$$

Following the lines of the algorithm of the successive derivatives, the expression (22) yields:

$$L_{2,i}(c) = -\int_{\gamma_0} \sum_j g_{ij}\iota_{X_1}\omega_j. \tag{23}$$

This is indeed the second step of a general recursive scheme which displays as follows:

Assume that all the k^{th}-first derivatives of the holonomy of the perturbed vector field vanish identically. This yields:

$$L_{k,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^{k-1}\iota_{X_1}\omega_j = 0. \tag{24}$$

The (*)-property yields new functions g_{ij}^k, R^k such that:

$$\sum_j g_{ij}^{k-1}\iota_{X_1}\omega_j = \sum_j g_{ij}^k df_j + dR^k. \tag{25}$$

This yields the following expression of the $(k+1)^{th}$-derivative of the holonomy of the perturbation:

$$L_{k+1,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^k\iota_{X_1}\omega_j. \tag{26}$$

From the general perturbation theory of Bautin's type (cf. [FY], [LTZ]), it now follows:

Theorem 4. *Let $X_0 + \epsilon X_1$ be an analytic perturbation of the vector field X_0. Assume that the vector field X_0 preserves $n-1$ functions $f = (f_1, \dots, f_{n-1})$ and that f defines an analytic submersion which displays the (*)-property. Assume that the perturbation X_1*

depends of finitely many parameters (typical example is given by a polynomial perturbation of fixed degree). Then, there exists a uniform bound to the number of isolated periodic orbits of $X_0 + \epsilon X_1$ *which intersects the transverse section* Σ *in the neighborhood of* 0.

References

[F1] Françoise, J.-P.: *Successive derivatives of a first-return map, application to quadratic vector fields.* Erg.Th. & Dynam. Sys. **16** (1996), 87–96.

[F2] Françoise, J.-P.: *Canonical partition functions of Hamiltonian systems and the stationary phase formula.* Commun. Math. Phys. **117** (1988), 37–47.

[FP] Françoise, J.-P.; Pons, R.: *Computer Algebra Methods and the stability of differential systems.* Random and computational dynamics **3**(4) (1995), 265–287.

[FY] Françoise, J.-P.; Yomdin, Y.: *Bernstein inequality and applications to analytic geometry and differential equations.* Journal of Functional Analysis **146** (1997), 185–206.

[LTZ] Lojasiewicz, S.; Tougeron, J.-Cl.; Zurro, M.-A.: *Eclatement des coefficients des series entiéres et deux théorèmes de Gabrielov.* Manuscripta Matematica **92** (1997), 325–337.