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Some Remarks on the Poisson-Nijenhuis and the Jacobi Structures

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Abstract. We study the relationship between two (compatible) Jacobi structures on a manifold M, using their associated homogeneous Poisson structures on $\mathbb{R}\times M$, in the case where these Poisson tensors are related by a Nijenhuis tensor.

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Introduction

All objects considered in this paper (manifolds, maps, differential forms, vector and tensor fields) are assumed to be differentiable of class C^∞.

Let M be a smooth manifold equipped with a *Poisson tensor* Λ and a *Nijenhuis tensor* N, that is, a tensor field of type $(1,1)$ whose *Nijenhuis torsion* $\tau(N)$ vanishes everywhere. The Nijenhuis torsion $\tau(N)$ of N is given by the following formula, where X and Y are vector fields on M,

$$\tau(N)(X,Y) = [NX, NY] - N([NX,Y] + [X,NY] - N[X,Y]).$$

We denote by $\Lambda^\# : T^*M \to TM$ the vector bundle map such that, for any $x \in M$, $\alpha, \beta \in T^*_xM$,

$$< \beta, \Lambda^\#(\alpha) >= \Lambda_x(\alpha, \beta).$$

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With Λ and N we can define a tensor field $\mathcal{R}(\Lambda, N)$ of type $(2,1)$, called the *Magri-Morosi concomitant* [Ma-Mo 84] of Λ and N, that is defined, for any pair of a 1-form α and a vector field X, by

$$\mathcal{R}(\Lambda, N)(\alpha, X) = (\mathcal{L}_{\Lambda^\#(\alpha)}N)X - \Lambda^\#(\mathcal{L}_X({}^tN\alpha)) + \Lambda^\#(\mathcal{L}_{NX}\alpha), \tag{1}$$

where ${}^tN$ is the transpose of $N : TM \to TM$.

Let Λ and N be, respectively, a Poisson tensor and a Nijenhuis tensor on M. The triple (M, Λ, N) is called a *Poisson-Nijenhuis manifold* [KSch-Ma 90] if

$$N\Lambda^\# = \Lambda^\#\, {}^tN \quad \text{and} \quad \mathcal{R}(\Lambda, N) = 0.$$

If (M, Λ, N) is a Poisson-Nijenhuis manifold, there exists a sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of Poisson tensors on M, with $\Lambda_k = N^k\Lambda$. Moreover, these Poisson tensors are pairwaise compatible, that is, $[\Lambda_l, \Lambda_k] = 0$, for all $l, k \in \mathbb{N}$.

Besides the Poisson-Nijenhuis structure, we are going to use, in what follows, the notion of Jacobi manifold.

A *Jacobi manifold* [Lich 78] is a triple (M, C, E) where C and E are respectively a bivector and a vector field on M, such that

$$[E, C] = 0 \quad \text{and} \quad [C, C] = 2E \wedge C.$$

When $E = 0$, the Jacobi manifold is a Poisson manifold. The *Jacobi bracket* of $f, g \in C^\infty(M, \mathbb{R})$ is given by

$$\{f, g\} = C(df, dg) + f(E.g) - g(E.f),$$

and it defines a *local Lie algebra structure* on $C^\infty(M, \mathbb{R})$.

If (M, C, E) is a Jacobi manifold and h is a nowhere vanishing function on M, then the pair $(hC, C^\#(dh) + hE) = (C_h, E_h)$ defines a new Jacobi structure on M, which is said to be *conformally equivalent* to (C, E).

With each Jacobi manifold (M, C, E), we may associate a homogeneous Poisson structure $(\Lambda, \frac{\partial}{\partial t})$ on $\mathbb{R} \times M$, with Λ given by

$$\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E), \tag{2}$$

where t is the canonical coordinate on $\mathbb{R}$.

This paper is divided into two sections. Section 1 is devoted to the subject of compatible Jacobi manifolds. We show how the compatibility is related with the Lichnerowicz-Jacobi cohomology and we present a way of generating compatible Jacobi structures on a manifold. In Section 2, we establish the conditions on the Poisson-Nijenhuis structure of $\mathbb{R} \times M$ to ensure the compatibility of the corresponding Jacobi structures on M.

1. Compatible Jacobi manifolds

The notion of compatibility of two Jacobi structures on a manifold was introduced in [NdC 98]. We recall that two Jacobi structures (C_1, E_1) and (C_2, E_2) on a manifold M are said to be *compatible* if $(C_1 + C_2, E_1 + E_2)$ is again a Jacobi structure on M.

Proposition 1.1. [NdC 98] *Let M be a manifold endowed with two Jacobi structures (C_1, E_1) and (C_2, E_2). Then, (C_1, E_1) and (C_2, E_2) are compatible if and only if*

$$[E_1, C_2] + [E_2, C_1] = 0 \quad \textit{and} \quad [C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1.$$

There are some equivalent ways of expressing the compatibility of two Jacobi structures on a manifold [NdC 98]. But the study of the compatibility of two Jacobi structures on a manifold, can also be done using the Lichnerowicz-Jacobi cohomology. If (M, C, E) is a Jacobi manifold, let us denote by $A^k(M)$ the space of skew-symmetric contravariant tensor fields of order k (k-tensors) on M and define the differential operator

$$\sigma : A^k(M) \to A^{k+1}(M), \quad \sigma(P) = -[C, P] + kE \wedge P. \tag{3}$$

The restriction of σ to the subspace

$$A^k_I(M) = \{P \in A^k(M) : \mathcal{L}_E P = 0\}$$

of the invariant k-tensors with respect to the vector field E is a *cohomology operator* on the Jacobi manifold and the resultant cohomology is called the *Lichnerowicz-Jacobi cohomology* of M [Le-Ma-Pa 97].

Proposition 1.2. *Two Jacobi structures (C_1, E_1) and (C_2, E_2) on a manifold M are compatible if and only if*

$$\sigma_1(C_2) = -\sigma_2(C_1) \quad \textit{and} \quad \sigma_1(E_2) = -\sigma_2(E_1),$$

where $\sigma_i, i = 1, 2$, are the cohomology operators of the Lichnerowicz-Jacobi cohomology of M, with respect to both Jacobi structures.

Proof. A direct computation using Proposition 1.1 and the definition (3) of the cohomology operators $\sigma_i, i = 1, 2$, gives the desired result. □

In [NdC 98] it was proved that two conformally equivalent Jacobi structures on M are compatible. Another way of obtaining compatible Jacobi structures uses the Lie derivative on the direction of some vector field.

Proposition 1.3. *Let X be a vector field on the Jacobi manifold (M, C, E) such that*

$$\mathcal{L}_X(\mathcal{L}_X C) = 0 \quad \textit{and} \quad \mathcal{L}_X(\mathcal{L}_X E) = 0.$$

Then the pair $(C_1, E_1) = (\mathcal{L}_X C, \mathcal{L}_X E)$ defines a new Jacobi structure on M which is compatible with (C, E).

Proof. With $E_1 = \mathcal{L}_X E = [X, E]$, we have $\mathcal{L}_{E_1} C = -\mathcal{L}_E(\mathcal{L}_X C)$, that is

$$\mathcal{L}_{E_1} C + \mathcal{L}_E C_1 = 0. \tag{4}$$

So,

$$\begin{aligned}\mathcal{L}_{E_1}C_1 &= \mathcal{L}_X(\mathcal{L}_{E_1}C)\\ &= -\mathcal{L}_X(\mathcal{L}_E C_1)\\ &= -\mathcal{L}_{[X,E]}(\mathcal{L}_X C)\\ &= -\mathcal{L}_{E_1}C_1,\end{aligned}$$

and

$$\mathcal{L}_{E_1}C_1 = 0. \tag{5}$$

On the other hand,

$$\begin{aligned}\mathcal{L}_X([C,C]) &= [\mathcal{L}_X C, C] + [C, \mathcal{L}_X C]\\ &= 2[C_1, C]\end{aligned}$$

and

$$\mathcal{L}_X(E \wedge C) = E_1 \wedge C + E \wedge C_1;$$

so,

$$[C_1, C] = E_1 \wedge C + E \wedge C_1. \tag{6}$$

Because $\mathcal{L}_X E_1 = \mathcal{L}_X C_1 = 0$, if we take the Lie derivative on the direction of X of both members of (6), we obtain

$$[C_1, C_1] = 2E_1 \wedge C_1. \tag{7}$$

The equalities (5) and (7) prove that (C_1, E_1) is a Jacobi structure, while (4) and (6) show the compatibility. □

We can also prove the following result.

Proposition 1.4. *Let X_1 and X_2 be two vector fields on the Jacobi manifold (M, C, E) such that*

$$[X_1, X_2] = 0, \quad [X_1, \mathcal{L}_{X_2}C] = 0, \quad [X_1, \mathcal{L}_{X_2}E] = 0$$

and

$$\mathcal{L}_{X_i}(\mathcal{L}_{X_i}C) = 0, \quad \mathcal{L}_{X_i}(\mathcal{L}_{X_i}E) = 0, \quad i = 1, 2.$$

Then

$$(C_1, E_1) = (\mathcal{L}_{X_1}C, \mathcal{L}_{X_1}E) \quad \text{and} \quad (C_2, E_2) = (\mathcal{L}_{X_2}C, \mathcal{L}_{X_2}E)$$

are compatible Jacobi structures on M.

Proof. We only have to check the compatibility. By Proposition 1.3, we know that the Jacobi structures (C, E) and (C_2, E_2) are compatible. Then,

$$[C, C_2] = E \wedge C_2 + E_2 \wedge C \quad \text{and} \quad [E, C_2] + [E_2, C] = 0. \tag{8}$$

If we take the Lie derivative, on the direction of X_1, of both members of equalities (8), we obtain

$$[C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1 \quad \text{and} \quad [E_1, C_2] + [E_2, C_1] = 0. \tag{9}$$

□

2. Homogeneous Poisson-Nijenhuis manifolds

Let M be a manifold endowed with two Jacobi structures (C_1, E_1) and (C_2, E_2). Take the corresponding homogeneous Poisson structures on $\mathbb{R} \times M$, $(\Lambda_i = \exp(-t)(C_i + \frac{\partial}{\partial t} \wedge E_i), \frac{\partial}{\partial t})$, $i = 1, 2$.

Proposition 2.1. [NdC 98] *The Jacobi structures (C_1, E_1) and (C_2, E_2) on M are compatible if and only if Λ_1 and Λ_2 are compatible Poisson tensors on $\mathbb{R} \times M$.*

Using this result, we want to study the relationship between two compatible Jacobi structures on M and their associated (compatible) homogeneous Poisson strucutres on $\mathbb{R}\times M$, in the case where these Poisson structures are related by a Nijenhuis tensor.

Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ such that $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$. Then, $\bar{N}$ is given by

$$\bar{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g \frac{\partial}{\partial t} \otimes dt, \tag{10}$$

where N is a $(1,1)$-tensor on M, Y is a vector field on M, γ is a 1-form on M and $g \in C^\infty(M, \mathbb{R})$. Reciprocally, if $\bar{N}$ is given by (10), then $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$.

Remarks

1. The image of a vector field on $\mathbb{R} \times M$ of the form $\bar{X} = \exp(-t)(f \frac{\partial}{\partial t} + X)$, where X is a vector field on M and $f \in C^\infty(M, \mathbb{R})$, by the $(1,1)$-tensor given by (10) is a vector field on $\mathbb{R} \times M$ of the same type of $\bar{X}$.
2. If Λ is the homogeneous Poisson tensor on $\mathbb{R} \times M$, given by (2), then $\bar{N}\Lambda$ is a homogeneous bivector on $\mathbb{R} \times M$: $\mathcal{L}_{\frac{\partial}{\partial t}}(\bar{N}\Lambda) = -\bar{N}\Lambda$

Proposition 2.2. *Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ given by (10). Then $\bar{N}$ is a Nijenhuis tensor on $\mathbb{R} \times M$ if and only if*

$$\begin{array}{rl} \text{i)} & \tau(N) = Y \otimes d\gamma; \\ \text{ii)} & \mathcal{L}_N \gamma = g d\gamma; \\ \text{iii)} & \mathcal{L}_Y N = -Y \otimes dg; \\ \text{iv)} & {}^tN(dg) = \mathcal{L}_Y \gamma + g dg. \end{array}$$

Proof. First, remark that if X is a vector field on M, then $\bar{N}X = NX + <\gamma, X> \frac{\partial}{\partial t}$ and that $\bar{N}(\frac{\partial}{\partial t}) = Y + g\frac{\partial}{\partial t}$. If X_1 and X_2 are vector fields on M, then

$$\begin{aligned} \tau(\bar{N})(X_1, X_2) &= \tau(N)(X_1, X_2) + (X_2. <\gamma, X_1> - X_1. <\gamma, X_2> \\ &\quad + <\gamma, [X_1, X_2]>)\bar{N}(\frac{\partial}{\partial t}) + ((NX_1).<\gamma, X_2> - (NX_2).<\gamma, X_1> \\ &\quad - <\gamma, [NX_1, X_2]> - <\gamma, [X_1, NX_2]> - <\gamma, N[X_1, X_2]>)\frac{\partial}{\partial t} \\ &= \tau(N)(X_1, X_2) - d\gamma(X_1, X_2)Y - (g d\gamma(X_1, X_2) - \mathcal{L}_N\gamma(X_1, X_2))\frac{\partial}{\partial t}. \end{aligned}$$

So, $\tau(\bar{N})(X_1, X_2) = 0$ if and only if

$$\tau(N)(X_1, X_2) = (Y \otimes d\gamma)(X_1, X_2) \tag{11}$$

and

$$\mathcal{L}_N \gamma(X_1, X_2) = g d\gamma(X_1, X_2). \tag{12}$$

Let X be a vector field on M. Then,

$$\begin{aligned} \tau(\bar{N})(X, \frac{\partial}{\partial t}) &= [NX, Y] + ((NX).g)\frac{\partial}{\partial t} - (Y. < \gamma, X >)\frac{\partial}{\partial t} \\ &\quad -N[X, Y] - < \gamma, [X, Y] > \frac{\partial}{\partial t} - (X.g)(Y + g\frac{\partial}{\partial t}) \\ &= -(\mathcal{L}_Y N)(X) - (Y \otimes dg)(X) \\ &\quad +(< {}^tNdg, X > - < \mathcal{L}_Y \gamma, X > - < gdg, X >)\frac{\partial}{\partial t}. \end{aligned}$$

So, $\tau(\bar{N})(X, \frac{\partial}{\partial t}) = 0$ if and only if

$$(\mathcal{L}_Y N)(X) = -(Y \otimes dg)(X) \tag{13}$$

and

$$< {}^tNdg - \mathcal{L}_Y \gamma - gdg, X >= 0. \tag{14}$$

Equalities (11), (12), (13) and (14) end the proof. □

Let us take the homogeneous Poisson tensor $\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E)$ on $(\mathbb{R} \times M)$.

Lemma 2.1. *With the notations of Proposition 2.2, $\bar{N}\Lambda = \Lambda\, {}^t\bar{N}$ if and only if*

$$\begin{aligned} &\text{i)} \quad NE = C^{\#}(\gamma) + gE; \\ &\text{ii)} \quad NC - C\, {}^tN = E \otimes Y + Y \otimes E; \\ &\text{iii)} \quad < \gamma, E >= 0. \end{aligned}$$

Proof. Let α be any 1-form on M. Then,

$$\begin{aligned} \bar{N}(\Lambda^{\#}(\alpha)) &= \exp(-t)(\bar{N}(C^{\#}(\alpha) - < \alpha, E > \frac{\partial}{\partial t})) \\ &= \exp(-t)(N(C^{\#}(\alpha)) - < \alpha, E > Y - (< \alpha, C^{\#}(\gamma) > + < \alpha, E > g)\frac{\partial}{\partial t}) \end{aligned}$$

and, taking account that

$${}^t\bar{N}(\alpha) = {}^tN\alpha + < \alpha, Y > dt,$$

we compute

$$\Lambda^{\#}({}^t\bar{N}\alpha) = \exp(-t)(C^{\#}({}^tN\alpha) + < \alpha, Y > E - < \alpha, NE > \frac{\partial}{\partial t})$$

and we conclude that $\bar{N}\Lambda^{\#}(\alpha) = \Lambda^{\#}({}^tN\alpha)$ if and only if

$$< \alpha, NE >=< \alpha, C^{\#}(\gamma) + gE > \tag{15}$$

and

$$(NC^{\#} - Y \otimes E)\alpha = (C^{\#}\,{}^tN + E \otimes Y)\alpha. \tag{16}$$

Equalities (15) and (16) give conditions i) and ii).

On the other hand,

$$\bar{N}(\Lambda^{\#}(dt)) = \exp(-t)(NE+ < \gamma, E > \frac{\partial}{\partial t}) \tag{17}$$

and, because ${}^t\bar{N}(dt) = \gamma + gdt$,

$$\Lambda^{\#}({}^t\bar{N}(dt)) = \exp(-t)(C^{\#}(\gamma) + gE). \tag{18}$$

Once the right members of (17) and (18) are equal, we obtain condition iii). □

Let us take the Magri-Morosi concomitant $\mathcal{R}(\Lambda, \bar{N})$ of Λ and $\bar{N}$, (1).

Lemma 2.2. *With the notations of Proposition* 2.2, $\mathcal{R}(\Lambda, \bar{N}) = 0$ *if and only if*

i) $NE + \mathcal{L}_Y E = gE - C^{\#}(dg)$
ii) $\mathcal{L}_Y C = gC - NC + Y \otimes E$
iii) $(\mathcal{L}_E N)X+ < \gamma, X > E = C^{\#}(i_X d\gamma) + (X.g)E$
iv) $\mathcal{R}(C, N)(df, X) = (X.(Y.f))E - (X.(E.f))Y- < \gamma, X > C^{\#}(df)$,

where X is any vector field on M and f is any function of $C^\infty(M, \mathbb{R})$.

Proof. Let f be any function of $C^\infty(M, \mathbb{R})$. The component of the vector field $\mathcal{R}(\Lambda,\bar{N})(df, \frac{\partial}{\partial t})$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(< df, C^{\#}(dg) + \mathcal{L}_Y E + NE - gE >). \tag{19}$$

The other components (without $\frac{\partial}{\partial t}$) of $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t})$, obtained from the computation of $< dh, \mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t}) >$, for any $h \in C^\infty(M, \mathbb{R})$, are

$$\exp(-t)(< dh, (-\mathcal{L}_Y C)^{\#}(df) + gC^{\#}(df) - N(C^{\#}(df)) + (E.f)Y >). \tag{20}$$

From (19) and (20), we conclude that $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t}) = 0$ gives conditions i) and ii) of the Lemma.

On the other hand, if X is any vector field on M, the component of the vector field $\mathcal{R}(\Lambda, \bar{N})(df, X)$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(< df, -(\mathcal{L}_E N)X- < \gamma, X > E + C^{\#}(i_X d\gamma) + (X.g)E >, \tag{21}$$

while the components without $\frac{\partial}{\partial t}$, obtained from the computation of $< dh, \mathcal{R}(\Lambda, \bar{N})(df, X) >$, for any $h \in C^\infty(M, \mathbb{R})$, are

$$\exp(-t)(< dh, \mathcal{R}(C, N)(df, X) + < \gamma, X > C^\#(df) + (X.(E.f))Y - (X.(Y.f))E > . \tag{22}$$

From (21) and (22), we conclude that $\mathcal{R}(\Lambda, \bar{N})(df, X) = 0$ gives conditions iii) and iv) of the Lemma. □

Theorem 2.1. *Let* $\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E)$ *and* $\bar{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt$, *be respectively the homogeneous Poisson tensor and the Nijenhuis tensor on* $\mathbb{R} \times M$. *Then, the triple* $(\mathbb{R} \times M, \Lambda, \bar{N})$ *is a Poisson-Nijenhuis manifold if and only if conditions* i), ii) *and* iii) *of Lemma* 2.1 *and* i) – iv) *of Lemma 2.2 hold. Moreover, the homogeneous Poisson tensor* $\bar{N}\Lambda$ *is given by* $\exp(-t)(C_1 + \frac{\partial}{\partial t} \wedge E_1)$, *where*

$$(C_1, E_1) = (gC - \mathcal{L}_Y C, gE - C^\#(dg) - \mathcal{L}_Y E) \tag{23}$$

is a Jacobi structure on M, *compatible with* (C, E).

Proof. We only have to find the expressions of C_1 and E_1. For any pair (f, h) of functions on M,

$$< dh, (\bar{N}\Lambda)^\#(df) >= \exp(-t) < dh, NC^\#(df) - (Y \otimes E)(df) > \tag{24}$$

and

$$< dh, (\Lambda^\#({}^t\bar{N})(df) >= \exp(-t) < dh, C^\#({}^tN(df)) + (E \otimes Y)(df) > . \tag{25}$$

Since $\bar{N}\Lambda = \frac{1}{2}(\bar{N}\Lambda + \Lambda\,{}^t\bar{N})$, we obtain from (24) and (25), using conditions ii) of Lemma 2.1 and ii) of Lemma 2.2,

$$\begin{aligned} (\bar{N}\Lambda)(df, dh) &= \exp(-t)(NC - Y \otimes E)(df, dh) \\ &= \exp(-t)(gC - \mathcal{L}_Y C)(df, dh). \end{aligned} \tag{26}$$

Also,

$$< dh, \bar{N}(\Lambda^\#(dt)) >= \exp(-t) < dh, NE > \tag{27}$$

and

$$< dh, \Lambda^\#({}^t\bar{N}(dt)) >= \exp(-t) < dh, C^\#(\gamma) + gE > . \tag{28}$$

From (27) and (28), and taking into account conditions i) of Lemma 2.1 and i) of Lemma 2.2, we conclude that

$$\begin{aligned} (\bar{N}\Lambda)(dt, dh) &= \exp(-t)NE(dt, dh) \\ &= \exp(-t)(gE - C^\#(dg) - \mathcal{L}_Y E)(dt, dh). \end{aligned} \tag{29}$$

□

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References

[KSch-Ma 90] Kosmann-Schwarzbach Y. and Magri F.: *Poisson-Nijenhuis structures.* Ann. Inst. Henri Poincaré **53** (1990), 35.

[Le-Ma-Pa 97] León M. de, Marrero J. C. and Padrón E.: *Lichnerowicz-Jacobi cohomology.* J. Phys. A: Math. Gen. **30** (1997), 6029.

[Lich 78] Lichnerowicz A.: *Les variétés de Jacobi et leurs algèbres de Lie associées.* J. Math. pures et appl. **57** (1990), 453.

[Ma-Mo 84] Magri F. and Morosi C.: *A Geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds.* Quaderno S19, Univ. of Milan, 1984.

[NdC 98] Nunes da Costa J. M.: *Compatible Jacobi manifolds: geometry and reduction.* J. Phys. A: Math. Gen. **31** (1998), 1025.