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Embedding Almost Lie Structures[1]

Paul Popescu **Marcela Popescu**

University of Craiova, Faculty of Mathematics and Informatics
13 Al.I.Cuza St., Craiova, 1100, Romania
e-mail: PaulPopescu@email.com

Abstract. The aim of the paper is to prove that there is a Lie pseudoalgebra which contains a given almost Lie structure and induces its structure. Under some regularity conditions, a very Lie algebroid exists.

1. Introduction

The almost Lie structure is a notion which includes both geometric notions: distributions (arbitrary vector subbundles of the tangent bundle) and Lie algebroids. This definition is obtained removing from the definition of a Lie algebroid the compatibility condition of the brackets and the Jacobi identity. The algebraic version of a Lie algebroid is a Lie pseudoalgebra and the algebraic version of an almost Lie structure is a pre-infinitesimal module. Using derivations, an interpretation of almost Lie structures is given in [5].

The geometric distributions of constant rank, generally non-integrable ones, are studied in subriemannian geometry, using additionally a Riemannian metric on the underlying vector bundle (see for example [3] and the bibliography therein). The origin of this notion can be found in the non-holonomic spaces of G. Vrănceanu [12]. Almost Lie structures allow singular non-integrable distributions on the anchor image, which are not always integrable.

The Lie algebroid is associated firstly by Pradines [9] with a Lie groupoid and it is studied by many other authors (see for example [1, 10] and the bibliography therein).

The algebraic versions of Lie algebroids and almost Lie structures are the Lie pseudoalgebras [9, 2] and preinfinitesimal modules [6] respectively.

Using a linear R-connection on an almost Lie structure, in the second section the derived almost Lie structures are constructed. In the third section some integrability and regularity

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conditions are studied. Using the inductive limit of the derived bundles, a Lie pseudoalgebra is obtained in the fourth section. In the same way as a geometric distribution is defined by mean of the Lie algebroid of the tangent bundle, it is proved that this Lie pseudoalgebra contains the given almost Lie structure and induces its structure. Under some regularity conditions, a very Lie algebroid exists. Notice that in [8] it is proved that the result is purely algebraic: if a pre-infinitesimal module allows a linear connection, then there is a Lie pseudoalgebra which contains it as a pre-infinitesimal submodule and induces its structure; moreover, there is an association of a Lie pseudoalgebra with a pre-infinitesimal module which has a functorial property.

2. The derived almost Lie structures

A *relative tangent space* (*RTS*) is a couple (θ, D), where $\theta = (R, q, M)$ is a vector bundle and $D : \theta \to \tau M$ is a vector bundle morphism, called an *anchor*. A triple (θ, D, L) is an *almost Lie structure* (ALS) if (θ, D) is an RTS and $L : \Gamma(\theta) \times \Gamma(\theta) \to \Gamma(\theta)$ is an *almost Lie map* (or a *bracket*), i.e. an $\mathbb{R}$-linear and skew symmetric map which enjoys the following property: $L(X, f \cdot Y) = (DX)(f) \cdot Y + f \cdot L(X, Y)$, $(\forall)X, Y \in \Gamma(\theta)$ and $f \in \mathcal{F}(M)$ ([4]). We denote by $\mathcal{J}(X, Y, Z) = L(L(X, Y), Z) + L(L(Y, Z), X) + L(L(Z, X), Y)$, which we call the *Jacobiator* of L.

If (θ, D, L) is an ALS, then

$$\mathcal{D} : \Gamma(\theta) \times \Gamma(\theta) \to \mathcal{X}(M),\ \mathcal{D}(X, Y) = [D(X), D(Y)] - D\left(L\left(X, Y\right)\right)$$

is an anchor map for $\theta \wedge \theta$, i.e. $(\theta \wedge \theta, \mathcal{D})$ is an RTS.

An *algebroid* is an ALS which has $\mathcal{D} = 0$ and a *Lie algebroid* is an algebroid which has $\mathcal{J} = 0$ (see [7]). The algebraic version of the Lie algebroid is the Lie pseudoalgebra [2]. A *Lie pseudoalgebra* over the real algebra $\mathcal{A}$ is an $\mathcal{A}$-module $\mathcal{M}$ together with an *anchor* $a : \mathcal{M} \to \mathcal{X}(M)$, which is a module morphism, and a bracket $[\cdot, \cdot] : \mathcal{M} \times \mathcal{M} \to \mathcal{D}er(\mathcal{A})$ which is an skew symmetric $\mathbb{R}$-bilinear map which enjoys the properties:

$$[X, u \cdot Y] = a(X)(u) \cdot Y + u \cdot [X, Y],$$

$$[a(X), a(Y)] = a\left([X, Y]\right), \tag{1}$$

$$\mathcal{J}(X, Y, Z) = \left[\left[X, Y\right], Z\right] + \left[\left[Y, Z\right], X\right] + \left[\left[Z, X\right], Y\right] = 0, \tag{2}$$

$(\forall)X, Y, Z \in \mathcal{M}, u \in \mathcal{A}$.

Consider now an ALS (θ, D, L) and a vector bundle ξ over the same base M as the base of θ. A *linear R-connection* on ξ (related to the RTS (θ, D)) is a map $\nabla : \Gamma(\theta) \times \Gamma(\xi) \to \Gamma(\xi)$, $\nabla(X, u) \overset{not.}{=} \nabla_X u$, such that the Koszul conditions hold, i.e. it is $\mathcal{F}(M)$-linear in the first argument:

$$\nabla_{X+Y} u = \nabla_X u + \nabla_Y u,\ \nabla_{fX} u = f \nabla_X u, \tag{3}$$

$(\forall)X, Y \in \Gamma(\theta),\ u \in \Gamma(\xi),\ f \in \mathcal{F}(M)$, and a derivation in the second argument:

$$\nabla_X\left(u + v\right) = \nabla_X u + \nabla_X v,\ \nabla_X\left(fu\right) = D(X)(f)u + f \nabla_X u, \tag{4}$$

$(\forall)X \in \Gamma(\theta)$, $u, v \in \Gamma(\xi)$, $f \in \mathcal{F}(M)$. Using the RTS $(\theta.D)$, a linear R-connection on θ is simply called a linear R-connection on θ. If an ALS is defined, the *curvature* of ∇ is defined by

$$\nabla_{X\wedge Y}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{L(X,Y)}Z \tag{5}$$

and for a linear R-connection on θ the *torsion* of ∇ is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - L(X,Y),\ (\forall)X,Y,Z \in \Gamma(\theta). \tag{6}$$

If (θ, D) is an RTS, it is easy to see that a linear R-connection ∇ on θ defines an almost Lie map L on θ according to the formula $L(X,Y) = \nabla_X Y - \nabla_Y X$, $(\forall)X,Y \in \Gamma(\theta)$.

While the torsion of a linear R-connection is a tensor on θ, the curvature is a linear R-connection on θ, related to the RTS $(\theta\wedge\theta, \mathcal{D})$. The R-linear connection $\nabla_{X\wedge Y}(U\wedge V) = \nabla_{X\wedge Y}U\wedge V + U\wedge\nabla_{X\wedge Y}V$ on $\theta\wedge\theta$ is entirely induced by ∇. The formula:

$$\mathcal{L}(X\wedge Y, U\wedge V) = \nabla_{X\wedge Y}(U\wedge V) - \nabla_{U\wedge V}(X\wedge Y), \tag{7}$$

$(\forall)X,Y,U,V \in \Gamma(\theta)$, allows us to define an almost Lie map on $\theta\wedge\theta$. Consider now the vector bundle $\mathcal{D}(\theta) = \theta\oplus(\theta\wedge\theta)$, which we call the *derived bundle* of θ, and let ∇ be a linear R-connection on θ. Then there is an anchor $\mathcal{D}' = D + \mathcal{D}$ on $\mathcal{D}(\theta)$, defined by $\mathcal{D}'(X + (Y\wedge Z)) = D(X) + \mathcal{D}(Y,Z)$. We define an almost Lie map $\mathcal{L}'$ on $\mathcal{D}(\theta)$ by:

$$\mathcal{L}'(X,Y) = L(X,Y) + X\wedge Y, \tag{8}$$

$$\mathcal{L}'(X\wedge Y, Z) = \mathcal{L}'(Z, X\wedge Y) = \nabla_{X\wedge Y}Z - \nabla_Z(X\wedge Y),$$

$$\mathcal{L}'(X\wedge Y, Z\wedge U) = \nabla_{X\wedge Y}(Z\wedge U) - \nabla_{Z\wedge U}(X\wedge Y).$$

$(\forall)X,Y,Z,U \in \Gamma(\theta)$. It is easy to see that the first term of the right hand side of the first and the second above formulas denotes the θ-component and the second term denotes the $\theta\wedge\theta$-component. We call the ALS $(\mathcal{D}(\theta), \mathcal{D}', \mathcal{L}')$ as the *derived* ALS of (θ, D, L), given by ∇.

Proposition 1. *The following properties hold true:*

1. $\mathcal{D}'(X,Y) = 0$, $(\forall)X,Y \in \Gamma(\theta)$.
2. *If the linear R-connection ∇ has no torsion, then $\mathcal{J}'(X,Y,Z) = 0$, $(\forall)X,Y,Z \in \Gamma(\theta)$, where $\mathcal{J}'$ denotes the Jacobiator of $\mathcal{L}'$.*

Proof. We have:
$\mathcal{D}'(X,Y) = [\mathcal{D}'(X), \mathcal{D}'(Y)] - \mathcal{D}'(\mathcal{L}'(X,Y)) = [D(X), D(Y)] - D(L(X,Y)) - \mathcal{D}(X,Y) = 0$.

For the second equality we have:
$\mathcal{L}'(\mathcal{L}'(X,Y),Z) = \mathcal{L}'(L(X,Y) + X\wedge Y, Z) = L(L(X,Y),Z) + L(X,Y)\wedge Z + \nabla_{X\wedge Y}Z - \nabla_Z(X\wedge Y) = L(L(X,Y),Z) + L(X,Y)\wedge Z + \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{L(X,Y)}Z - \nabla_Z X\wedge Y - X\wedge\nabla_Z Y$ $= -\nabla_Z L(X,Y) + \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_X Y\wedge Z - \nabla_Y X\wedge Z + \nabla_Z X\wedge Y - X\wedge\nabla_Z Y$. So, $\mathcal{L}'(\mathcal{L}'(X,Y),Z) = -\nabla_Z L(X,Y) + \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_X Y\wedge Z - \nabla_Y X\wedge Z + \nabla_Z X\wedge Y - X\wedge\nabla_Z Y$.

Writing the analogous expressions for $\mathcal{L}'(\mathcal{L}'(Y,Z),X)$ and $\mathcal{L}'(\mathcal{L}'(Z,X),Y)$ and summing the second equality is obtained. □

The ALS $(\mathcal{D}(\theta), \mathcal{D}', \mathcal{L}') = (\mathcal{D}^{(1)}(\theta), \mathcal{D}^{(1)}, \mathcal{L}^{(1)})$ and the linear R-connection ∇ on θ define a torsion free linear R-connection $\overline{\nabla}$ on $\mathcal{D}(\theta)$, according to the formulas:

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \frac{1}{2} X \wedge Y,\ \overline{\nabla}_X (Y \wedge Z) = \nabla_X (Y \wedge Z), \\ \overline{\nabla}_{X \wedge Y} Z &= \nabla_{X \wedge Y} Z,\ \overline{\nabla}_{X \wedge Y} (Z \wedge U) = \nabla_{X \wedge Y} (Z \wedge U), \end{aligned}$$

$(\forall) X, Y, Z, U \in \Gamma(\theta)$. The above R-connection can be used in order to define the *second derived ALS* of (θ, D, L) as being the derived ALS $(\mathcal{D}^{(2)}(\theta), \mathcal{D}^{(2)}, \mathcal{L}^{(2)})$ of the (first) derived ALS $(\mathcal{D}^{(1)}(\theta), \mathcal{D}^{(1)}, \mathcal{L}^{(1)})$.

The order $p \in \mathbb{N}$ *derived ALS* (or the *p-derived ALS*), denoted by $(\mathcal{D}^{(p)}(\theta)$, $\mathcal{D}^{(p)}$, $\mathcal{L}^{(p)})$, is obtained inductively as the derived ALS of the ALS $(\mathcal{D}^{(p-1)}(\theta)$, $\mathcal{D}^{(p-1)}$, $\mathcal{L}^{(p-1)})$.

3. First regularity conditions for relative tangent spaces and almost Lie structures

Definition 1. *An* RTS (θ, D) *is closed if the following condition is satisfied: for every* X, $Y \in \Gamma(\theta)$ *there is* $Z \in \Gamma(\theta)$ *such that* $[D(X), D(Y)] = D(Z)$.

Using a partition of unity it can be shown that an RTS which is locally closed (i.e. for every $x \in M$ there is an open neighborhood U of x such that the restriction θ_U is closed) is necessarily closed.

Proposition 2. *Let* (θ, D, L) *be an ALS such that the RTS* (θ, D) *is closed and there is a vector subbundle* $\mu \subset \theta$ *which enjoys the property* $\mathcal{D}(X, Y) = 0$, $(\forall) X, Y \in \Gamma(\mu)$. *Then there is an algebroid* (θ, D, L') *such that* $L'(X, Y) = L(X, Y)$, $(\forall) X, Y \in \Gamma(\mu)$.

Proof. It suffices to show that the property holds for trivial bundles, then using a partition of unity a global algebroid can be constructed. Thus consider a base $\{X_1, \ldots, X_k\}$ of the $\mathcal{F}(M)$-module $\Gamma(\mu)$, a base $\{X_1, \ldots, X_k, Y_1, \ldots, Y_l\}$ of the $\mathcal{F}(M)$-module $\Gamma(\mu)$ and the local functions $\{f^p_{i\alpha},\ f^\gamma_{i\alpha},\ f^p_{\alpha\beta},\ f^\gamma_{\alpha\beta}\}$ such that $[D(X_i), D(X_j)] = D(L(X, Y))$, $[D(X_i), D(Y_\alpha)] = D(f^p_{i\alpha} X_p + f^\gamma_{i\alpha} Y_\gamma)$, $[D(Y_\alpha), D(Y_\beta)] = D(f^p_{\alpha\beta} X_p + f^\gamma_{\alpha\beta} Y_\gamma)$. Using the formulas: $L'(u^i X_i, v^j X_j) = L(u^i X_i, v^j X_j)$, $L'(u^i X_i, v^\alpha Y_\alpha) = -L'(v^\alpha Y_\alpha,\ u^i X_i) =$ $(u^i D(X_i)(v^\alpha) +\ u^i v^\gamma f^\alpha_{i\gamma}) Y_\alpha -\ (v^\alpha D(Y_\alpha)(u^i) +\ u^j v^\alpha f^i_{j\alpha}) X_i$, $L'(u^\alpha Y_\alpha, v^\beta Y_\beta) = (u^\beta D(Y_\beta)(v^\alpha) - v^\beta D(Y_\beta)(u^\alpha) +\ u^\beta v^\gamma f^\alpha_{\beta\gamma}) Y_\alpha + u^\beta v^\gamma f^i_{\beta\gamma} X_i$, an AL map L' can be defined on μ. □

Corollary 1. *If an RTS* (θ, D) *is closed then there is an AL map* L *such that* (θ, D, L) *is an algebroid.*

Using Propositions 1 and 2 the following result holds:.

Proposition 3. *If an ALS* (θ, D, L) *has a closed RTS* (θ, D), *then the derived ALS* $(\mathcal{D}(\theta), \mathcal{D}'$, $\mathcal{L}')$ *has a closed RTS* $(\mathcal{D}(\theta), \mathcal{D}')$ *and there is an AL map* $\mathcal{L}''$ *on* $\mathcal{D}(\theta)$ *such that* $(\mathcal{D}(\theta), \mathcal{D}', \mathcal{L}'')$ *is an algebroid and moreover* $\mathcal{L}''(X, Y) = \mathcal{L}'(X, Y)$, $\mathcal{J}''(X, Y, Z) = 0$, $(\forall) X, Y, Z \in \Gamma(\theta)$.

The following result concerning high derived ALS follows:

Theorem 1. *Let (θ, D, L) be an ALS such that the n-derived ALS $(\mathcal{D}^{(n)}(\theta),\ \mathcal{D}^{(n)},\ \mathcal{L}^{(n)})$ has a closed RTS $(\mathcal{D}^{(n)}(\theta), \mathcal{D}^{(n)})$. Then there is an AL map $\mathcal{L}''$ on $\mathcal{D}^{(n+1)}(\theta)$ such that $(\mathcal{D}^{(n+1)}(\theta), \mathcal{D}^{(n+1)}, \mathcal{L}'')$ is an algebroid and $\mathcal{L}^{(n+1)}(X,Y) = \mathcal{L}''(X,Y)$, $\mathcal{J}''(X,Y,Z) = 0$, $(\forall)X,Y,Z \in \Gamma(\mathcal{D}^{(n)}(\theta))$.*

As in the hypothesis of the above Theorem, for an ALS (θ, D, L) consider the $\mathcal{F}(M)$-submodule $\mathcal{S}''$ of $\mathcal{D}^{(n+1)}(\theta)$ generated by $\{\mathcal{J}''(X,Y,Z) \mid (\forall)X,Y,Z \in \Gamma(\mathcal{D}^{(n+1)}(\theta))\}$. The quotient module $\mathcal{D}_0^{(n+1)}(\theta) = \mathcal{D}^{(n+1)}(\theta)/\mathcal{S}''$ has a Lie pseudo-algebra structure and according to the above theorem the canonical projection $\mathcal{D}^{(n+1)}(\theta) \to \mathcal{D}^{(n+1)}(\theta)/\mathcal{S}''$ restricted to $\mathcal{D}^{(k)}(\theta)$, $0 \le k \le n$ is injective. Let us denote by $[\cdot,\cdot]^0$ the bracket of this Lie pseudo-algebra. Consider the following sequence of vector spaces: $W^1 = \Gamma(\theta)$, $W^2 = [W^1, W^1]^0$ and $W^k = [W^1, W^{k-1}]^0$, $k \le n+1$. Since $[X, fX] = X(f) \cdot X$, $(\forall)X \in \mathcal{X}(M), f \in \mathcal{F}(M)$, it follows that $W^1 \subset W^2 \subset \cdots \subset W^{n+1}$. Let us denote by $W_1 = W^1$, $W_2 = W^2/W^1$, ..., $W_{n+1} = W^{n+1}/W^n$, $W_i = \{0\}$, $i > n+1$. Then $W_* = \bigoplus_{i=1}^{\infty} W_i$ inherits a graded Lie algebra structure, i.e. $[W_i, W_j] \subset W_{i+j}$. The shaves restricted to every point induce a Lie algebra structure of a Carnot group (see [3]). As a regularity condition, it can be asked that these Lie algebras have the same dimension in every point, defining a Lie algebra bundle structure (compare with [3]).

Given an RTS (θ, D) consider the following sequence of sheaves:

$$\mathcal{X}^1 = D(\Gamma(\theta)),\ \mathcal{X}^2 = \left[\mathcal{X}^1, \mathcal{X}^1\right],\ \ldots,\ \mathcal{X}^k = \left[\mathcal{X}^1, \mathcal{X}^{k-1}\right], \ldots.$$

Notice that $\mathcal{X}^1 \subset \mathcal{X}^2 \subset \cdots \subset \mathcal{X}^k \subset \cdots$ as local sheaves.

Proposition 4. *If (θ, D, L) is an ALS, then $\mathcal{D}^{(n)}\left(\Gamma\left(\mathcal{D}^{(n)}(\theta)\right)\right) = \mathcal{X}^{n+1}$, $(\forall)n \ge 1$. The n-derived ALS $(\mathcal{D}^{(n)}(\theta),\ \mathcal{D}^{(n)},\ \mathcal{L}^{(n)})$ has a closed RTS $(\mathcal{D}^{(n)}(\theta), \mathcal{D}^{(n)})$ iff $\mathcal{X}^{n+1}$ is involutive (i.e., it is closed under brackets).*

Proof. According to the definition of the derived ALS it suffices to show that the brackets of k arbitrary local fields from $D(\Gamma(\theta))$ belong to $\mathcal{X}^k$. It can be proved that $[\mathcal{X}^p, \mathcal{X}^q] \subset \mathcal{X}^{p+q}$, then the general case follows using the induction according the length k and the Jacobi identity for vector fields. □

We give now an example.

Consider the vector fields of $\mathcal{X}(\mathbb{R}^4)$:

$$X_1 = x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2},\ X_2 = x^2 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^4},$$

which define a trivial vector bundle θ having the base $\mathbb{R}^2$ and the fibre type $\mathbb{R}^2$. The inclusion of the base $\{X_1, X_2\}$ in $\mathcal{X}(\mathbb{R}^4)$ defines an anchor $D : \theta \to \tau\mathbb{R}^2$. Notice that the distribution defined by $\{X_1, X_2\}$ is singular. A straightforward computation shows that:

$$X_{12} = [X_1, X_2] = -x^1 \frac{\partial}{\partial x^1} + \left(x^3 - x^2\right) \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4},$$

$$X_{112}=[X_1,[X_1,X_2]]=-2x^4\frac{\partial}{\partial x^1}-2x^3\frac{\partial}{\partial x^2}=-2X_1,$$

$$X_{212}=[X_2,[X_1,X_2]]=\left(x^2-x^3\right)\frac{\partial}{\partial x^2}+\left(2x^2-x^3\right)\frac{\partial}{\partial x^3}+2x^1\frac{\partial}{\partial x^4},$$

$$X_{1112}=[X_1,[X_1,[X_1,X_2]]]=0,\ X_{2112}=[X_2,[X_1,[X_1,X_2]]]=2X_{12},$$

$$X_{1212}=[X_1,[X_2,[X_1,X_2]]]=2X_{12},$$

$$X_{2212}=[X_2,[X_2,[X_1,X_2]]]=\left(x^3-x^2\right)\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3},$$

$$X_{11112}=X_{21112}=0,\ X_{11212}=X_{12112}=-4X_1,$$

$$X_{21212}=X_{22112}=2X_{212},\ X_{12212}=-2x^3\frac{\partial}{\partial x^2},$$

$$X_{22212}=\left(x^2-x^3\right)\frac{\partial}{\partial x^2}+\left(2x^2-x^3\right)\frac{\partial}{\partial x^3},$$

$$\begin{aligned}X_{111112}&=X_{121112}=X_{112112}=X_{111212}=\\X_{112212}&=X_{211112}=X_{221112}=0,\end{aligned}$$

$$X_{122112}=X_{121212}=X_{212112}=X_{211212}=4X_{12},$$

$$X_{122212}=X_{222112}=X_{221212}=X_{212212}=2X_{222212}=2X_{22212}.$$

Thus $\mathcal{X}^6$ is involutive. According to Proposition 4 it follows that the RTS $(\mathcal{D}^{(5)}(\theta),\,\mathcal{D}^{(5)})$ is closed.

4. A Lie pseudoalgebra associated with an almost Lie structure

Using inductively the n-derived ALS' $(\mathcal{D}^{(n)}(\theta),\,\mathcal{D}^{(n)},\,\mathcal{L}^{(n)})$ we can consider $(\mathcal{D}^{(\infty)}(\theta),\,\mathcal{D}^{(\infty)},\,\mathcal{L}^{(\infty)})$, where $\mathcal{D}^{(\infty)}(\theta)$ is a vector bundle which has an infinite dimensional fibre. We prove that the submodule of the finite degree sections on this vector bundle is a Lie pseudoalgebra over $\mathcal{F}(M)$. In fact, this Lie pseudoalgebra over $\mathcal{F}(M)$ is a sheave of modules. Adding some regularity conditions, a Lie algebroid can be obtained.

Firstly we prove:

Proposition 5. *Consider an ALS (θ,D,L) and a linear R-connection ∇ on θ which is torsion free. Let us denote by $\theta\oplus(\theta\wedge\theta)$ the first derived bundle, as $(\theta\oplus(\theta\wedge\theta))\oplus((\theta\oplus(\theta\wedge\theta))\,\bar{\wedge}\,(\theta\oplus(\theta\wedge\theta)))$ the second derived bundle and by $\bar{\nabla}$ the torsion free R-linear connection induced on the second derived bundle. Then $\bar{\nabla}_{X\bar{\wedge}Y}Z=0,(\forall)X,Y\in\Gamma(\theta)$.*

Proof. $\bar{\nabla}_{X\bar{\wedge}Y}Z=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{\bar{L}(X,Y)}Z=\bar{\nabla}_X\left(\nabla_YZ+\frac{1}{2}Y\wedge Z\right)-\bar{\nabla}_Y\left(\nabla_XZ+\frac{1}{2}X\wedge Z\right)-\bar{\nabla}_{L(X,Y)+X\wedge Y}Z=\nabla_X\nabla_YZ+\frac{1}{2}X\wedge\nabla_YZ+\frac{1}{2}\nabla_XY\wedge Z+\frac{1}{2}Y\wedge\nabla_XZ-\nabla_Y\nabla_XZ+\frac{1}{2}Y\wedge\nabla_XZ+\frac{1}{2}\nabla_YX\wedge Z+\frac{1}{2}X\wedge\nabla_YZ-\nabla_{L(X,Y)}Z-\frac{1}{2}L(X,Y)\wedge Z-\nabla_{X\wedge Y}Z=0.$ □

For $X,Y\in\Gamma(\theta)$ let us denote by $D=\mathcal{D}^{(0)}$, $L=\mathcal{L}^{(0)}$, $\mathcal{D}=\mathcal{D}^{(1)}$, $\mathcal{L}=\mathcal{L}^{(1)}$ and $\bar{D}=\mathcal{D}^{(2)}$, $\bar{L}=\mathcal{L}^{(2)}$ the anchors and the almost Lie maps on $\theta=\mathcal{D}^{(0)}(\theta)$, $\mathcal{D}(\theta)=\mathcal{D}^{(1)}(\theta)$ and $\mathcal{D}^{(2)}(\theta)$ respectively and by $\wedge_1=\wedge$ and $\wedge_2=\bar{\wedge}$. Then $\mathcal{L}^{(2)}(X,Y)=\mathcal{L}^{(0)}(X,Y)+X\wedge_1Y+X\wedge_2Y$.

According to Proposition 1 we have $\mathcal{D}^{(2)}(X \wedge_2 Y) = \mathcal{D}^{(1)}(X,Y) = 0$ and $\mathcal{D}^{(2)}(\mathcal{L}^{(2)}(X,Y)) = \left[\mathcal{D}^{(0)}(X), \mathcal{D}^{(0)}(Y)\right], (\forall)X,Y \in \Gamma(\theta)$.

For $X,Y \in \Gamma(\theta)$ we have $\mathcal{L}^{(\infty)}(X,Y) = L(X,Y) + X \wedge_1 Y + \cdots + X \wedge_n Y + \cdots$. Generally, for $0 \leq p \leq q \leq r$ and $X_p \in \Gamma\left(\mathcal{D}^{(p)}(\theta)\right)$, $X_q \in \Gamma\left(\mathcal{D}^{(q)}(\theta)\right)$ we have:

$$\begin{aligned}\mathcal{L}^{(\infty)}(X_p,Y_q) = \mathcal{L}^{(q)}(X_p,Y_q) + X_p \wedge_{q+1} Y_q + \cdots + X_p \wedge_n Y_q + \cdots = \\ \mathcal{L}^{(r)}(X_p,Y_q) + X_p \wedge_{r+1} Y_q + \cdots + X_p \wedge_n Y_q + \cdots. \qquad (9)\end{aligned}$$

According to the first statement of Proposition 1 we have $\mathcal{D}^{(n)}(X_p,Y_q) = 0$ for $n > \max(p,q)$, thus $\mathcal{D}^{(n)}(X_p \wedge_n Y_q) = 0$ for $n > 1 + \max(p,q)$.

We denote as $\Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ the $\mathcal{F}(M)$-submodule of sections $X_\infty = X_0 + X_1 + \cdots \in \Gamma\left(\mathcal{D}^{(\infty)}(\theta)\right)$ which has the property that there is an $n \in \mathbb{N}$ such that $\mathcal{D}^{(p)}(X_p) = 0$, $(\forall)p > n$; the smallest n which has this property is called the *degree* of X_∞ and it is denoted as $\deg X_\infty$. The anchor of X_∞ is $\mathcal{D}^{(\infty)}(X_\infty) = \sum_{i=0}^{n} \mathcal{D}^{(i)}(X_i)$, where $n = \deg X_\infty$. Notice that

$$\mathcal{L}^{(\infty)}(X_\infty, Y_\infty) = \sum_{p,q\in\mathbb{N}} \mathcal{L}^{(\infty)}(X_p,Y_q). \qquad (10)$$

Proposition 6. *If $X_\infty, Y_\infty \in \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ then $\mathcal{L}^{(\infty)}(X_\infty,Y_\infty) \in \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ and* $\deg\left(\mathcal{L}^{(\infty)}(X_\infty,Y_\infty)\right) \leq \max(\deg(X_\infty), \deg(X_\infty))$.

Proof. It suffices to prove that $\mathcal{D}^{(\infty)}\left(\mathcal{L}^{(\infty)}(X_p,Y_q)\right) = 0$ for $p,q > \max(\deg(X_\infty), \deg(X_\infty))$. Indeed, for a sufficiently large $n \in \mathbb{N}$ we have $\mathcal{D}^{(\infty)}\left(\mathcal{L}^{(\infty)}(X_p,Y_q)\right) = \mathcal{D}^{(n)}\left(\mathcal{L}^{(n)}(X_p,Y_q)\right) = \left[\mathcal{D}^{(n)}(X_p), \mathcal{D}^{(n)}(X_q)\right] = 0$. □

Proposition 7. *If $X_\infty, Y_\infty \in \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ then*

$$\mathcal{D}^{(\infty)}\left(\mathcal{L}^{(\infty)}(X_\infty,Y_\infty)\right) = \left[\mathcal{D}^{(\infty)}(X_\infty), \mathcal{D}^{(\infty)}(Y_\infty)\right].$$

Proof. Fix $p,q \in \mathbb{N}$. For a sufficiently large $n \in \mathbb{N}$ we have $\mathcal{D}^{(\infty)}(\theta)\left(\mathcal{L}^{(\infty)}(X_p,Y_q)\right) = \mathcal{D}^{(n)}\left(\mathcal{L}^{(n)}(X_p,Y_q)\right) = \left[\mathcal{D}^{(n)}(X_p), \mathcal{D}^{(n)}(X_q)\right]$. According to (10) the conclusion follows. □

For an $X_\infty \in \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ we denote as $N(X_\infty)$ the set of $n \in \mathbb{N}$ such that $X_k = 0$, $(\forall)0 \leq k \leq n$. It is obvious that $X_\infty = 0$ iff $N(X_\infty) = \mathbb{N}$.

Proposition 8. *If $X_\infty, Y_\infty, Z_\infty \in \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)$ then $\mathcal{J}^{(\infty)}(X_\infty,Y_\infty,Z_\infty) = 0$.*

Proof. Since $\mathcal{J}^{(\infty)}$ is additive, it suffices to show that $\mathcal{J}^{(\infty)}(X_p,Y_q,Z_r) = 0$, $(\forall)p,q,r \in \mathbb{N}$. We prove in fact that every $n \in \mathbb{N}$ is in $N(\mathcal{J}^{(\infty)}(X_p,Y_q,Z_r))$.

For $n > 1 + \max(p,q,r)$ consider $\mathcal{L}^{(\infty)}(X_p,X_q) = \mathcal{L}^{(n)}(X_p,X_q) + \mathcal{L}'_{n+1}(X_p,X_q)$, where $\mathcal{L}'_{n+1}(X_p,X_q) = X_p \wedge_{n+1} X_q + \cdots$. It is clear that $n \in N\left(\mathcal{L}'_{n+1}(X_p,X_q)\right)$. We have:

$$\begin{aligned}\mathcal{L}^{(\infty)}\left(\mathcal{L}^{(\infty)}(X_p,X_q), X_r\right) &= \mathcal{L}^{(\infty)}\left(\mathcal{L}^{(n)}(X_p,X_q), X_r\right) + \\ +\mathcal{L}^{(\infty)}\left(\mathcal{L}'_{n+1}(X_p,X_q), X_r\right) &= \mathcal{L}^{(n)}\left(\mathcal{L}^{(n)}(X_p,X_q), X_r\right) + \\ &\quad \mathcal{L}''_{n+1}\left(\mathcal{L}^{(n)}(+X_p,X_q), X_r\right) + \mathcal{L}^{(\infty)}\left(\mathcal{L}'_{n+1}(X_p,X_q), X_r\right),\end{aligned}$$

where $\mathcal{L}''_{n+1}\left(\mathcal{L}^{(n)}\left(X_p, X_q\right), X_r\right) = \left(\mathcal{L}^{(n)}\left(X_p, X_q\right)\right) \wedge_{n+1} X_r + \cdots$. It follows that $n \in N\left(\mathcal{L}''_{n+1}\left(\mathcal{L}^{(n)}_{n+1}\left(X_p, X_q\right), X_r\right)\right)$.

Using the construction of $\mathcal{L}^{(\infty)}$, the definition of $\mathcal{L}'_{n+1}\left(X_p, X_q\right)$ and Proposition 5, the fact that $n \in \left(\mathcal{L}^{(\infty)}\left(\mathcal{L}'_{n+1}\left(X_p, X_q\right), X_r\right)\right)$ follows. Using Proposition 1 we have $\mathcal{J}^{(n)}(X_p, Y_q, Z_r) = 0$. It follows that $n \in N\left(\mathcal{J}^{(\infty)}(X_p, Y_q, Z_r)\right)$, thus $\mathcal{J}^{(\infty)}(X_p, Y_q, Z_r) = 0$. □

The following result follows:

Theorem 2. *The couple* $\left(\mathcal{F}(M), \Gamma_0\left(\mathcal{D}^{(\infty)}(\theta)\right)\right)$ *is a Lie pseudoalgebra.*

For a $p \in \mathbb{N}$, consider the following sequence of sheaves:

$$\mathcal{X}^1 = \Gamma((\mathcal{D}^{(p)}(\theta)),\ \mathcal{X}^2 = \mathcal{L}^{(\infty)}\left(\mathcal{X}^1, \mathcal{X}^1\right),\ \ldots,\ \mathcal{X}^k = \mathcal{L}^{(\infty)}\left(\mathcal{X}^{k-1}, \mathcal{X}^{k-1}\right), \ldots .$$

We have:

$$\mathcal{X}^1 \subset \mathcal{X}^2 \subset \cdots \subset \mathcal{X}^k \subset \cdots \tag{11}$$

as local sheaves.

Proposition 9. *If the sequence* (11) *is stationary, i.e.* $\mathcal{X}^n = \mathcal{X}^{n+1} = \cdots = \mathcal{X}_0$ *for some* $n, p \in \mathbb{N}$, *then there is an algebroid* $(\xi, a, [\cdot,\cdot])$ *over the same base* M *as* θ, *which has the module of sections isomorphic (as Lie pseudoalgebras) with* $(\mathcal{X}_0, \mathcal{D}^{(\infty)}, \mathcal{L}^{(\infty)})$.

Proof. In the case when the sequence is stationary, using (9) it follows that $\mathcal{X}_0$ is finite generated and according to a theorem of Swan [11] we have $\mathcal{X}_0 = \Gamma(\xi)$ for a suitable vector bundle ξ over the base M. In this case ξ has a Lie algebroid structure inherited from $\mathcal{X}_0$ and the conclusion follows. □

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