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A Domain Decomposition Method for Helmholtz Scattering Problems

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1 Introduction

We present a study of iterative nonoverlapping domain decomposition methods (DDMs) for the harmonic scattering wave equation in the 3D case. We introduce some new nonlocal transmission conditions at subdomain interfaces in order to obtain an exponential rate of convergence. This work is a natural continuation of the work by Despres [Des91]. We present numerical results for a mixed finite element approximation. The parallel performance of the method on a tightly coupled machine and a loosely connected network is also shown.

2 Domain decomposition methods

A model problem

We study the scattering scalar Helmholtz equation in three dimensions. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, Γ its boundary, and $\boldsymbol{n}$ the outgoing normal to Γ. The problem to solve is:

$$\begin{cases} (a) & -\nabla(\frac{1}{\mu}\nabla u) - \omega^2 \epsilon u = f \quad \text{on} \quad \Omega \\ (b) & \frac{1}{\mu}\frac{\partial u}{\partial n} + i\omega\sqrt{\frac{\epsilon}{\mu}}u = 0 \quad \text{on} \quad \Gamma \\ (c) & u = 0 \quad \text{on} \quad \partial F \end{cases} \tag{1}$$

The boundary condition *(b)* plays an essential role and can be interpreted as a first-order absorbing boundary condition, where μ and ϵ are two positive parameters piecewise C^1. We know that for every f in $L^2(\Omega)$, (1) has a unique weak solution in $H^1(\Omega)$.

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Domain decomposition methods

We apply the DDM concept to the Helmholtz scattering problem. The originality in our work is the introduction of some new nonlocal transmission conditions at the interfaces between subdomains in order to obtain an exponential rate of convergence.

Let us give a brief presentation of the method. The general idea is to split the domain Ω into several subdomains $(\Omega_k)_{k\in I}$. The solution is the limit of the following iterative process. We denote as u_k^n the restriction of the approximate solution to the domain Ω_k at step n, u_k^n being the solution of the following problem:

$$\begin{cases} \text{Find} \quad u_k^{n+1} \in H^1(\Omega_k) & \\ \nabla(\dfrac{1}{\mu_k}\nabla u_k^{n+1}) - \omega^2 \epsilon_k u_k^{n+1} = f_k & \text{in} \quad \Omega_k \\ \dfrac{1}{\mu_k}\dfrac{\partial u_k^{n+1}}{\partial n_k} + i\omega\sqrt{\dfrac{\epsilon_k}{\mu_k}} u_k^{n+1} = 0 & \text{on} \quad \Gamma_k \\ u_k^{n+1} = 0 & \text{on} \quad \partial F_k \\ \dfrac{1}{\mu_k}\dfrac{\partial u_k^{n+1}}{\partial n_k} + i\mathbf{T_{jk}} u_k^{n+1} = -\dfrac{1}{\mu_j}\dfrac{\partial u_j^n}{\partial n_j} + i\mathbf{T_{kj}} u_j^n = g_{kj} & \text{on} \quad \Sigma_{jk} \quad (*), \end{cases} \tag{2}$$

where $\mathbf{T_{kj}}$ and $\mathbf{T_{jk}}$ are continuous linear operators, for which $T_{kj} = T_{jk} = T$ and

$$\mathbf{T_{kj}} : \mathbf{H^{1/2}}(\Sigma_{\mathbf{kj}}) \longrightarrow \mathbf{H^{-1/2}}(\Sigma_{\mathbf{kj}})$$

is a symmetric isomorphism between $H^{1/2}(\Sigma_{kj})$ and $H^{-1/2}(\Sigma_{kj})$. We call equation $(*)$ on Σ_{kj} a transmission condition. The following theorem ensures that u_k^{n+1} is well defined at each step n.

Theorem 2.1 *Let $f_k \in L^2(\Omega_k)$ and $g_{kj} \in H^{-1/2}(\Sigma_{kj})$, $\mu_k \in L^\infty(\Omega_k)$ and $\epsilon_k \in L^\infty(\Omega_k)$ and piecewise C^1. Then problem (2) has an unique weak solution $u_k \in H^1(\Omega_k)$.*

Proof see [Gha96]. ■

In the following theorem, we prove that the iterative process (2) is convergent.

Theorem 2.2 *Under the hypothesis $\dfrac{1}{\mu_k}\dfrac{\partial u_k^0}{\partial n_k} \in H^{-1/2}(\partial\Omega_k)$, $\forall k \in I$, $(\mu_k, \epsilon_k) \in L^\infty(\Omega_k)^2$, piecewise C^1 and $(\dfrac{1}{\mu_k}, \dfrac{1}{\epsilon_k}) \in L^\infty(\Omega_k)^2$, the solution of equations (2), u_k^n converges in $H^1(\Omega_k)$ to u_k, the solution on Ω_k.*

Proof see [Gha96]. ■

Geometrical convergence

For the sake of clarity, we show the convergence in a homogeneous medium. The iterative process (2) is written as

$$x^{n+1} = \mathcal{A}x^n, \tag{3}$$

where x^n is the sequence defined on the interface (Γ_k, Σ_{kj}) by

$$x^n = (x^n_{\Gamma_k}, x^n_{\Sigma_{kj}}), \left\{ \begin{array}{l} x^n_{\Gamma_k} \in L^2(\Gamma_k) \\ x^n_{\Sigma_{kj}} \in L^2(\Sigma_{kj}). \end{array} \right.$$

More precisely, $V = \oplus_k \left[\oplus_{j \neq k} L^2(\Sigma_{kj}) \oplus L^2(\Gamma_k) \right]$,

$$\begin{array}{rcl} \mathcal{A}: \quad V & \longrightarrow & V \\ x^n = (x^n_{\Sigma_{kj}}, x^n_{\Gamma_k}) & \longrightarrow & x^{n+1} = \mathcal{A}x^n, \end{array}$$

such that:

$$x^n_{\Sigma_{kj}} = (\mathbf{S}^*)^{-\mathbf{1}} \frac{\partial e_k}{\partial n_k} + i\mathbf{S}e_k,$$
$$x^n_{\Gamma_k} = \frac{\partial e_k}{\partial n_k} + i\omega e_k,$$

where e_k is the unique solution of the problem.

$$\left\{ \begin{array}{llll} \Delta e_k + \omega^2 e_k = 0 & \text{in} & \Omega_k & (1) \\ (\mathbf{S}^*)^{-\mathbf{1}} \dfrac{\partial e_k}{\partial n_k} + i\mathbf{S}e_k = x^n_{\Sigma_{kj}} & \text{on} & \Sigma_{kj} & (2) \\ \dfrac{\partial e_k}{\partial n_k} + i\omega e_k = x^n_{\Gamma_k} & \text{on} & \Gamma_k & (3). \end{array} \right. \tag{4}$$

x^{n+1} is constructed in the following way:

$$x^{n+1}_{\Sigma_{kj}} = \mathcal{A}x^n_{\Sigma_{kj}} = -(\mathbf{S}^*)^{-\mathbf{1}} \frac{\partial e_j}{\partial n_j} + i\mathbf{S}e_j,$$
$$x^{n+1}_{\Gamma_k} = \mathcal{A}x^n_{\Gamma_k} = 0.$$

Some properties of the $\mathcal{A}$ operator are:

- $||\mathcal{A}|| \leq 1$,
- if some eigenvalues of $\mathcal{A}$ are close to 1, then convergence is slow [GJC95].

For achieving geometrical convergence, it is necessary to use a relaxation method. The fourth $(*)$ equation in (2) is replaced by:

$$\frac{1}{\mu_k} \frac{\partial u^{n+1}_k}{\partial n_k} + iT_{kj}u^{n+1}_k = r(-\frac{1}{\mu_j} \frac{\partial u^n_j}{\partial n_j} + iT_{kj}u^n_j) + (1-r)(\frac{1}{\mu_k} \frac{\partial u^n_k}{\partial n_k} + iT_{kj}u^n_k) \quad on \quad \Sigma_{kj},$$

where r is the relaxation parameter and belongs to]0,1[. As a result, we have

$$x^{n+1} = r\mathcal{A}x^n + (1-r)x^n. \tag{5}$$

Theorem 2.3 *Assume* $\dfrac{\partial u^0_k}{\partial n_k} \in H^{-1/2}(\partial\Omega_k)$, $\forall k \in I$. *If the interfaces* Σ_{kj} *do not intersect, we get an exponential rate of convergence for the relaxed iterative process:*

$$\exists \epsilon > 0 \quad \text{such that} \quad ||(1-r)Id + r\mathcal{A}||_{\mathcal{L}(V,V)} \leq \sqrt{1 - \epsilon^2 r(1-r)} < 1. \tag{6}$$

Proof We assume the following identity:

$$\exists \epsilon > 0, \quad \forall x \in V, \quad ||(I - \mathcal{A})x||_V \geq \epsilon ||x||_V. \tag{7}$$

So, let $x \in V$ such that $||x||_V = 1$. From

$$||(I - \mathcal{A})x||_V^2 = ||\mathcal{A}x||_V^2 + ||x||_V^2 - 2Re < \mathcal{A}x, x > \geq \ (\epsilon ||x||_V)^2.$$

we deduce that

$$2Re < \mathcal{A}x, x > \leq \ 2 - \epsilon^2,$$

and

$$||(r\mathcal{A} + (1-r)I)x||_V^2 \leq (1-r)^2 ||x||^2 + r^2 ||\mathcal{A}x||^2 + 2r(1-r)Re < \mathcal{A}x, x >,$$

$$||(r\mathcal{A} + (1-r)I)x||_V^2 \leq (1-r)^2 + r^2 + 2r(1-r)(1 - \epsilon^2/2) = 1 - r(1-r)\epsilon^2.$$

Finally, we have:

$$\forall x \in V, \qquad ||(r\mathcal{A} + (1-r)I)\frac{x}{||x||}||_V \leq \sqrt{1 - r(1-r)\epsilon^2}$$

$$\forall x \in V, \qquad ||(r\mathcal{A} + (1-r)I)x||_V \leq \sqrt{1 - r(1-r)\epsilon^2}||x||_V.$$

■

Assumption (7) is proved if the bijectivity of the $Id - \mathcal{A}$ operator is obtained. Then:

$$\forall x \in V, \quad x = (I - \mathcal{A})^{-1}(I - \mathcal{A})x, \quad ||x|| \leq ||(I - \mathcal{A})^{-1}|| ||(I - \mathcal{A})x||, \tag{8}$$

and

$$\frac{1}{||(I - \mathcal{A})^{-1}||} = \epsilon.$$

First, we show that $\underline{I - \mathcal{A} \text{ is injective}}$.

If $x \in V$, is such that $x = \mathcal{A}x$, the field e solution of

$$\begin{cases} \omega^2 e_k + \Delta e_k = 0 & \text{in} \quad \Omega_k \\ \dfrac{\partial e_k}{\partial n_k} + i\omega e_k = x_k & \text{on} \quad \Gamma_k \\ (S^*)^{-1}\dfrac{\partial e_k}{\partial n_k} + iSe_k = x_{kj} & \text{on} \quad \Sigma_{kj} \end{cases}$$

satisfies

$$(S^*)^{-1}\frac{\partial e_k}{\partial n_k} + iSe_k = x_{kj} = (\mathcal{A}x)_{kj} = -(S^*)^{-1}\frac{\partial e_j}{\partial n_j} + iSe_j \quad \text{on} \quad \Sigma_{kj},$$

$$(S^*)^{-1}\frac{\partial e_j}{\partial n_j} + iSe_j = x_{jk} = (\mathcal{A}x)_{jk} = -(S^*)^{-1}\frac{\partial e_k}{\partial n_k} + iSe_k \quad \text{on} \quad \Sigma_{kj}$$

and

$$\frac{\partial e_k}{\partial n_k} + i\omega e_k = x_k = (\mathcal{A}x)_k = 0 \qquad \text{on} \qquad \Gamma_k,$$

then $e = (e_k)$ is such that $e_k \in H^1(\Omega_k)$ and

$$\begin{cases} \omega^2 e + \Delta e = 0 & \text{in} \quad \Omega - \cup\Gamma_{kj} \cup \Sigma_{kj} \\ \dfrac{\partial e_k}{\partial n_k} + i\omega e_k = 0 & \text{on} \quad \Gamma_k \\ e_k = e_j, \qquad \dfrac{\partial e_k}{\partial n_k} = -\dfrac{\partial e_j}{\partial n_j} & \text{on} \quad \Sigma_{kj}, \end{cases}$$

so, e is solution of the Helmholtz equation in Ω with $\dfrac{\partial e}{\partial n} + i\omega e = 0 \quad \text{on} \quad \partial\Omega,$
and we deduce $e = 0$ and $x = 0$.

Second, we show that $\underline{I - \mathcal{A} \text{ is surjective}}$.

Given a g in V where $g_{/\Gamma_k} = g_k$, $g_{/\Sigma_{jk}} = g_{kj}$, $(g_k, g_{kj}) \in (H^{-1/2}(\Gamma_k), H^{-1/2}(\Sigma_{kj}))$, we find an $x \in L^2$ such that:

$$(I - \mathcal{A})x = g,$$

which imply the existence of e belonging $H^1(\Omega)$, and satisfying the Helmholtz equation on $\Omega - \cup\Gamma_k \cup \Sigma_{kj}$ and

$$\begin{cases} \left[(S^*)^{-1}\dfrac{\partial e_k}{\partial n_k} + iSe_k\right] - \left[-(S^*)^{-1}\dfrac{\partial e_j}{\partial n_j} + iSe_j\right] = g_{kj} & \text{on} \quad \Sigma_{kj} \\ \dfrac{\partial e_k}{\partial n_k} + i\omega e_k = g_k & \text{on} \quad \Gamma_k. \end{cases} \tag{9}$$

Finally, the field e must satisfy the Helmholtz equation in each subdomain and the jump of equations (9). We note that the jump of the trace needs to belong to the space $H^{1/2}(\Sigma_{kj})$ and its normal derivative needs to belong to the space $H^{-1/2}(\Sigma_{kj})$. These conditions are satisfied if the interfaces Σ_{kj} do not intersect. We get the solution e using potential theory (see [KC93]). By defining

$$\begin{cases} x_{kj} = (S^*)^{-1}\dfrac{\partial e_k}{\partial n_k} + iSe_k \quad \text{on} \quad \Sigma_{kj} \\ x_k = \dfrac{\partial e_k}{\partial n_k} + i\omega e_k = g_k \quad \text{on} \quad \Gamma k, \end{cases} \tag{10}$$

we can easily show that $(x_k,\, x_{kj}) \in (L^2(\Gamma_k), L^2(\Sigma_{kj}))$. Finally, we obtain

$$(I - \mathcal{A})x = g.$$

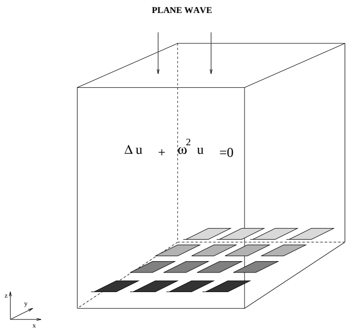
In conclusion, the geometrical convergence with a nonlocal operator is proved given a particular decomposition. The generalisation of the proof given any decomposition is an open problem.

We implement this method using the mixed hybrid finite element method [CR89]. We perform several tests to study the improvement of the convergence due to the following transparent-like operators:

$$T_p = \omega(I + \frac{c_m}{\omega^2}\Delta_{\Sigma_{kj}})^{1/2}, \tag{11}$$

where $c_m = \dfrac{1}{\sqrt{\epsilon_k.\mu_k} + \sqrt{\epsilon_j.\mu_j}}$, $\Delta_{\Sigma_{kj}}$ is the Laplace-Beltrami operator and ω is the frequency of the problem.

Figure 1 Computational problem

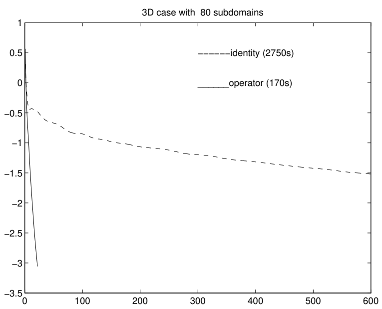


3 Numerical results

We choose a cubic domain. The scattering problem with slits located at the center of the computational domain in (3D) case is solved. The source is a plane wave (Fig. 1).

We compare the convergence of the DDM obtained with the transparent-like operator T_p and the identity operator [Des91]. Figure 2 shows the fast convergence with the new transparent transmission operator.

Figure 2 Convergence of the DDM with 80 subdomains



Our computations are done on a CRAY C90 computer. If we need 10^{-3} precision, for example, the method that does not use the nonlocal operator requires 1000 iterations and 2750 s execution time. The method with the T_p operator requires only 150 iterations and 170 s excution time. In Figure 3 on the right, the discretization step h varies for a fixed operator T_p. When the step h is fine enough (40 points per wavelength) then the convergence curves (in the log scale) confirm the exponential rate of convergence.

The same experiment is carried out with the identity operator (see Fig. 3 on the left) with mesh refinement. For fixed h, we still have exponential convergence but the corresponding rate depends on h and degenerates when h tends to 0.

Figure 3 Convergence of the DDM with identity operator and with T_p operator

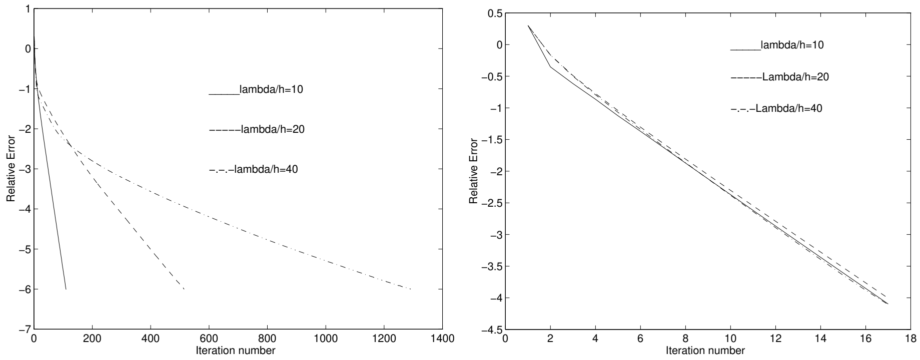


Table 1 CRAY T3D (above), SUN (below)

p	2	4	8	16
$T_s(s)$	1045	288	92	28
$T_{//}(s)$	552	103	15	2.8
$S(p)$	1.89	2.89	5.78	10.03

p	2	4	6	9	12
T_s (s)	3585	2523	465	241	167
$T_{//}$ (s)	2589	752	127	41	28
$S(p)$	1.38	2.56	3.67	5.7	5.94

4 Parallel version of the DDM

The domain decomposition algorithm can be parallelised naturally. After the discretization of the method, we have to solve several independent problems, at each step n. The solution of each subdomain can be calculated on each processor of a parallel computer. Between two iteration steps, it receives the value of both the trace and the flux of the solution, as evaluated by the processors which compute the solution of the neighbouring subdomains. The parallelization tool is PVM (Parallel Virtual Machine). Our computations are done on a heterogeneous network of workstations (SUN SPARC), and on a multiprocessor computer (CRAY T3D). The achieved speed-up $S(p)$ is defined by the ratio between the sequential execution time T_s obtained with an optimal sequential version of the method and the parallel execution time $T_{//}$ on p processors. Both tables show the various achieved speed-ups on both platforms.

We obtain a good computational performance with the CRAY T3D. The performance degrades if we increase the processor number using the heterogeneous network of workstations. Not all machines have the same computing power and it might happen that the faster machines are waiting for slower machines. This is not the case for a multiprocessor CRAY T3D, because all the processors are equivalent and the interconnecting network is optimized for parallelism.

Finally, we remark that with a parallel version, we can solve a large size problem on a distributed memory machine, which cannot solved on a sequential machine because of its memory limitations.

5 Conclusion

The conclusions of the theoretical studies are the following: we prove that the iterative process of the domain decomposition method with nonlocal transmission conditions converges. Given a particular decomposition, we obtain a geometrical convergence. The generalisation to any kind of decomposition remains an open problem.
The numerical results of the implementation of the DDM show better convergence with nonlocal transmission operators. They also show that the rate of convergence does not depend on the discretisation step. Finally, we obtain a good performance of the parallel version of the method. The communication time between processors is very small compared to the computation time.

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