SOBOLEV SPACES OF INFINITE ORDER WITH RAPIDLY DECREASING SEQUENCE

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This paper is a continuation of the research in the theory of Banach spaces of infinitely differentiable functions carried out by the author [1]–[5] and Yu. A. Dubinskii [6]–[8].

Let a Sobolev space of infinite order

$$(1)\quad W^\infty\{a_n,p,r\}_{(0,a)} \equiv \Big\{u(x)\in C^\infty(0,a)\colon \varrho(u)=\sum_{n=0}^\infty a_n\|D^nu(x)\|_r^p<\infty\Big\}$$

be defined by a rapidly decreasing sequence $\{a_n\}$, i.e.,

$$(2)\qquad 0<a_{n+1}\le a_n^q<1,\quad q>1,\quad n=0,1,\dots.$$

In (1) we suppose that $1\le p<\infty$, $r\ge 1$, $\|\cdot\|_r$ is the norm in the Lebesgue space $L_r(0,a)$, and that D^nu is the n-th order derivative of the function $u(x)$.

It is easy to verify that for any sequence $\{a_n\}$ satisfying the condition (2) the series

$$\sum_{n=0}^\infty \frac{M_n^c}{M_{n+1}^c}.$$

converges. Here $\{M_n^c\}$ denotes a logarithmically convex regularization of the sequence

$$M_n=\begin{cases} a_n^{-1/p} & \text{if } a_n\ne 0\\ \infty & \text{if } a_n=0\end{cases}$$

(the definition of $\{M_n^c\}$ can be found in [9]). The convergence ensures the nontriviality of the space

$$(3)\qquad \mathring{W}^\infty\{a_n,p,r\}_{(0,a)}\equiv\{u(x)\in C_0^\infty(0,a)\colon \varrho(u)<\infty\},$$

i.e., the space defined by (3) contains a function $u(x) \not\equiv 0$ for any $x \in [0, a]$.

We will consider a problem of finding conditions for sequences $\{b_n\}$ and $\{d_n\}$ such that there exists a function $u(x) \in W^\infty\{a_n, p, r\}_{(0,a)}$ satisfying the following equalities

$$D^n u(0) = b_n, \quad D^n u(a) = d_n, \quad n = 0, 1, \dots \tag{4}$$

(clearly it is sufficient to consider the case $d_n = 0$ for all n since the case when $d_n \not\equiv 0$ is analogous).

Definition 1. The indices $\{n_i\}$ such that

$$M_{n_i} = M^c_{n_i}$$

are called fundamental indices for convex logarithmical regularization of the sequence $\{M_n\}$.

Definition 2. The sequence $\{M_n\}$ is called almost-logarithmically convex if for all its fundamental indices the following condition is satisfied

$$\sup_i (n_{i+1} - n_i) = K < \infty.$$

If $K = 1$, then the sequence $\{M_n\}$ is logarithmically convex. It was established earlier (see [5]) that if the space (1) is defined by a sequence $\{a_n\}$ satisfying the condition (2) and the sequence $\{a_n^{-1}\}$ is logarithmically convex, then for the existence of a function $u(x) \in W^\infty\{a_n, p, r\}_{(0,a)}$ with the properties (4) it is necessary and sufficient that

$$\sum_{n=0}^{\infty} a_{n+1}^{1/r} a_{n+1}^{1-1/r} |b_n|^p < \infty. \tag{5}$$

Here the condition of logarithmical convexity of the sequence $\{a_n^{-1}\}$ is essential, since without it the condition (5) is not sufficient.

Indeed, let us consider a sequence

$$a_n = \begin{cases} 2^{-2^n}, & n = 2k \\ 2^{-2^{n-1}\cdot 3}, & n = 2k+1, \ k = 0, 1, \dots . \end{cases} \tag{6}$$

The sequence satisfies the condition (2) with $q = 4/3$. It is easy to see that the sequence $\{a_n^{-1}\}$ is not logarithmically convex:

$$a_{2k+1}^{-1} = 2^{2^{2k}\cdot 3} > \sqrt{a_{2k}^{-1} a_{2k+2}^{-1}} = 2^{2^{2k}\cdot 2{,}5}.$$

However it is almost-logarithmically convex (the even indices are fundamental). Therefore as it has been shown in [2] the spaces $W^\infty\{a_n,p,r\}_{(0,a)}$ and $W^\infty\{(M_n^c)^{-1},p,r\}_{(0,a)}$ coincide (in the sense that they have the same elements).

By definition of convex regularization via logarithms, for n, $n_i < n \leq n_{i+1}$, we have

$$M_n^c = M_{n_i}^{(n_{i+1}-n)/(n_{i+1}-n_i)} M_{n_{i+1}}^{(n-n_i)/(n_{i+1}-n_i)} \tag{7}$$

(for our case $n_i = 2i$, $n_{i+1} - n_i = 2$ for all i). Hence

$$(M_n^c)^{-1} = \begin{cases} 2^{-2^n}, & n = 2k \\ 2^{-2^{n-1}\cdot 2{,}5}, & n = 2k+1,\ k = 0,1,\dots . \end{cases}$$

Besides, this sequence also satisfies the condition (2) with exactly the same q. According to the above-mentioned theorem, for the existence of a function $u(x) \in W^\infty\{(M_n^c)^{-1},p,r\}_{(0,a)}$ satisfying the condition (4) it is necessary and sufficient that the series

$$\sum_{n=0}^{\infty} |b_n|^p (M_{n+1}^c)^{-1/r} (M_n^c)^{-(1-1/r)} \tag{8}$$

converges.

Using the equalities

$$(M_{2k}^c)^{-1} = a_{2k}, \qquad (M_{2k+1}^c)^{-1} = a_{2k+1}\cdot 2^{2^{2k-1}},$$

we can select such a sequence $\{b_n\}$ that the series (5) converges and the series (8) diverges. For instance, for $r < \infty$ it is sufficient to assume

$$b_{2k+1} = 0, \quad b_{2k} = (2^{2^{2k}\cdot 2{,}5})^{\frac{1}{rp}} (2^{2^{2k}})^{\frac{r-1}{rp}}, \quad k = 0,1,\dots .$$

Thus the condition (5) ceases to be sufficient.

Lemma 1. *If a sequence $\{a_n\}$ satisfies the condition* (2), *then the sequence $\{M_n = a_n^{-1}\}$ is almost-logarithmically convex.*

Proof. If $q \geq 2$, then the sequence $\{M_n\}$ is logarithmically convex, i.e.,

$$M_{n+1} \leq \sqrt{M_n M_{n+2}}.$$

Indeed, it follows from (2) that $M_n^q \leq M_{n+1}$ and $M_{n+1}^q \leq M_{n+2}$, and $q/2 \geq 1$. Hence we have

$$\sqrt{M_n M_{n+2}} \geq \sqrt{M_n M_{n+1}^q} \geq M_n^{1/2} M_{n+1} \geq M_{n+1}.$$

For $1 < q < 2$ it is sufficient to consider the case

$$a_n^2 < a_{n+1} \leq a_n^q \tag{9}$$

Let $\gamma_{n_i,n}$, $\gamma_{n_i,n_{i+1}}$ be angular coefficients of the straight lines defined by the points (n_i, N_{n_i}), (n, N_n) and (n_i, N_{n_i}), $(n_{i+1}, N_{n_{i+1}})$ correspondingly, with $N_{n+1} \geq qN_n$. By definition of the convex logarithmic regularization, we have for all $n_i < n \leq n_{i+1}$, $i = 1, 2, \dots,$

$$\gamma_{n_i,n} = \frac{N_n - N_{n_i}}{n - n_i} \geq \gamma_{n_i,n_{i+1}} = \frac{N_{n_{i+1}} - N_{n_i}}{n_{i+1} - n_i}.$$

Using (9) we establish the following inequality

$$\frac{N_n - N_{n_i}}{n - n_i} \geq \frac{N_{n_i}(q^{n_{i+1}-n_i} - 1)}{n_{i+1} - n_i}.$$

which can not be true when $n_{i+1} - n_i \to \infty$ and the condition (6) holds. Hence $\sup_i (n_{i+1} - n_i) = K < \infty$, and the proof is complete. □

Lemma 2. *If a sequence $\{a_n\}$ satisfies the condition* (2)*, then for the sequence $\{M_n^c\} = \{(a_n^{-1})^c\}$ the inequality*

$$(M_n^c)^q \leq M_{n+1}^c, \quad n = 0, 1, \dots, \tag{10}$$

is true with the same q.

Proof. By the definition of the convex regularization by means of logarithms it follows that

$$a_n^{-1} \geq M_n^c \tag{11}$$

and for $n_i < n \leq n_{i+1}$

$$M_n^c = M_{n-1}^c (M_{n_{i+1}} M_{n_i}^{-1})^{1/(n_{i+1}-n_i)} \tag{12}$$

(see formula (7)). From (2) and (11) we obtain

$$a_{n_{i+1}-1}^{-q} < a_{n_{i+1}}^{-1} = M_{n_{i+1}} = M_{n_{i+1}}^c,$$

$$(M^c_{n_{i+1}-1})^q < M^c_{n_{i+1}} = M_{n_{i+1}}. \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}(M_{n_{i+1}} M_{n_i}^{-1})^{1/(n_{i+1}-n_i)} &= M^c_{n_{i+1}} (M^c_{n_{i+1}-1})^{-1} \\ &> (M^c_{n_{i+1}-1})^q (M^c_{n_{i+1}-1})^{-1} \\ &= (M^c_{n_{i+1}-1})^{q-1}\end{aligned} \tag{14}$$

Since the sequence $\{M^c_n\}$ is monotonically increasing, from (12) and (14) we obtain for $n_i < n \le n_{i+1}$

$$M^c_n > M^c_{n-1} (M^c_{n_{i+1}-1})^{q-1} \ge (M^c_{n-1})^q.$$

The validity of the inequality (10) is established. □

Based on Lemmas 1 and 2, the fact that the spaces $W^\infty\{a_n, p, r\}_{(0,a)}$ and $W^\infty\{(M^c_n)^{-1}, p, r\}_{(0,a)}$ coincide, we can prove for the case of an almost-logarithmically convex sequence $\{a_n^{-1}\}$ the following theorem:

Theorem 1. *If $\{a_n\}$ is a sequence satisfying the condition (2), then for a function $u(x)$ to exist in the space (1) and fulfil the condition (4) it is necessary and sufficient that*

$$\sum_{n=0}^{\infty} |b_n|^p (M^c_n)^{-(1-1/r)} (M^c_{n+1})^{-1/r} < \infty. \tag{15}$$

Remark 1. If in the condition (2) we set $q \ge 2$, then the conditions (15) and (5) coincide.

Remark 2. Theorem 1 is also true in the case when the function $u(x)$ is defined on the half-line $\mathbb{R}^+$.

Lemmas 1 and 2, and the embedding theorems established by the author previously in [5] enable us to obtain a criteria for embedding and compact embedding of spaces defined by a sequence $\{a_n\}$ that satisfies the condition (2).

Theorem 2. *Let $\{a_n\}$ be a sequence for which the condition (2) holds. Then for the embedding*

$$W^\infty\{a_n, p, r\}_{(G)} \hookrightarrow W^\infty\{c_n, p, r\}_{(G)}, \tag{16}$$

where G is a region in $\mathbb{R}^1$, it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}} c_n M^c_n = K < \infty. \tag{17}$$

Theorem 3. *If the sequence $\{a_n\}$ satisfies the condition* (2) *and the region G is a finite interval or a circle, then the embedding* (16) *is compact if and only if*

$$\lim_{n\to\infty} c_n M_n^c = 0. \tag{18}$$

Remark 3. If $q \geq 2$ in the inequality (2), then the conditions established above for the embedding (17) and for the compact embedding (18) coincide with the ones obtained previously in [5].

Remark 4. The sequence $\{c_n\}$ in Theorems 3 and 4 does not have to be rapidly decreasing.

References

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