NORMAL STRUCTURE AND MODULUS OF U-CONVEXITY IN BANACH SPACES

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Let X be a normed linear space, and let

$$S(X) = \{x \in X : \|x\| = 1\}$$

be the unit sphere of X. Brodskii and Milman [1] introduced the following geometric concept:

Definition 1. A bounded, convex subset K of a Banach space X is said to have *normal structure* if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that

$$\sup\{\|x_0 - y\| : y \in H\} < d(H),$$

where $d(H) = \sup\{\|x-y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded, convex subset of X has normal structure. A Banach space X is said to have *w-normal structure* if every weakly compact set K in X that contains more than one point has normal structure. It is clear that if X is reflexive, then the normal structure and the w-normal structure coincide.

Kirk [6] proved that if a weakly compact subset K of X has normal structure, then every nonexpansive mapping on K has a fixed point. It is well known that uniformly convex Banach spaces [2], uniformly smooth spaces [8], the Banach spaces which are uniformly convex in every direction [9], the nearly uniformly convex spaces [3], U-spaces [4], and the Banach spaces with $J(X) < \frac{3}{2}$, where $J(X) = \sup\{\|y - x\| \wedge \|y + x\| : x, y \in S(X)\}$ [4], have normal structure.

Gao and Lau [4] showed that if X fails to have w-normal structure, then X satisfies geometric conditions given in the following lemma:

Lemma. *Let X be a Banach space without w-normal structure, then for any $0 < \varepsilon < 1$ there exist x_1, x_2, x_3, in $S(X)$ satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - 1| < \varepsilon$*;*

(ii) $\big|\|x_1 - x_2\| - 1\big| < \varepsilon$, $\big|\|x_3 - (-x_1)\| - 1\big| < \varepsilon$*;*

(iii) $\frac{1}{2}\|x_1 + x_2\| > 1 - \varepsilon$, $\frac{1}{2}\|x_3 + (-x_1)\| > 1 - \varepsilon$.

In this paper we give a new class of Banach spaces that have normal structure.

For $x \in S(X)$, let ∇_x denote the set of norm 1 supporting functionals of $S(X)$ at x.

Definition 2. Let X be a Banach space. The *modulus of u-convexity* of X is defined as $u(\varepsilon) = \inf\big\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X)$, and $\langle f, x - y\rangle \geq \varepsilon$ for some $f \in \nabla_x\big\}$, $0 \leq \varepsilon \leq 2$.

Proposition 1. *$u(\varepsilon)$ is an increasing function mapping $[0, 2]$ into $[0, 1]$.*

Proof. Let $\varepsilon_1 < \varepsilon_2$. Then $\langle f, x - y\rangle \geq \varepsilon_2$ implies $\langle f, x - y\rangle \geq \varepsilon_1$. So, $u(\varepsilon_1) \leq u(\varepsilon_2)$. □

Let $\delta(\varepsilon) = \inf\big\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), \|x - y\| \geq \varepsilon\big\}$, $0 \leq \varepsilon \leq 2$, be the *modulus of convexity* of X.

Proposition 2. *$u(\varepsilon) \geq \delta(\varepsilon)$.*

Proof. $\forall\varepsilon > 0$, and $f \in \nabla_x$, $\langle f, x - y\rangle \geq \varepsilon$ implies $\|x - y\| \geq \varepsilon$. □

In [7], Lau introduced the following notion in order to study the Chebyshev subset of X:

Definition 3. A Banach space X is called a *U-space* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in S(X)$

$$\left\|\frac{x + y}{2}\right\| > 1 - \delta \quad \Rightarrow \quad \langle f, y\rangle > 1 - \varepsilon, \quad \forall f \in \nabla_x.$$

Theorem 1. *Let X be a Banach space, and let $u(\varepsilon)$ be the modulus of u-convexity of X, then X is a U-space if and only if $u(\varepsilon) > 0$ for every $\varepsilon > 0$.*

Proof. "$\Rightarrow$" Let $\varepsilon > 0$. By Definition 3, there exists a $\delta > 0$, such that if $x, y \in S(X)$, and $\langle f, y\rangle \leq 1 - \varepsilon$ for some $f \in \nabla_x$, then we have $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$. That is, if $x, y \in S(X)$ and $\langle f, x - y\rangle \geq \varepsilon$ for some $f \in \nabla_x$, then $1 - \|\frac{1}{2}(x + y)\| \geq \delta$.

So, by Definition 2, $u(\varepsilon) = \inf\{1-\frac{1}{2}\|x+y\| : x, y \in S(X), \text{ and } \langle f, x-y\rangle \geq \varepsilon, \text{ for some } f \in \nabla_x\} > \delta > 0$.

"$\Leftarrow$" Let $\varepsilon > 0$. Take δ such that $0 < \delta \leq u(\varepsilon)$. Let $x, y \in S(X)$. If there exists a $f \in \nabla_x$ such that $\langle f, y\rangle \leq 1-\varepsilon$, i.e., if $\langle f, x-y\rangle \geq \varepsilon$ for some $f \in \nabla_x$, then we have $1 - \|\frac{1}{2}(x+y)\| \geq u(\varepsilon)$ by Definition 2. Thus, $\|\frac{1}{2}(x+y)\| \leq 1-u(\varepsilon) \leq 1-\delta$.

This means that for $x, y \in S(X)$

$$\left\|\frac{x+y}{2}\right\| > 1-\delta \quad \Rightarrow \quad \langle f, y\rangle > 1-\varepsilon, \quad \forall f \in \nabla_x.$$

So, X is a U-space by Definition 3. □

A normed linear space is called *uniformly nonsquare* [5] if there exists a $\delta > 0$ such that for $x, y \in S(X)$ either $\|\frac{1}{2}(x+y)\| \leq 1-\delta$ or $\|\frac{1}{2}(x-y)\| \leq 1-\delta$.

Theorem 2. *For any Banach space X, if $u(1) > 0$, then X is uniformly nonsquare.*

Proof. $u(1) = \inf\{1-\frac{1}{2}\|x+y\| : x, y \in S(X), \text{ and } \langle f, x-y\rangle \geq 1, \text{ for some } f \in \nabla_x\} > 0$ implies that if x, $y \in S(X)$, and $1-\frac{1}{2}\|x-y\| < u(1)$, then we have $\langle f, x-y\rangle < 1$, $\forall f \in \nabla_x$; i.e., if x, $y \in S(X)$, and $\frac{1}{2}\|x+y\| > 1-u(1)$, then we have $\langle f, x-y\rangle < 1$, $\forall f \in \nabla_x$.

If X is not a uniformly nonsquare space, then for above $u(1) > 0$ there exist x, $y \in S(X)$ such that both $\frac{1}{2}\|x+y\| > 1-u(1)$ and $\frac{1}{2}\|x-y\| > 1-u(1)$. So, both $\langle f, x-y\rangle < 1$ and $\langle f, x+y\rangle < 1$, $\forall f \in \nabla_x$. Hence, both $\langle f, y\rangle > 0$ and $\langle f, -y\rangle > 0$. This is a contradiction. □

Theorem 3. *For any Banach space X, if there exists a $\delta > 0$ such that $u(\frac{1}{2}-\delta) > 0$, then X has normal structure, and therefore X has the fixed point property.*

Proof. The inequality $u(\frac{1}{2}-\delta) > 0$ implies $u(1) > 0$, which in turn implies that X is uniformly nonsquare by Theorem 2, and this implies that X is reflexive [5]. So the normal structure and w-normal structure coincide.

Since $u(\frac{1}{2}-\delta) > 0$, we have that $\langle f, x-y\rangle < \frac{1}{2}-\delta$, for every $f \in \nabla_x$ and for every $x, y \in S(X)$ such that $\frac{1}{2}\|x+y\| > 1-u(\frac{1}{2}-\delta)$.

Suppose that X does not have normal structure. Then X does not have w-normal structure either. Then there exist points $x_1, x_2, x_3 \in S(X)$ satisfying the three conditions in Lemma with respect to $\varepsilon = \min\{u(\frac{1}{2}-\delta), \delta\}$.

Let f be any supporting functional of $x_1 \in S(X)$, i.e., $f \in \nabla_{x_1}$.

Since

$$\begin{aligned}
&\tfrac{1}{2}\|x_1 + x_2\| > 1 - \varepsilon > 1 - u\big(\tfrac{1}{2} - \delta\big),\\
&\tfrac{1}{2}\|x_1 - x_3\| > 1 - \varepsilon > 1 - u\big(\tfrac{1}{2} - \delta\big),\\
&\ \|x_2 - x_3\| \le 1 + \varepsilon \le 1 + \delta,
\end{aligned}$$

we have

$$\begin{aligned}
&\langle f, x_1 - x_2\rangle < \tfrac{1}{2} - \delta,\\
&\langle f, x_1 + x_3\rangle < \tfrac{1}{2} - \delta,\\
&\langle f, x_2 - x_3\rangle \le \|x_2 - x_3\| < 1 + \delta.
\end{aligned}$$

Therefore

$$\begin{aligned}
2 = 2\langle f, x_1\rangle &= \langle f, x_1 - x_2\rangle + \langle f, x_1 + x_3\rangle + \langle f, x_2 - x_3\rangle\\
&< \tfrac{1}{2} - \delta + \tfrac{1}{2} - \delta + 1 + \delta = 2 - \delta.
\end{aligned}$$

Since $\delta > 0$, this is impossible. $\square$

From Theorem 1 and Theorem 3 we have the following corollaries:

Corollary 1 ([4]). *U-spaces have normal structure.*

Since uniformly convex spaces and uniformly smooth spaces are both U-spaces [4], we have

Corollary 2 ([2], [8]). *Both uniformly convex spaces and uniformly smooth spaces have normal structure.*

Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any subset K as in Definition 1, there exists $x \in K$ such that

$$\sup\{\|x - y\| \colon y \in K\} < c \operatorname{diam} K.$$

By using the ultraproduct, Theorem 3 can be refined to the case of uniform normal structure.

Theorem 4. *For any Banach space X, if there exists a δ, $\frac{1}{2} > \delta > 0$, such that $u\big(\frac{1}{2} - \delta\big) > 0$, then X has uniform normal structure.*

References

[1] M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk SSSR(N.S.) **59** (1948), 837–840.

[2] M. M. Day, *Normed linear spaces*, second edition, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[3] D. van Dulst and B. Sims, *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type* (KK), Banach Spaces Theory and its Applications, Proceedings, Bucharest, Lecture Notes in Math., vol. 991, Springer-Verlag, 1983, pp. 35–43.

[4] J. Gao and K. S. Lau, *On two classes of Banach spaces with normal structure*, Studia Math. **99** (1991), no. 1, 41–56.

[5] R. C. James, *Uniformly nonsquare Banach spaces*, Ann. Math. **80** (1964), 542–550.

[6] W. A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly **72** (1965), 1004–1006.

[7] K. S. Lau, *Best approximation by closed sets in Banach spaces*, J. Approx. Theory **23** (1978), 29–36.

[8] B. Turret, *A dual view of a theorem of Baillon*, Lecture Notes in Pure and Appl. Math., vol. 80, Marcel Dekker, 1982, pp. 279–286.

[9] V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. **87** (1971), 5–33.