WEIGHTED INEQUALITIES FOR ONE-SIDED MAXIMAL FUNCTION IN ORLICZ CLASSES

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Let Φ be a class of functions $\varphi\colon \mathbb{R}^1 \to \mathbb{R}^1$ which are nonnegative, even, increasing on $[0,\infty)$, and such that $\varphi(0+) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$.

A function φ is called quasiconvex if there exists a Young function ω and a constant $c > 1$ such that the chain of inequalities

$$\omega(t) \le \varphi(t) \le \omega(ct), \quad t \ge 0$$

holds. We can associate every quasiconvex function φ with its complementary function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\ge 0}(st - \varphi(s)).$$

Further, for every quasiconvex function φ we define the number $p(\varphi)$ as follows:

$$\frac{1}{p(\varphi)} = \inf\{\alpha\colon \varphi^{\alpha} \text{ is quasiconvex}\}.$$

Let g be a positive locally integrable function on the real line. And let the one-sided Hardy-Littlewood maximal functions $M_g^+ f$ and $M_g^- f$ be defined for $f \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x,x+h)} \int_x^{x+h} |f(y)| g(y)\ \mathrm{d}y,$$
$$M_g^- f(x) = \sup_{h>0} \frac{1}{g(x-h,x)} \int_{x-h}^{x} |f(y)| g(y)\ \mathrm{d}y,$$

where $g(x,x+h) = \int_x^{x+h} g(y)\ \mathrm{d}y$.

The author thanks the organizers of the conference for the invitation and support. He thanks the European Mathematical Society for supporting his travel expenses. He also thanks the Mathematical Institute of the Academy of Sciences of the Czech Republic for its hospitality.

Let w be a weight (nonnegative measurable function) in $\mathbb{R}^1$. We say that $w \in A_p^+(g)$, $p \in (1,\infty)$, if

$$\left(\frac{1}{g(a,c)}\int_a^b w(x)g(x)\,\mathrm{d}x\right)\left(\frac{1}{g(a,c)}\int_b^c w(x)^{-1/(p-1)}g(x)\,\mathrm{d}x\right)^{p-1} \le K$$

with some K independent of $a<b<c$.

We say that $w \in A_1^+(g)$ if $M_g^- w(x) \le Kw(x)$ for a.e. x. Similarly, $w \in A_p^-(g)$, $p \in (1,\infty)$, if

$$\left(\frac{1}{g(a,c)}\int_b^c w(x)g(x)\,\mathrm{d}x\right)\left(\frac{1}{g(a,c)}\int_a^b w(x)^{-1/(p-1)}g(x)\,\mathrm{d}x\right)^{p-1} \le K$$

for all $a<b<c$ and $w \in A_1^-(g)$ if $M_g^+ w(x) \le Kw(x)$.

Recently, weighted inequalities for the operator $M_g^+ f$ have been extensively studied [1]–[6].

In particular, the following characterization has been proved.

Theorem ([1], [2], [3], [4], [6])**.** *Let g and w be positive, locally integrable functions on the real line. Let $1<p<\infty$. Then the following conditions are equivalent:*

(i) *There exists a constant $K>0$ such that*

$$\int_{\{x:\ M_g^+ f(x)>\lambda\}} w(x)g(x)\,\mathrm{d}x \le K\lambda^{-p}\int_{-\infty}^{+\infty}|f(x)|^p g(x)\,\mathrm{d}x$$

holds for all $\lambda>0$ and every $f \in L^p(w)$.

(ii) *The weight w satisfies $A_p^+(g)$.*

(iii) *There exists a constant $K>0$ such that for every $f \in L^p(w)$*

$$\int_{-\infty}^{+\infty}\left(M_g^+ f(x)\right)^p w(x)g(x)\,\mathrm{d}x \le K\int_{-\infty}^{+\infty}|f(x)|^p w(x)g(x)\,\mathrm{d}x.$$

If $p=1$ then (i) *is equivalent to saying that w satisfies $A_1^+(g)$.*

The analogous theorem holds for $M_g^- f$ and the corresponding $A_p^-(g)$ classes.

In this paper we present generalization of these results to Orlicz classes. Some results in this direction were obtained in [4], [5]. First we give general weak type inequalities.

Theorem 1. *Let $\varphi \in \Phi$ and w_i $(i = 1, 2, 3, 4)$ be weight functions on the real line. Then the following conditions are equivalent:*

(i) *There exists $c_1 > 0$ such that for all f and λ we have the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x \\ \le c_1 \int_{-\infty}^{+\infty} \varphi\big(c_1 |f(x)| w_3(x)\big) w_4(x) g(x)\ \mathrm{d}x.$$

(ii) *There are constants $c_2 > 0$ and $\varepsilon > 0$ such that*

$$\int_b^c \widetilde{\varphi}\Big(\varepsilon \frac{\int_a^b \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x}{\lambda g(a,c) w_3(x) w_4(x)}\Big) w_4(x) g(x)\ \mathrm{d}x \\ \le c_2 \int_a^b \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x$$

for every $\lambda > 0$ and all numbers $a < b < c$.

In particular cases various criteria for one-weight inequalities have quite a simple form.

Our results are as follows.

Theorem 2. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(i) *There is a constant $c > 0$ such that the inequality*

$$\varphi(\lambda) \int_{\{x:\ M_g^+ f(x) > \lambda\}} w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)|) w(x) g(x)\ \mathrm{d}x$$

is fulfilled for every $\lambda > 0$ and f.

(ii) *w is quasiconvex and $w \in A^+_{p(\varphi)}(g)$.*

Theorem 3. *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi\big(\lambda w(x)\big) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)| w(x)) g(x)\ \mathrm{d}x$$

holds for every $\lambda > 0$ and f.

(ii) *φ is quasiconvex, $w^{p(\varphi)} \in A^+_{p(\varphi)}(g)$, and $w^{-p(\widetilde{\varphi})} \in A^-_{p(\widetilde{\varphi})}(g)$.*

Theorem 4. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi\Big(\frac{\lambda}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi\Big(c \frac{|f(x)|}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x$$

holds for every $\lambda > 0$ and f.
(ii) *φ is quasiconvex and $w \in A^-_{p(\widetilde{\varphi})}(g)$.*

Now we give a complete characterization of weights for which various kinds of strong type inequalities hold.

Theorem 5. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\big(M_g^+ f(x)\big) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)|) w(x) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$ and $w \in A^+_{p(\varphi)}(g)$.*

Theorem 6. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\big(M_g^+ f(x) w(x)\big) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)| w(x)) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$, $w^{p(\varphi)} \in A^+_{p(\varphi)}(g)$ and $w^{-p(\widetilde{\varphi})} \in A^-_{p(\widetilde{\varphi})}(g)$.*

Theorem 7. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\bigg(\frac{M_g^+ f(x)}{w(x)}\bigg) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi\bigg(\frac{|f(x)|}{w(x)}\bigg) w(x) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$ and $w \in A^-_{p(\widetilde{\varphi})}(g)$.*

Analogous theorems hold for $M_g^- f$.

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