ON THE APPROXIMATION NUMBERS OF HARDY-TYPE OPERATORS

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Introduction

While attending last year's Edinburgh Workshop on elliptic PDE's and related areas of harmonic analysis Michael Solomyak suggested that David Edmunds, Desmond Evans and I should try to develop the results of our paper [1] so that they might be more easily applied and their connections with related results of his and Birman in various papers, in particular in [2], be more readily established. I shall describe our so far limited success in this direction. It is necessary first to rehearse some of the definitions and theorems of [1].

1. Preliminaries (see [1])

We are concerned with the problem of estimating the approximation numbers $a_n(T)$ of the weighted Hardy operator T from $L^p(0,\infty)$ into itself given by

$$Tf(x) = v(x)\int_0^x f(t)u(t)\ \mathrm{d}t,$$

where $u \in L^{p'}_{\mathrm{loc}}(0,\infty)$, $v \in L^p(0,\infty)$, $1/p + 1/p' = 1$ and

$$a_n(T) = \inf\{\|T - P_n\| \colon P_n \in \mathcal{F}_n\},$$

where $\mathcal{F}_n$ is the set of bounded linear operators on $L^p(0,\infty)$ of rank less then n.

1.1. Definitions. If $I = (a,b)$ is an interval in $\mathbb{R}^+ = (0,\infty)$ we make the following definitions:

$$\ell(I;f) = \int_I\int_I \left| v(x)v(y)\int_x^y f(t)u(t)\ \mathrm{d}t\right|^p \mathrm{d}x\,\mathrm{d}y, \quad f \in L^p(I),$$

$$L(I;u,v)=\sup\left\{\frac{\ell(I;f)}{\mu(I)}\colon \|f\|_{L^p(I)}\le 1\right\}^{1/p},\quad \text{where}\quad \mathrm{d}\mu=v^p(x)\ \mathrm{d}x,$$

$$A_c=\sup_{a\le s\le c}\|u\|_{L^{p'}(s,c)}\|v\|_{L^p(a,s)},$$

$$B_c=\sup_{c\le s\le b}\|u\|_{L^{p'}(c,s)}\|v\|_{L^p(s,b)},$$

$$J(I)=\max(A_c,B_c),\quad \text{where}\quad \mu(a,c)=\frac{1}{2}\mu(I)$$

$$\text{or}\quad J(I)=\inf_{a\le c\le b}\max(A_c,B_c),$$

$$K(I)=\sup\{\|F-F_I\|_{L^p(I,\mu)}\colon \|f\|_{L^p(I)}\le 1\},$$

$$\text{where}\quad F(x)=\int_0^x f(t)u(t)\ \mathrm{d}t\quad\text{and}\quad F_I=\frac{1}{\mu(I)}\int_I F(x)\ \mathrm{d}\mu(x).$$

1.2. Lemma. *For $1\le p<\infty$, $p\ne 2$,*

$$\frac{1}{2}J(I)\le K(I)\le\begin{cases}2J(I) & (p=1)\\ 2p^{1/p}p'^{1/p'}J(I) & (p>1),\end{cases}$$

and if $p=2$, then

$$\frac{1}{\sqrt{2}}J(I)\le K(I)\le 2J(I).$$

Also

$$K(I)=L(I)/\sqrt{2}\quad (p=2),$$

$$\frac{1}{2}L(I)\le K(I)\le L(I)\quad (p\ne 2).$$

1.3. Definition. The sequence defined by

$$c_0=0,\quad c_{k+1}=\inf\{t\colon L(c_k,t)>\varepsilon\}$$

is called the (ε,L)-sequence.

There are two possibilities:

(i) The sequence is finite of length $N+1$ so that

$$0=c_0<c_1<\cdots<c_N<c_{N+1}=\infty$$

and

$$L(c_k,c_{k+1})=\varepsilon\ (k=0,1,\dots,N-1),\quad L(c_N,c_{N+1})\le\varepsilon.$$

(ii) The sequence is infinite and then $L(c_k,c_{k+1})=\varepsilon$ for all k.

1.4. Theorem. (i) *T is bounded if and only if $L(0,\infty)<\infty$.*

(ii) *If $N+1$ is the length of the (ε,L)-sequence, then*

$$a_{N+2}(T)\le\varepsilon,\quad a_N(T)\ge\frac{1}{4}\varepsilon\quad \text{if}\quad p\ne 2, 1\le p<\infty$$

and

$$a_{N+2}(T)\le\frac{1}{\sqrt{2}}\varepsilon,\quad a_N(T)\ge\frac{1}{\sqrt{2}}\varepsilon\quad \text{if}\quad p=2.$$

2. A connection with $\int_0^x u(t)v(t)\,\mathrm{d}t$

We need two lemmas. Throughout we suppose that $u\in L^{p'}_{\mathrm{loc}}(0,\infty)$, $v\in L^p(0,\infty)$.

2.1. Lemma. *If the weights u_0, v_0 are non-negative and constant on I, then $L(I;u_0,v_0)=\alpha_p|I|u_0v_0$, where $\alpha_p=L((0,1);1,1)$.*

2.2. Lemma. *If $v(x)>0$ almost everywhere and u is non-negative, then $L(I;u,v)=L(I^*;w,1)$, where $\mu(x)=\int_0^x v^p$, $I^*=(\mu(a),\mu(b))$ and $w=(u\circ\mu^{-1})/(v\circ\mu^{-1})^{p-1}\in L^{p'}_{\mathrm{loc}}(I^*)$ and $\int_{I^*}w=\int_I uv$.*

The proofs follow easily by direct substitution.

2.3. Definition. For an (ε,L)-sequence $c_0<c_1<\cdots<c_{N+1}=\infty$ define $N(x,\varepsilon)=\max\{k\colon c_{k+1}\le x\}$.

Then we have

2.4. Theorem. *The equality*

$$\lim_{\varepsilon\to+0}N(x,\varepsilon)\varepsilon=\alpha_p\int_0^x uv$$

holds provided u and v are continuous and strictly positive on $[0,\infty)$ and $0\le x<\infty$.

The proof follows from the definition of the Riemann integral and properties of L, and Lemma 2.1, using Lemma 2.2 to transpose the problem to one in which u, v are replaced by w, 1, and $\mathbb{R}^+$ by $[0,\mu(\infty))$.

2.5. Corollary. *The same conclusion holds if $u\in L^{p'}_{\mathrm{loc}}(0,\infty)$, $v\in L^p(0,\infty)$, $v>0$ almost everywhere, and $\operatorname*{ess\,inf}_{0\le t\le x}\dfrac{u(t)}{v(t)^{p-1}}>0$.*

This follows by an argument depending on the density of the continuous functions in $L^p(0,x)$, $L^{p'}(0,x)$ and on continuity on $L^p(0,x)\times L^{p'}(0,x)$ of the functionals involved.

2.6. Corollary. *If* $\liminf_{N\to\infty} Na_N(T) < \infty$ *then* $uv \in L^1(0,\infty)$.

Proof. By Theorem 1.4, $\alpha_p \int_0^x uv = \lim_{\varepsilon\to+0} \varepsilon N(x,\varepsilon) \le \liminf_{\varepsilon\to+0} \varepsilon N(\infty,\varepsilon) \le 4\liminf_{N\to\infty} Na_N(T)$. □

3. Restriction to the case of non-decreasing w

Now suppose u/v^{p-1} is non-decreasing, i.e., w is non-decreasing on the interval $[0,\mu(\infty))$. We have

3.1. Lemma. *If w is non-decreasing then*

$$\sup_{a\le x\le b}\{w(x)(b-x)\} \le p^{1/p}{p'}^{1/p'} \sup_{a\le y\le b}\left\{\left(\int_a^y w^{p'}\right)^{1/p'}(b-y)^{1/p}\right\}$$

and

$$\sup_{a\le y\le b}\left\{\left(\int_a^y w^{p'}\right)^{1/p'}(b-y)^{1/p}\right\} \le \left(\frac{p}{p'}\right)^{1/p'} \sup_{a\le x\le b}\{w(x)(b-x)\}.$$

Proof. Since w is non-decreasing,

$$w(x)(y-x)^{1/p'} \le \left(\int_x^y w^{p'}\right)^{1/p'} \le \left(\int_a^y w^{p'}\right)^{1/p'} \quad \text{if} \quad x\le y,$$

and so

$$\begin{aligned}\sup_{a\le y\le b}\left\{\left(\int_a^y w^{p'}\right)^{1/p'}(b-y)^{1/p}\right\} &\ge w(x)\sup_{x\le y\le b}\{(y-x)^{1/p'}(b-y)^{1/p}\}\\ &= \left(\frac{1}{p}\right)^{1/p}\left(\frac{1}{p'}\right)^{1/p'} w(x)(b-x).\end{aligned}$$

Conversely,

$$\left(\int_a^y w^{p'}\right)^{1/p'} \le \left[\sup_{a\le x\le b}\{w(x)(b-x)\}\right]\left[\int_a^y \frac{1}{(b-x)^{p'}}\,\mathrm{d}x\right]^{1/p'},$$

and

$$\int_a^y \frac{1}{(b-x)^{p'}}\,\mathrm{d}x \le \frac{1}{p'-1}\frac{1}{(b-y)^{p'-1}}.$$

Thus,

$$\sup_{a\le y\le b}\left\{\left(\int_a^y w^{p'}\right)^{1/p'}(b-y)^{1/p}\right\}\le\left(\frac{p}{p'}\right)^{1/p'}\sup_{a\le x\le b}\{w(x)(b-x)\}.$$

□

3.2. Corollary. *$L(I;u,v)=O(\int_I uv)$ uniformly in I provided u/v^{p-1} is non-decreasing.*

Proof. By Lemma 1.2, we have $L(I;u,v)\le c(p)J(I;u,v)$. It suffices to use Lemma 3.1 and the fact that $\int_{I^*} w=\int_I uv$. □

The proof of the next theorem is then a simple consequence of Theorem 2.4 and Corollary 2.6.

3.3. Theorem. *If u/v^{p-1} is non-decreasing on $[0,\infty)$ then*

$$\lim_{\varepsilon\to+0}\varepsilon N(\infty,\varepsilon)=\alpha_p\int_0^\infty uv$$

(with ∞ a possible value for each).

3.4. Corollary. *If $p\ne 2$, then*

$$\frac{1}{4}\le\liminf_{N\to\infty}\frac{Na_N(T)}{\alpha_p\int_0^\infty uv}\le\limsup_{N\to\infty}\frac{Na_N(T)}{\alpha_p\int_0^\infty uv}\le 1$$

and if $p=2$, then

$$\lim_{N\to\infty}Na_N(T)=\frac{\alpha_p}{\sqrt{2}}\int_0^\infty uv.$$

This follows directly from Theorem 1.4 (if correctly interpreted for the ∞ case). Note it is well-known from Sturm-Liouville theory that $\alpha_2=1/\pi$.

A simple example. Let $u(x)=\mathrm{e}^{Ax}$, $v(x)=\mathrm{e}^{-Bx}$, where $A<B$. Then for $p=2$,

$$a_N\sim\frac{1}{N\pi\sqrt{2}}\int_0^\infty \mathrm{e}^{(A-B)x}\,\mathrm{d}x=\frac{1}{N(B-A)\pi\sqrt{2}}$$

as $N\to\infty$, and for $p\ne 2$,

$$\frac{B-A}{4\alpha_p}\le\liminf_{N\to\infty}\frac{a_N}{N}\le\limsup_{N\to\infty}\frac{a_N}{N}\le\frac{B-A}{\alpha_p}$$

I should like to conclude by thanking the organizers of this workshop for their invitation here and for many past kindness.

References

[1] D. E. Edmunds, W. D. Evans and D. J. Harris, *Approximation numbers of certain Volterra integral operators*, J. London Math. Soc. **37**, no. 2, 471–489.

[2] M. Birman and M. Solomyak, *Estimates for the singular numbers of integral operators*, Uspekhi Mat. Nauk **32**, no. 1, 17–84 ((Russian)); English translation in Russian Math. Surveys **32** (1977), no. 1.