ESTIMATES FOR FELLER SEMIGROUPS GENERATED BY PSEUDODIFFERENTIAL OPERATORS

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It is well known that the generator A of a Feller semigroup $\{T_t\}_{t\geq 0}$ on $C_\infty(\mathbb{R}^n)$ satisfies the positive maximum principle. Under mild regularity assumptions, Ph. Courrège proved in [3] that every operator satisfying the positive maximum principle is on $C_0^\infty(\mathbb{R}^n)$ of the form

$$Au(x) = -q(x, D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} q(x, \xi)\widehat{u}(\xi)\, \mathrm{d}\xi, \tag{0.1}$$

where $q\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function such that for every fixed $x \in \mathbb{R}^n$ the function $q(x, \cdot)\colon \mathbb{R}^n \to \mathbb{C}$ is negative definite. An interesting problem is to find sufficient conditions on $q(x, \xi)$ which ensure that $-q(x, D)$ —defined on $C_0^\infty(\mathbb{R}^n)$ by (0.1)—extends to a generator of a Feller semigroup. Using the Lévy-Khinchine representation, A can be decomposed into the sum of a second-order differential operator and a non-local operator. We are interested in the case where no differential operator term occurs. In a series of papers the first-named author gave some sufficient conditions on $q(x, \xi)$ under which $-q(x, D)$ extends to a generator of a Feller semigroup $\{T_t\}_{t\geq 0}$, cf. [10], [11] and the references given there. Using a martingale problem approach, W. Hoh obtained similar results under somewhat weaker assumptions, see [6] and the references there. The essential idea in [10] was to decompose $q(x, \xi)$ into a sum

$$q(x, \xi) = q(x_0, \xi) + \big(q(x, \xi) - q(x_0, \xi)\big) \equiv q_1(\xi) + q_2(x, \xi) \tag{0.2}$$

and to treat $q_2(x, \xi)$ as a perturbation of the continuous negative definite function $q_1(\xi)$. Observe that $-q_1(D)$ itself generates a Feller semigroup (subsequently denoted by $\{S_t\}_{t\geq 0}$) which is even a convolution semigroup.

In this paper we will compare the semigroups $\{T_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$, that is, we want to see how the perturbation $q_2(x, \xi)$ can give estimates for $T_t u - S_t u$ for certain functions u. In the first section we recall some notation and definitions and in the second section we give precise conditions on $q(x,\xi)$, $q_1(\xi)$, and $q_2(x,\xi)$, such that $-q(x,D)$ extends to a generator of a Fellerian or sub-Markovian semigroup. These results are taken from [10], [11], [6], and [8]. In Section 3 we prove the following L^2-estimate for $T_tu - S_tu$,

$$\|T_tu - S_tu\|_0 \le \frac{t^{1-\rho}}{1-\rho}\widetilde{d}_1 \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1}\|u\|_{a^2,1-\rho}, \tag{0.3}$$

where $\widetilde{d}_1$ is a constant, the functions ϕ_α, $|\alpha| \le n+1$, express the *smallness* of $q_2(x,\xi)$ with respect to $q_1(\xi)$, and $\|\cdot\|_{a^2,1-\rho}$ is an appropriate anisotropic Sobolev norm. Section 4 contains a pointwise estimate,

$$|T_tu(x) - S_tu(x)| \le d_2'\, t\, \|u\|_{n+3} + d_3'\, t^2\, \|u\|_{n+5}, \tag{0.4}$$

where d_2', d_3' are constants depending again on the functions ϕ_α, cf. Corollary 4.3 below.

Since $\{T_t\}_{t\ge 0}$ is generated by the pseudodifferential operator $-q(x,D)$, it is reasonable to consider the operator $e^{-tq(x,D)}$ with symbol $e^{-tq(x,\xi)}$ as an approximation of T_t, see Maslov, Shishmareyev [16] or Kumano-go [15, Sect. 7.4], and as general references [9], [18]. In terms of a symbolic calculus for (classical) pseudodifferential operators this approach is the most natural. In our setting, however, no symbolic calculus is available, for, in general, $\xi \mapsto q(x,\xi)$ fails to be differentiable. Thus the methods of [9], [15], [16], [18] cannot be used here. It is still possible to study the pseudodifferential operator $e^{-tq(x,D)}$, especially its mapping properties, in dependence of the properties of $q(x,\xi)$. This is done in Section 5, Theorem 5.6 and its corollaries. It turns out that $e^{-tq(x,D)}$ maps $L^2(\mathbb{R}^n)$ continuously into the anisotropic Sobolev space $H^{a^2,k}(\mathbb{R}^n)$, where k increases with the smoothness of $x \mapsto q_2(x,\xi)$. Since $\bigcap_{k\in\mathbb{N}} H^{a^2,k}(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n)$, the operator $e^{-tq(x,D)}$ maps $L^2(\mathbb{R}^n)$ into $C_b^\infty(\mathbb{R}^n)$ whenever $x \mapsto q_2(x,\xi)$ is of class C^∞. Moreover, we can show that

$$|T_tu(x) - e^{-tq(x,D)}u(x)| \le t\,\gamma'\|u\|_{n+1} \tag{0.5}$$

holds.

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1. Some notation

Let us recall some frequently used notation and definitions. We will need the following function spaces:

$B_b(\mathbb{R}^n)$ bounded Borel measurable functions;

$C^m(\mathbb{R}^n)$ m-times continuously differentiable functions, $0 \le m \le \infty$; for $m = 0$ we write $C(\mathbb{R}^n)$;

$C_0^m(\mathbb{R}^n)$ the elements of $C^m(\mathbb{R}^n)$ with compact support;

$C_b^m(\mathbb{R}^n)$ the elements of $C^m(\mathbb{R}^n)$ that are bounded with all their partial derivatives up to the order m;

$C_u^m(\mathbb{R}^n)$ the elements of $C_b^m(\mathbb{R}^n)$ which are uniformly continuous with all their partial derivatives up to the order m;

$C_\infty(\mathbb{R}^n)$ continuous functions vanishing at infinity;

$L_p(u)$ the usual Lebesgue space, $1 \le p \le \infty$;

$H^s(\mathbb{R}^n)$ the usual L^2-Sobolev space of the order $s \in \mathbb{R}$.

The norm in $L_p(u)$ is denoted by $\|\cdot\|_{L^p}$ but for $p = 2$ we use $\|\cdot\|_0$ and $(\cdot,\cdot)_0$ for the corresponding inner product. In $H^s(\mathbb{R}^n)$ we write $\|\cdot\|_s$ for the norm and for the inner product $(\cdot,\cdot)_s$. The supremum-norm on $B_b(\mathbb{R}^n)$ is denoted by $\|\cdot\|_\infty$.

The *Fourier transform* of a function u is defined as

$$\widehat{u}(\xi) = \big(\mathbf{F}_{x\to\xi}\, u(x)\big)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, \mathrm{d}x, \tag{1.1}$$

and of a measure μ as

$$\widehat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \mu(\mathrm{d}x). \tag{1.2}$$

The *maximum* of $a, b \in \mathbb{R}$ is $a \vee b$, the *minimum* is $a \wedge b$.

A *Feller semigroup* is a family of linear contractions $T_t\colon C_\infty(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n)$, $t \ge 0$, satisfying

$$T_t T_s u = T_{t+s} u \quad \text{and} \quad T_0 u = u, \tag{1.3}$$

$$\lim_{t\to 0} \|T_t u - u\|_\infty = 0, \tag{1.4}$$

and

$$0 \le u \le 1 \quad \text{implies} \quad 0 \le T_t u \le 1. \tag{1.5}$$

Similarly, a family of linear contractions $T_t \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $t \geq 0$, is a *sub-Markovian semigroup* if

$$T_t T_s u = T_{t+s} u \quad \text{and} \quad T_0 u = u, \tag{1.6}$$

$$\lim_{t\to 0} \|T_t u - u\|_0 = 0, \tag{1.7}$$

and

$$0 \leq u \leq 1 \text{ a.e.} \quad \text{implies} \quad 0 \leq T_t u \leq 1 \text{ a.e.} \tag{1.8}$$

The *infinitesimal generator* A of a Fellerian or sub-Markovian semigroup is given by the strong limit

$$Au = \lim_{t\to 0} \frac{T_t u - u}{t} \tag{1.9}$$

on

$$D(A) = \big\{ u \colon \text{the limit (1.9) exists in the norm sense} \big\}. \tag{1.10}$$

Generators of Feller semigroups are closely related to continuous negative definite functions. A function $\psi \colon \mathbb{R}^n \to \mathbb{C}$ is *negative definite* if $\psi(0) \geq 0$ and $\xi \to e^{-t\psi(\xi)}$ is positive definite for all $t > 0$. Many properties of negative definite functions are given in [1]. We recall some results for *real-valued* continuous negative definite functions $\psi \colon \mathbb{R}^n \to \mathbb{R}$. Because of the continuity and subadditivity of $\sqrt{\psi}$ we have

$$0 \leq \psi(\xi) \leq c_\psi (1 + |\xi|^2). \tag{1.11}$$

Since $\sqrt{\psi}$ is subadditive, the following *Peetre-type inequality* holds

$$\frac{1}{2\,(1+\psi(\xi-\eta))} \leq \frac{1+\psi(\xi)}{1+\psi(\eta)} \leq 2\,(1+\psi(\xi-\eta)). \tag{1.12}$$

We write $\big(\{X_t\}_{t\geq 0}, \mathbb{P}^x\big)_{x\in\mathbb{R}^n}$ for the *Markov process* associated with the Feller semigroup $\{T_t\}_{t\geq 0}$, and $\mathbb{E}^x(\cdot)$ for the expectation w.r.t. $\mathbb{P}^x$.

Finally, we assume some knowledge of the multi-index notation. Further standard notation that is not explained here can be found in [12].

2. On a class of pseudodifferential operators

In this section we introduce the pseudodifferential operators we will work with later on and we also recall some results from [10] and [6].

Let $a^2\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function such that

$$a^2(\xi) \geq \kappa_0 \, |\xi|^{r_0} \quad \text{for large } |\xi| \tag{2.1}$$

with some $0 < r_0 \leq 2$ and

$$1 + a^2(\xi) \leq \kappa_1(1 + |\xi|^2). \tag{2.2}$$

Note that (2.1) is a non-trivial assumption whereas (2.2) is a restatement of (1.11) with $\kappa_1 \simeq c_\psi$, which follows from the continuity and subadditivity of $a = \sqrt{a^2}$.

Furthermore, let $q\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for a fixed $x \in \mathbb{R}^n$ the function $\xi \mapsto q(x,\xi)$ is negative definite. We assume that $q(x,\xi)$ has a decomposition

$$q(x,\xi) = q(x_0,\xi) + (q(x,\xi) - q(x_0,\xi)) \equiv q_1(\xi) + q_2(x,\xi), \tag{2.3}$$

where $q_1 : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function. Note that therefore $q(x,\xi) \geq 0$ and $q_1(\xi) \geq 0$ hold for all $x, \xi \in \mathbb{R}^n$. The crucial idea is to consider $q_2(x,\xi)$ as a perturbation of $q_1(\xi)$, and to view objects (like semigroups or stochastic processes) related to $q(x,\xi)$ as perturbations of the corresponding objects related to $q_1(\xi)$. For such $q(x,\xi)$ we define on $C_0^\infty(\mathbb{R}^n)$ the pseudodifferential operator $q(x,D)$ by

$$q(x,D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} q(x,\xi)\widehat{u}(\xi) \, \mathrm{d}\xi. \tag{2.4}$$

For $q(x,\cdot)$ is continuous and negative definite, it is polynomially bounded and $q(x,D)u(x)$ is well-defined for every $x \in \mathbb{R}^n$. In a similar manner we define the operators $q_1(D)$ and $q_2(x,D)$. It is well known that $-q_1(D)$ has extensions to a generator of a symmetric Fellerian semigroup and also to a generator of a symmetric sub-Markovian semigroup. Since these semigroups coincide on $L^2(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$, we denote both by $\{S_t\}_{t\geq 0}$. We need some additional assumptions on $q(x,\xi)$ in order to ensure that $-q(x,D)$ extends to a generator of a Fellerian, resp., sub-Markovian semigroup. In this paper we state all conditions w.r.t. the fixed reference function $a^2(\xi)$; different set-ups are considered in [14], [17], [19].

Basically there are two types of conditions: boundedness conditions in terms of a^2 and ellipticity conditions in terms of a^2. We list some of these below and state the appropriate combinations in subsequent theorems.

Ellipticity conditions.

$$1 + q_1(\xi) \geq \gamma_1 \,(1 + a^2(\xi)) \quad \text{for all } \xi \in \mathbb{R}^n; \tag{2.5}$$

$$1 + q(x,\xi) \geq \gamma_2 \,(1 + a^2(\xi)) \quad \text{for all } x, \xi \in \mathbb{R}^n; \tag{2.6}$$

$$q(x,\xi) \geq \widetilde{\gamma}(x) a^2(\xi) \quad \text{for all } x \in \mathbb{R}^n, \ |\xi| \geq \rho > 0. \tag{2.7}$$

Here, $\gamma_1, \gamma_2 > 0$ are constants and $\widetilde{\gamma}(x)$ is a continuous, strictly positive function.

Note that it is possible (and in our situation even equivalent) to assume for all $x \in \mathbb{R}$ and large $|\xi|$ that $q_1(\xi) \geq \gamma_1' \, a^2(\xi)$ or $q(x,\xi) \geq \gamma_2' \, a^2(\xi)$ rather than (2.5) or (2.6).

Boundedness conditions.

$$1 + q_1(\xi) \leq \gamma_0 \,(1 + a^2(\xi)) \quad \text{for all } \xi \in \mathbb{R}^n; \tag{2.8}$$

$$\begin{cases} q(x,\xi) \leq \gamma'(\xi) \leq c\,(1 + |\xi|^2) \quad \text{for all } x, \xi \in \mathbb{R}^n \\ \text{with a continuous function } \gamma' \colon \mathbb{R}^n \to \mathbb{R} \ \text{s.t. } \gamma'(0) = 0; \end{cases} \tag{2.9}$$

$$\begin{cases} q_2(\cdot,\xi) \in C^m(\mathbb{R}^n) \quad \text{and} \quad |\partial_x^\alpha q_2(x,\xi)| \leq \phi_\alpha(x)\,(1 + a^2(\xi)) \\ \text{for all } x, \xi \in \mathbb{R}^n \text{ and all } \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq m, \\ \text{with functions } \phi_\alpha \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n); \end{cases} \tag{2.10.m}$$

$$\begin{cases} q(\cdot,\xi) \in C^m(\mathbb{R}^n) \quad \text{and} \quad |\partial_x^\alpha q(x,\xi)| \leq \gamma_3\,(1 + a^2(\xi)) \\ \text{for all } x, \xi \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}_0^\alpha, \ |\alpha| \leq m. \end{cases} \tag{2.11.m}$$

Note that (2.8) is equivalent to the more naturally looking condition $q_1(\xi) \leq \gamma_0' \,(1 + a^2(\xi))$ but (2.8) suits our purposes better. Sometimes we also need

$$q(x,0) = 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{2.12}$$

Theorem 2.1 ([10]). *Suppose that (2.5), (2.8), and (2.10.m) hold for some sufficiently large m. If $\sum_{|\alpha| \leq m} \|\phi_\alpha\|_{L^1}$ is small compared to γ_1, then $-q(x,D)$ extends to a generator of a Feller semigroup $\{T_t\}_{t \geq 0}$.*

If $-q(x,D)$ is symmetric, then it also extends to a generator of a sub-Markovian semigroup $\{T_t\}_{t \geq 0}$, and we have the estimate

$$\|T_t u\|_{L^\infty} \leq c' \,\frac{e^{ct}}{t^{n/2r_0}} \|u\|_{L^1} \tag{2.13}$$

for all $u \in C_\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Remark 2.2. (i) The largeness condition on m and the smallness condition on $\sum_{|\alpha|\le m}\|\phi_\alpha\|_{L^1}$ are explicitly given in [10]. Here we can do without the exact and rather technical conditions. One should, however, note that these conditions depend *only* on the dimension n and the constant r_0 of (2.1).

(ii) If $\|\phi_0\|_{L^\infty} < \gamma_1$, it is easy to see that the assumptions of Theorem 2.1 imply (2.6) with a suitable constant γ_2.

For $\lambda \in \mathbb{R}$ we put $q_\lambda(x,D) \equiv q(x,D) + \lambda\,\mathrm{id}$.

Corollary 2.3 ([11]). *Under the assumptions of Theorem* 2.1 *there exists a* $\lambda_1 \ge 0$ *such that for all* $\lambda \ge \lambda_1$ *the operator* $-q_\lambda(x,D)$ *extends to a generator of a Feller semigroup and also to a generator of a sub-Markovian semigroup.*

Note that in Corollary 2.3 no symmetry condition is required.

Theorem 2.4 ([6]). *Suppose that* (2.7), (2.12), *and* (2.11.m) *hold for some* $m > 2\,(\frac{n}{r_0}\vee 2)+n+2$. *If* $q(x,D)$ *maps* $C_0^\infty(\mathbb{R}^n)$ *into* $C_\infty(\mathbb{R}^n)$, *then* $-q(x,D)$ *has an extension that generates a Feller semigroup.*

Note that, e.g., the condition (2.10.m) with $m > n-1+n/r_0$ implies $q(x,D)\big(C_0^\infty(\mathbb{R}^n)\big) \subset C_\infty(\mathbb{R}^n)$.

If $q(x,D)$ is symmetric, Theorem 2.4 also guarantees the existence of a sub-Markovian semigroup.

Theorem 2.5 ([8]). *Suppose that* (2.5), (2.8), *and* (2.10.m) *hold for a sufficiently large* m. *If* $\sum_{|\alpha|\le m}\|\phi_\alpha\|_{L^1}$ *is small w.r.t.* γ_1, *and if, in addition,* (2.9) *and* (2.13) *hold, then for any* $\lambda > 0$ *the pseudodifferential operator* $-q_\lambda(x,D)$ *extends to a generator of a strong Feller semigroup.*

In view of the above theorems we may always assume that the pseudodifferential operator $-q(x,D)|C_0^\infty(\mathbb{R}^n)$ with symbol $-q(x,\xi)$ as in (2.3) extends to a generator of a Fellerian and a sub-Markovian semigroup. We will denote both semigroups by $\{T_t\}_{t\ge 0}$. For the rest of the paper we will make the following *General Assumption.*

General Assumption. *Whenever we consider the operator* $q(x,D) = q_1(D) + q_2(x,D)$, *the symbols will fulfil* (2.5) *and* (2.8) *with a fixed continuous negative definite function* a^2 *satisfying* (2.1) *and* (2.2).

For our estimates we need a class of anisotropic Sobolev spaces associated with the function a^2. We put for $s \ge 0$

$$\|u\|_{a^2,s}^2 = \int_{\mathbb{R}^n} (1+a^2(\xi))^{2s}|\widehat{u}(\xi)|^2\,\mathrm{d}\xi, \tag{2.14}$$

and

$$H^{a^2,s}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \colon \|u\|_{a^2,s} < \infty\}. \tag{2.15}$$

The space $H^{a^2,s}(\mathbb{R}^n)$ is a Hilbert space with the inner product $(\cdot,\cdot)_{a^2,s}$, and we clearly have $H^{a^2,0}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Moreover, it is possible to define the spaces $H^{a^2,-s}(\mathbb{R}^n)$, $s \geq 0$, in the usual way using the P. Lax' negative norm (with some obvious modifications), cf. [20], and to identify $H^{a^2,-s}(\mathbb{R}^n)$ with the dual of $H^{a^2,s}$. In particular, (2.14) with $s \leq 0$ holds for $u \in L^2(\mathbb{R}^n)$.

Under our *General Assumption* $q_1(D)$ is a closed operator on $L^2(\mathbb{R}^n)$ with domain $H^{a^2,1}(\mathbb{R}^n)$ and we have

$$\gamma_1^s \|u\|_{a^2,s} \leq \|(1+q_1(D))^s u\|_0 \leq \gamma_2^s \|u\|_{a^2,s} \tag{2.16}$$

for all $u \in H^{a^2,s}(\mathbb{R}^n)$.

Proposition 2.6 ([10])**.** *Under the assumptions of Theorem* 2.1 $q(x,D)$ *is a closed operator on* $L^2(\mathbb{R}^n)$ *with domain* $H^{a^2,1}(\mathbb{R}^n)$*. In particular, if* $-q_\lambda(x,D)$ *defined on* $C_0^\infty(\mathbb{R}^n)$ *extends to a generator of a sub-Markovian semigroup, then the domain of this extension is* $H^{a^2,1}(\mathbb{R}^n)$*.*

For the operator $q_2(x,D)$ we have the following mapping property.

Lemma 2.7. *Suppose that* (2.10.m) *holds with some* $m \geq n+1$*. Then* $q_2(x,D)$ *maps* $H^{a^2,s}(\mathbb{R}^n)$ *continuously into* $H^{a^2,s-1}(\mathbb{R}^n)$ *for all* $s \in \mathbb{R}$ *s.t.* $|s-1| < (m-n)/2$*. Moreover, we have the following estimate*

$$\|q_2(x,D)u\|_{a^2,s-1} \leq d_1(n,s,\kappa_1) \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1} \, \|u\|_{a^2,s}. \tag{2.17}$$

In the proof of Lemma 2.7 we will use the Peetre-type inequality (1.12), and the inequality

$$|\widehat{q}_2(\xi,\eta)| \leq \widetilde{\gamma}_{k,n} \sum_{|\alpha|\leq k} \|\phi_\alpha\|_{L^1} \, (1+|\xi|^2)^{-k/2} \, (1+a^2(\eta)), \tag{2.18}$$

which follows from (2.10.m) for any $k \leq m$, see [10]. Here $\widehat{q}_2(\xi,\eta)$ stands for the Fourier transform w.r.t. the first variable, i.e.,

$$\widehat{q}_2(\xi,\eta) = \big(\mathbf{F}_{x\mapsto\xi}\, q_2(x,\eta)\big)(\xi,\eta).$$

Proof of Lemma 2.7 (cf. also [10, Lemma 3.1]). For $u, v \in C_0^\infty(\mathbb{R}^n)$ we find

$$\left|\big(q_2(x,D)u,v\big)_0\right| = \Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{q}_2(\xi-\eta,\eta)\widehat{u}(\eta)\overline{\widehat{v}(\xi)}\ \mathrm{d}\eta\ \mathrm{d}\xi\Big|$$

$$\leq \widetilde{\gamma}_{m,n} \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-m/2}\,(1+a^2(\eta))|\widehat{u}(\eta)|\,|\widehat{v}(\xi)|\ \mathrm{d}\eta\ \mathrm{d}\xi$$

$$\leq 2^{|s-1|}\,\widetilde{\gamma}_{m,n} \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-m/2}(1+a^2(\xi-\eta))^{|s-1|}$$
$$\times\ (1+a^2(\eta))^s|\widehat{u}(\eta)|\,(1+a^2(\xi))^{1-s}|\widehat{v}(\xi)|\ \mathrm{d}\eta\ \mathrm{d}\xi$$

$$\leq 2^{|s-1|}\,\kappa_1^{|s-1|}\,\widetilde{\gamma}_{m,n} \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{|s-1|-m/2}$$
$$\times\ (1+a^2(\eta))^s|\widehat{u}(\eta)|\,(1+a^2(\xi))^{1-s}|\widehat{v}(\xi)|\ \mathrm{d}\eta\ \mathrm{d}\xi$$

$$\leq 2^{|s-1|}\,\kappa_1^{|s-1|}\,\widetilde{\gamma}_{m,n} \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n} (1+|\tau|^2)^{|s-1|-m/2}\ \mathrm{d}\tau\ \|u\|_{a^2,s}\,\|v\|_{a^2,1-s},$$

that is,

$$\left|\big(q_2(x,D)u,v\big)_0\right| \leq d_1(n,s,\kappa_1) \sum_{|\alpha|\leq m} \|\phi_\alpha\|_{L^1}\,\|u\|_{a^2,s}\,\|v\|_{a^2,-(s-1)},$$

which, in turn, gives (2.17). □

3. An L^2-comparison of semigroups

In this section we assume that $-q(x,D)$ generates a sub-Markovian and a Fellerian semigroup, both denoted by $\{T_t\}_{t\geq 0}$, and by $\{S_t\}_{t\geq 0}$ we denote the corresponding semigroups generated by $-q_1(D)$. The semigroup $\{S_t\}_{t\geq 0}$ is a convolution semigroup, i.e.,

$$S_t u(x) = \int_{\mathbb{R}^n} u(x+y)\,\mu_t(\mathrm{d}y), \tag{3.1}$$

where μ_t is determined by

$$\widehat{\mu}_t(\xi) = (2\pi)^{-n/2}\,e^{-tq_1(\xi)}. \tag{3.2}$$

Moreover, we assume that our *General Assumption* and (2.10.m) hold for some $m > n+1$.

Note that, in order to construct a semigroup by starting with $-q(x,D)$, it might be necessary to enlarge m or to impose a smallness condition on $\sum_{|\alpha|\le m}\|\phi_\alpha\|_{L^1}$.

Now, however, our point of view is that $-q(x,D)$ does already generate the semigroup $\{T_t\}_{t\ge 0}$, and we want to study the effect of $q_2(x,D)$ viewed as a perturbation of $q_1(D)$. Our aim is to compare $\{T_t\}_{t\ge 0}$ with $\{S_t\}_{t\ge 0}$ for small $t>0$. We will need the elementary estimate

$$e^{-x} \le \frac{1}{1+x} \quad \text{for all } x \ge 0. \tag{3.3}$$

Theorem 3.1. *Let $\{T_t\}_{t\ge 0}$, $\{S_t\}_{t\ge 0}$ and $q(x,\xi)$ be as above, in particular assume (2.10.m) for some $m > n+1$. Then we have for $0 < t \le 1$ and $0 \le \rho < 1$*

$$\|T_t u - S_t u\|_0 \le \frac{t^{1-\rho}}{1-\rho} d_1(n,1,\kappa_1)\,\frac{\gamma_0^{1-\rho}}{\gamma_1} \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1}\,\|u\|_{a^2,1-\rho}. \tag{3.4}$$

If $\rho = 0$, we have for all $t \ge 0$

$$\|T_t u - S_t u\|_0 \le t\, d_1(n,1,\kappa_1)\,\frac{\gamma_0}{\gamma_1} \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1}\,\|u_{a^2,1}\|. \tag{3.5}$$

Proof. For $u \in H^{a^2,s}(\mathbb{R}^n)$ we see that

$$\begin{aligned} \|T_t u - S_t u\|_0 &= \Big\| \int_0^t T_{t-s} q_2(x,D) S_s u \,\mathrm{d}s \Big\|_0 \\ &\le \int_0^t \|T_{t-s}\|_{L^2-L^2} \|q_2(x,D) S_s u\|_0 \,\mathrm{d}s \\ &\le d_1(n,1,\kappa_1) \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1} \int_0^t \|S_s u\|_{a^2,1}\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

To estimate $\|S_s u\|_{a^2,1}$ we assume first that $0 < t \le 1$ and $0 \le \rho < 1$. Then we obtain for $s \leq t$

$$
\begin{aligned}
\|S_s u\|_{a^2,1}^2 &\leq \frac{1}{\gamma_1^2}\,\|(1+q_1(D))S_s u\|_0^2 \\
&\leq \frac{1}{\gamma_1^2}\int_{\mathbb{R}^n}(1+q_1(\xi))^2\big(e^{-sq_1(\xi)}\big)^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&\leq \frac{1}{\gamma_1^2}\int_{\mathbb{R}^n}(1+q_1(\xi))^2\frac{1}{(1+sq_1(\xi))^2}\,|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&\leq \frac{1}{s^{2\rho}}\,\frac{1}{\gamma_1^2}\int_{\mathbb{R}^n}(1+q_1(\xi))^{2(1-\rho)}|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&\leq \frac{1}{s^{2\rho}}\,\frac{\gamma_0^{2(1-\rho)}}{\gamma_1^2}\int_{\mathbb{R}^n}(1+a^2(\xi))^{2(1-\rho)}|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&= \frac{1}{s^{2\rho}}\,\frac{\gamma_0^{2(1-\rho)}}{\gamma_1^2}\,\|u\|_{a^2,1-\rho}^2
\end{aligned}
\tag{3.7}
$$

Combining (3.6) and (3.7) we conclude that

$$
\begin{aligned}
\|T_t u - S_t u\|_0 &\leq d_1(n,1,\kappa_1)\,\frac{\gamma_0^{1-\rho}}{\gamma_1}\sum_{|\alpha|\leq n+1}\|\phi_\alpha\|_{L^1}\int_0^t\frac{\mathrm{d}s}{s^\rho}\,\|u\|_{a^2,1-\rho} \\
&= \frac{t^{1-\rho}}{1-\rho}\,d_1(n,1,\kappa_1)\,\frac{\gamma_0^{1-\rho}}{\gamma_1}\sum_{|\alpha|\leq n+1}\|\phi_\alpha\|_{L^1}\,\|u\|_{a^2,1-\rho},
\end{aligned}
$$

and the first part of the assertion is established. By a similar calculation we get

$$
\begin{aligned}
\|S_s\|_{a^2,1}^2 &\leq \frac{1}{\gamma_1^2}\int_{\mathbb{R}^n}(1+q_1(\xi))^2\big(e^{-sq_1(\xi)}\big)^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&\leq \frac{1}{\gamma_1^2}\int_{\mathbb{R}^n}(1+q_1(\xi))^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&\leq \frac{\gamma_0^2}{\gamma_1^2}\int_{\mathbb{R}^n}(1+a^2(\xi))^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \\
&= \frac{\gamma_0^2}{\gamma_1^2}\,\|u\|_{a^2,1}^2,
\end{aligned}
$$

which implies (3.5). $\square$

By (2.1) we have $H^{r_0}(\mathbb{R}^n) \subset H^{a^2,1}(\mathbb{R}^n)$ and $H^{r_0(1-\rho)}(\mathbb{R}^n) \subset H^{a^2,1-\rho}(\mathbb{R}^n)$. For $n = 1$ and $r_0(1-\rho) < 1/2$, that is, $\rho > (2r_0-1)/(2r_0)$, it follows that the indicator function of any bounded interval belongs to $H^{a^2,1-\rho}(\mathbb{R}^1)$. In this case Theorem 3.1 gives a direct L^2-comparison of the transition functions of the stochastic processes corresponding to $\{T_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$, respectively.

4. Some pointwise estimates

In [13] it was proved that for any Feller semigroup $\{T_t\}_{t\geq 0}$ that extends to a map from $C_b(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$ the operators $T_t|C_0^\infty(\mathbb{R}^n)$ itself are pseudodifferential operators with the symbols $(2\pi)^{n/2}\lambda_t(x,\xi)$ where

$$\lambda_t(x,\xi) = \mathbb{E}^x\big(e^{i\xi(X_t - x)}\big). \tag{4.1}$$

The expectation is to be understood with respect to the Feller process $\big(\{X_t\}_{t\geq 0}, \mathbb{P}^x\big)_{x\in\mathbb{R}^n}$ associated with $\{T_t\}_{t\geq 0}$. Thus we have on $C_0^\infty(\mathbb{R}^n)$

$$T_t u(x) = \int_{\mathbb{R}^n} e^{ix\xi}\lambda_t(x,\xi)\widehat{u}(\xi)\,d\xi. \tag{4.2}$$

In [8] some sufficient criteria were given for a Feller semigroup generated by a pseudodifferential operator $-q_\lambda(s,D)$ to be strongly Feller, i.e., to map $B_b(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$, see Theorem 2.5. Clearly, for these (strong) Feller semigroups, (4.1) and (4.2) are valid. If $-q(x,D)$ maps $C_u^2(\mathbb{R}^n)$ into $C_u(\mathbb{R}^n)$, we have

$$q(x,\xi) = (2\pi)^{n/2}\,\frac{d}{dt}\lambda_t(x,\xi)\Big|_{t=0}, \tag{4.3}$$

and for $u \in C_u^2(\mathbb{R}^n)$

$$\frac{d}{dt}T_t u = T_t A u = A T_t u, \tag{4.4}$$

where A denotes the extension of $-q(x,D)|C_0^\infty(\mathbb{R}^n)$ onto $C_u^2(\mathbb{R}^n)$.

By a result of Courrège [2] the operator $-q_1(D)$ always fulfils this additional hypothesis. Using this result it is easy to find examples of operators $-q(x,D)$ with variable coefficients satisfying this assumption, too.

Until now, however, we do not know a simple *general* criterion in order to decide whether $-q(x,D)$ maps $C_u^2(\mathbb{R}^n)$ into $C_u(\mathbb{R}^n)$ or not, see also [4].

In this section we will assume that (4.1)–(4.4) hold.

Lemma 4.1. *Let $\{T_t\}_{t\geq 0}$ be a Fellerian semigroup generated by the pseudodifferential operator $-q(x,D)$. We assume the General Assumption,* (2.10.0), *and* (4.1)–(4.4). *Then we have*

$$\left|\lambda_t(x,\xi) - e^{-tq(x,\xi)}\right| \leq 2t\left(\gamma_0 + \|\phi_0\|_{L^\infty}\right)\left(1 + a^2(\xi)\right). \tag{4.5}$$

Proof. Using Taylor's formula in t, we obtain from (4.3) and (4.4)

$$\begin{aligned}
&\left|\lambda_t(x,\xi) - e^{-tq(x,\xi)}\right| \\
&\qquad = t\left|\int_0^1 \left(-e^{-ix\xi}\,T_{\rho t}\left(e^{i(\cdot,\xi)}q(\cdot,\xi)\right)(x,\xi) + q(x,\xi)e^{-\rho t q(x,\xi)}\right)\,\mathrm{d}\rho\right| \\
&\qquad = t\left|\int_0^1 \left(-e^{-ix\xi}\int_{\mathbb{R}^n} e^{iy\xi}q(y,\xi)\,p_{\rho t}(x,\mathrm{d}y) + q(x,\xi)e^{-\rho t q(x,\xi)}\right)\,\mathrm{d}\rho\right| \\
&\qquad \leq t\int_0^1 \left(\int_{\mathbb{R}^n} q(y,\xi)\,p_{\rho t}(x,\mathrm{d}y) + q(x,\xi)e^{-\rho t q(x,\xi)}\right)\,\mathrm{d}\rho \\
&\qquad \leq t\left(\sup_{y\in\mathbb{R}^n} q(y,\xi) + q(x,\xi)\right),
\end{aligned}$$

where $p_t(x,\mathrm{d}y)$ denotes the transition kernel of the associated Feller process, i.e., we have for $u \in C_b(\mathbb{R}^n)$

$$T_t u(x) = \mathbb{E}^x(u(X_t)) = \int_{\mathbb{R}^n} u(y)\,p_t(x,\mathrm{d}y).$$

From (2.8) and (2.10.0) we deduce that

$$\sup_{y\in\mathbb{R}^n} q(y,\xi) + q(x,\xi) \leq 2\,(\gamma_0 + \|\phi_0\|_{L^\infty})\,(1 + a^2(\xi)),$$

and we are done. □

In the next theorem we denote by $D_F(q_1(D))$ and $D_F(q(x,D))$ the domain of the extension of $-q_1(D)$ and $-q(x,D)$, which generate the Feller semigroups $\{S_t\}_{t\geq 0}$ and $\{T_t\}_{t\geq 0}$, respectively.

Theorem 4.2. *Suppose that the* General Assumption, $(2.10.n+3)$, *and* (4.1)–(4.4) *hold. Since $C_0^\infty(\mathbb{R}^n) \subset D_F(q_1(D)) \cap D_F(q(x,D))$, we have for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$|T_t u(x) - S_t u(x)| \leq d_2\,t\,\|(1 + a^2)\,\widehat{u}\|_{L^1} + d_3\,t^2\,\|(1 + a^2)^2\,\widehat{u}\|_{L^1} \tag{4.6}$$

with the constants

$$d_2 = (2\pi)^{-n}\,\widetilde{\gamma}_{n+1,n} \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-(n+1)/2}\,\mathrm{d}\xi \tag{4.7}$$

and

$$\begin{aligned} d_3 =& 2(2\pi)^{-n}\kappa_1\,(\gamma_0 + \|\phi_0\|_{L^\infty})\,\widetilde{\gamma}_{n+3,n} \\ &\times \sum_{|\alpha|\le n+3} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-(n+1)/2}\,\mathrm{d}\xi. \end{aligned} \tag{4.8}$$

The constant $\widetilde{\gamma}_{k,n}$ is taken from (2.18)*, and κ_1 is taken from* (2.2)*.*

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$ we get

$$\begin{aligned} T_t u(x) - S_t u(x) &= \int_0^t T_{t-s}(q_1(D) - q(x,D))S_s u(x)\,\mathrm{d}s \\ &= -\int_0^t T_{t-s} q_2(x,D) S_s u(x)\,\mathrm{d}s, \end{aligned} \tag{4.9}$$

which yields

$$\begin{aligned} &T_t u(x) - S_t u(x) = \\ &= (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\xi}\lambda_{t-s}(x,\xi)\,\widehat{q}_2(\xi-\eta,\eta)\,e^{-sq_1(\eta)}\,\widehat{u}(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}s \\ &= (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\xi}e^{-(t-s)q(x,\xi)}\,\widehat{q}_2(\xi-\eta,\eta)\,e^{-sq_1(\eta)}\,\widehat{u}(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad + (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\xi}\big(\lambda_{t-s}(x,\xi) - e^{-(t-s)q(x,\xi)}\big) \\ &\qquad \times\, \widehat{q}_2(\xi-\eta,\eta)\,e^{-sq_1(\eta)}\,\widehat{u}(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}s \\ &\equiv I_1 + I_2. \end{aligned}$$

As before, $\widehat{q}_2(\xi,\eta)$ stands for the Fourier transform w.r.t. the first variable. We will now estimate I_1. For $s \le t$ we have $e^{-(t-s)q(x,\xi)} \le 1$, leading to

$$\begin{aligned} |I_1| &\le (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\widehat{q}_2(\xi-\eta,\eta)|\,|\widehat{u}(\eta)|\,\Big|\int_0^t e^{-sq_1(\eta)}\,\mathrm{d}s\Big|\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\le (2\pi)^{-n}\,t\,\widetilde{\gamma}_{n+1,n} \sum_{|\alpha|\le n+1} \|\phi_\alpha\|_{L^1} \\ &\qquad \times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-(n+1)/2}(1+a^2(\eta))|\widehat{u}(\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\le d_2\,t\,\|(1+a^2)\,\widehat{u}\|_{L^1}. \end{aligned}$$

Next we turn to I_2. Using (4.5), we find that

$$
\begin{aligned}
|I_2| &\leq (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left|\lambda_{t-s}(x,\xi) - e^{-(t-s)q(x,\xi)}\right| \\
&\qquad \times |\widehat{q}_2(\xi-\eta,\eta)|\, e^{-sq_1(\eta)}\, |\widehat{u}(\eta)| \, \mathrm{d}\eta \, \mathrm{d}\xi \, \mathrm{d}s \\
&\leq 2(2\pi)^{-n} \left(\gamma_0 + \|\phi_0\|_{L^\infty}\right) \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (t-s)(1+a^2(\xi))|\widehat{q}_2(\xi-\eta,\eta)| \\
&\qquad \times e^{-sq_1(\eta)} |\widehat{u}(\eta)| \, \mathrm{d}\eta \, \mathrm{d}\xi \, \mathrm{d}s \\
&\leq 2(2\pi)^{-n}\, \widetilde{\gamma}_{n+3,n} \left(\gamma_0 + \|\phi_0\|_{L^\infty}\right) \\
&\qquad \times \sum_{|\alpha|\leq n+3} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-(n+3)/2} \\
&\qquad \times (1+a^2(\xi))(1+a^2(\eta))|\widehat{u}(\eta)| \int_0^t (t-s) e^{-sq_1(\eta)} \, \mathrm{d}s \, \mathrm{d}\eta \, \mathrm{d}\xi \\
&\leq (2\pi)^{-n}\, \widetilde{\gamma}_{n+3,n} \left(\gamma_0 + \|\phi_0\|_{L^\infty}\right) \\
&\qquad \times \sum_{|\alpha|\leq n+3} \|\phi_\alpha\|_{L^1} t^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-(n+3)/2} \\
&\qquad \times \frac{1+a^2(\xi)}{1+a^2(\eta)} \,(1+a^2(\eta))^2\, |\widehat{u}(\eta)| \, \mathrm{d}\eta \, \mathrm{d}\xi \\
&\leq d_3\, t^2\, \|(1+a^2)^2\, \widehat{u}\|_{L^1},
\end{aligned}
$$

where we have used (1.12) and (2.2). □

Corollary 4.3. *In the situation of Theorem* 4.2 *we have*

$$
|T_t u(x) - S_t u(x)| \leq d_2'\, t\, \|u\|_{n+3} + d_3'\, t^2\, \|u\|_{n+5}. \tag{4.10}
$$

Proof. Since $1 + a^2(\eta) \leq \kappa_1(1+|\eta|^2)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} (1+a^2(\eta))|\widehat{u}(\eta)| \, \mathrm{d}\eta \\
&\qquad \leq \kappa_1 \int_{\mathbb{R}^n} (1+|\eta|^2)|\widehat{u}(\eta)| \, \mathrm{d}\eta \\
&\qquad \leq \kappa_1 \int_{\mathbb{R}^n} (1+|\eta|^2)^{-(n+1)/2}(1+|\eta|^2)^{(n+3)/2}\, |\widehat{u}(\eta)| \, \mathrm{d}\eta \\
&\qquad \leq \kappa_1 \Big(\int_{\mathbb{R}^n} (1+|\eta|^2)^{-(n+1)/2} \, \mathrm{d}\eta \Big)^{1/2} \|u\|_{n+3},
\end{aligned}
$$

and analogously we get

$$\int_{\mathbb{R}^n} (1 + a^2(\eta))^2 \, |\widehat{u}(\eta)| \; \mathrm{d}\eta \le \kappa_1^2 \Big(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{-(n+1)/2} \; \mathrm{d}\eta \Big)^{1/2} \, \|u\|_{n+5},$$

which proves the corollary. □

5. Towards a symbolic calculus

In the theory of pseudodifferential operators it is useful to have a *symbolic calculus*, i.e., a calculus reducing operations on the level of operators to manipulations on the level of functions, namely the symbols. Differentiability properties w.r.t. the second variable of the symbol are essential when establishing such a calculus. In general, the symbols considered here are not differentiable in the second variable ξ. Therefore, a symbolic calculus does not seem to be available. In [7] W. Hoh, however, made a remarkable attempt to develop such a symbolic calculus for pseudodifferential operators with symbols $q(x,\xi)$ which are for fixed x negative definite functions of ξ.

One of the most interesting problems in a symbolic calculus is the comparison of the semigroup $\{T_t\}_{t\ge 0}$ generated by $-q(x,D)$ with the one-parameter family of pseudodifferential operators with symbols $(2\pi)^{n/2} e^{-tq(x,\xi)}$, $t \ge 0$,

$$e^{-tq(x,D)} u(x) = \int_{\mathbb{R}^n} e^{ix\xi} \, e^{-tq(x,\xi)} \, \widehat{u}(\xi) \; \mathrm{d}\xi. \tag{5.1}$$

Note that $e^{-tq(x,D)}$ *does not mean* the exponential, i.e., the semigroup, of and related to the operator $q(x,D)$ but something that could be called *symbolic exponential*. Of course, in the constant coefficient case, where $q(x,\xi)$ is independent of x, both notions of exponentials coincide.

In order to estimate the difference between the proper and the symbolic exponentials we need some auxiliary results.

Lemma 5.1. *For all $x \ge 0$ and $m \in \mathbb{N}$ we have*

$$e^{-x} \le e\Big(\frac{m}{e}\Big)^m \Big(\frac{1}{1+x}\Big)^m \le e\Big(\frac{m}{e}\Big)^m \, \frac{1}{1+x^m}. \tag{5.2}$$

(Note that by Stirling's formula $e\big(\frac{m}{e}\big)^m \le m!$.)

Proof. By elementary calculus we determine the maximum of $f(x) = (1+x)$, $e^{-x/m}$. This proves the first inequality, whereas the second is obvious. □

Let us recall the chain rule for higher-order derivatives, cf. [5, 0.430].

Lemma 5.2. *Let $f, g \in C^N(\mathbb{R})$. For any $1 \le k \le N$ we have*

$$\partial_x^k (f \circ g)(x) = \sum_{j=1}^{k} f^{(j)}(g(x)) \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_\ell=j \\ \alpha_1+2\alpha_2+\ldots+\ell\alpha_\ell=k}} \frac{k!}{\alpha_1!\ldots\alpha_\ell!} \prod_{i=1}^{\ell} \Big(\frac{g^{(i)}(x)}{i!}\Big)^{\alpha_i}. \tag{5.3}$$

We want to estimate $(x, \xi) \mapsto e^{-tq(x,\xi)}$ and its derivatives w.r.t. the variable x.

Lemma 5.3. *Suppose that the General Assumption,* (2.6) *and* (2.10.m) *hold for some $m \in \mathbb{N}$, and let $N \le m$. Then for all $k \in \mathbb{Z}$, $k \ge -1$, and $\beta \in \mathbb{N}_0^n$ with $|\beta| = 1$ there exist functions $\Phi_{N\beta,j} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $1 \le j \le N$, such that*

$$\big|\partial_x^{N\beta} e^{-tq(x,\xi)}\big| \le \frac{1}{t^k}\Big(\sum_{j=1}^{N} e\Big(\frac{k+j}{\gamma_2\, e}\Big)^{k+j} \Phi_{N\beta,j}(x)\Big)\Big(\frac{1}{1+a^2(\xi)}\Big)^k \tag{5.4}$$

holds for all $0 < t \le 1$ and $x, \xi \in \mathbb{R}^n$.

Proof. By Lemma 5.2 we have

$$\partial_x^{N\beta} e^{-tq(x,\xi)} = \sum_{j=1}^{N} (-t)^j\, e^{-tq(x,\xi)} \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_\ell=j \\ \alpha_1+2\alpha_2+\ldots+\ell\alpha_\ell=N}} \frac{N!}{\alpha_1!\ldots\alpha_\ell!} \prod_{i=1}^{\ell} \Big(\frac{\partial_x^{i\beta} q_2(x,\xi)}{i!}\Big)^{\alpha_i}, \tag{5.5}$$

where we have used the fact that $\partial_x^{i\beta} q(x,\xi) = \partial_x^{i\beta}(q_1(\xi) + q_2(x,\xi)) = \partial_x^{i\beta} q_2(x,\xi)$. From Lemma 5.1 we obtain

$$e^{-tq(x,\xi)} \le e\Big(\frac{m}{e}\Big)^m \Big(\frac{1}{1+tq(x,\xi)}\Big)^m$$

for all $m \in \mathbb{N}_0$ and for $t \le 1$ we have because of (2.6)

$$e^{-tq(x,\xi)} \le \frac{1}{t^m}\, e\Big(\frac{m}{\gamma_2\, e}\Big)^m \Big(\frac{1}{1+a^2(\xi)}\Big)^m.$$

If $m = j + k$, $j \ge 1$ and $k \ge -1$, then

$$\big|(-t)^j e^{-tq(x,\xi)}\big| \le \frac{1}{t^k}\, e\Big(\frac{k+j}{\gamma_2\, e}\Big)^{k+j} \Big(\frac{1}{1+a^2(\xi)}\Big)^{k+j}.$$

By (2.10.m) we have $\left|\partial_x^{i\beta} q_2(x,\xi)\right| \le \phi_{i\beta}(x)(1+a^2(\xi))$ with $\phi_{i\beta} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $1 \le i \le N$, hence

$$\Big|\prod_{i=1}^{\ell}\Big(\frac{\partial_x^{i\beta} q_2(x,\xi)}{i!}\Big)^{\alpha_i}\Big| \le \prod_{i=1}^{\ell}\Big(\frac{\phi_{i\beta}(x)}{i!}\Big)^{\alpha_i}(1+a^2(\xi))^{\alpha_i}.$$

For $t \le 1$ and $k \ge -1$ we have

$$\begin{aligned}\big|\partial_x^{N\beta} e^{-tq(x,\xi)}\big| &\le \frac{1}{t^k}\sum_{j=1}^{N} e\Big(\frac{k+j}{\gamma_2\, e}\Big)^{k+j}\Big(\frac{1}{1+a^2(\xi)}\Big)^{k+j}\\ &\quad\times \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_\ell=j\\ \alpha_1+2\alpha_2+\ldots+\ell\alpha_\ell=N}} \frac{N!}{\alpha_1!\ldots\alpha_\ell!}\prod_{i=1}^{\ell}\Big(\frac{\phi_{i\beta}(x)}{i!}\Big)^{\alpha_i}(1+a^2(\xi))^{\alpha_i}\\ &= \frac{1}{t^k}\Big(\sum_{j=1}^{N} e\Big(\frac{k+j}{\gamma_2\, e}\Big)^{k+j}\Phi_{N\beta,j}(x)\Big)\Big(\frac{1}{1+a^2(\xi)}\Big)^{k},\end{aligned}$$

where

$$\Phi_{N\beta,j}(x) = \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_\ell=j\\ \alpha_1+2\alpha_2+\ldots+\ell\alpha_\ell=N}} \frac{N!}{\alpha_!\ldots\alpha_\ell!}\prod_{i=1}^{\ell}\Big(\frac{\phi_{i\beta}(x)}{i!}\Big)^{\alpha_i}, \tag{5.6}$$

which by our assumptions belongs to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. ☐

Corollary 5.4. *In the situation of Theorem* 5.3 *we have*

$$\big|\partial_x^{N\beta}\big(e^{-tq(x,\xi)} - e^{-tq_1(\xi)}\big)\big| \le \frac{1}{t^k}\,\Psi_{N\beta,k}(x)\Big(\frac{1}{1+a^2(\xi)}\Big)^{k} \tag{5.7}$$

for all $0 < t \le 1$, $k > -1$, $N = 0,1,\ldots,m$, *and* $\beta \in \mathbb{N}_0^n$, $|\beta| = 1$, *and a suitable function* $\Psi_{N\beta,k} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. Since $e^{-tq_1(\xi)}$ is independent of x, we have only to consider the case $N = 0$, i.e., the case when there is no derivative at all. Observing that $q(x,\xi) = q_1(\xi) + q_2(x,\xi)$, we get

$$\begin{aligned}\big|e^{-tq(x,\xi)} - e^{-tq_1(\xi)}\big| &= e^{-t(q_1(\xi)\wedge q(x,\xi))}\,\big|e^{-t(q-q_1\wedge q)(x,\xi)} - e^{-t(q_1-q_1\wedge q)(x,\xi)}\big|\\ &= e^{-t(q_1(\xi)\wedge q(x,\xi))}\,\big(1-e^{-t\,|q_2(x,\xi)|}\big)\\ &\le t\,|q_2(x,\xi)|\big(e^{-tq(x,\xi)} \vee e^{-tq_1(\xi)}\big).\end{aligned}$$

By (2.10.0), (2.5), (2.6), and Lemma 5.1 we have for $t \leq 1$

$$\begin{aligned}\left|e^{-tq(x,\xi)} - e^{-tq_1(\xi)}\right| &\leq t\,\phi_0(x)(1+a^2(\xi))\big(e^{-tq(x,\xi)} \vee e^{-tq_1(\xi)}\big)\\ &\leq t\,\phi_0(x)\big(1+a^2(\xi)\big)\,e\Big(\frac{k+1}{e}\Big)^{k+1}\Big(\frac{1}{1+tq(x,\xi)} \vee \frac{1}{1+tq_1(\xi)}\Big)^{k+1}\\ &\leq \frac{1}{t^k}\,\phi_0(x)\,e\Big(\frac{k+1}{e}\Big)^{k+1}\Big(\frac{1}{\gamma_2} \vee \frac{1}{\gamma_1}\Big)^{k+1}\Big(\frac{1}{1+a^2(\xi)}\Big)^{k},\end{aligned}$$

and the assertion follows with

$$\Psi_{N\beta,k}(x) = \begin{cases} e\Big(\frac{k+1}{(\gamma_1\wedge\gamma_2)\,e}\Big)^{k+1}\phi_0(x) & \text{if } N=0\\ \sum_{j=1}^N e\Big(\frac{k+j}{\gamma_2\,e}\Big)^{k+j}\Phi_{N\beta,j}(x) & \text{if } N>0,\end{cases} \tag{5.8}$$

and $\Phi_{N\beta,j}$ as in (5.6). □

From Theorem 5.3 and Corollary 5.4 we get the following result.

Corollary 5.5. *Let $q(x,\xi)$ be as in Theorem 5.3 and $N \leq m$. Then we have for all $0 < t \leq 1$, $x, \xi \in \mathbb{R}^n$, and $k \geq -1$*

$$\left|\mathbf{F}_{x\mapsto\eta}\big(e^{-tq(x,\xi)} - e^{-tq_1(\xi)}\big)(\eta,\xi)\right| \leq \frac{1}{t^k}\,C\,\Big(\frac{1}{1+a^2(\xi)}\Big)^{k}\Big(\frac{1}{1+|\eta|^2}\Big)^{N/2} \tag{5.9}$$

with the constant $C = C(N,n,k,\Phi_\bullet,\phi_\bullet)$ given by

$$\begin{aligned}C = (n+1)^{\frac{N}{2}-1}\Bigg(&e\Big(\frac{k+1}{(\gamma_1\wedge\gamma_2)\,e}\Big)^{k+1}\|\phi_0\|_{L^1}\\ &+\sum_{j=1}^{N}\sum_{|\beta|=1} e\Big(\frac{k+j}{\gamma_2\,e}\Big)^{k+j}\|\Phi_{N\beta,j}\|_{L^1}\Bigg).\end{aligned} \tag{5.10}$$

Proof. For $N \in \mathbb{N}$ we obtain from Hölder's inequality

$$(1+|\eta|^2)^{\frac{N}{2}} \leq (n+1)^{\frac{N}{2}-1}\,(1+|\eta_1|^N + \ldots + |\eta_n|^N). \tag{5.11}$$

Therefore, we need only to estimate $\left|\eta^{N\beta}\int_{\mathbb{R}^n} e^{-ix\eta}\,(e^{-tq(x,\eta)} - e^{-tq_1(\eta)})\,\mathrm{d}x\right|$ for some $\beta \in \mathbb{N}_0^n$, $|\beta| = 1$, and $N \in \mathbb{N}_0$. We proceed as in [10, Lemma 2.1] to obtain

$$\begin{aligned}
&\Big|\eta^{N\beta}\int_{\mathbb{R}^n} e^{-ix\eta}\big(e^{-tq(x,\eta)}-e^{-tq_1(\eta)}\big)\,\mathrm{d}x\Big|\\
&\qquad\qquad = \Big|\int_{\mathbb{R}^n}\big(\partial_x^{N\beta}e^{-ix\eta}\big)\big(e^{-tq(x,\eta)}-e^{-tq_1(\eta)}\big)\,\mathrm{d}x\Big|\\
&\qquad\qquad = \Big|\int_{\mathbb{R}^n} e^{-ix\eta}\,\partial_x^{N\beta}\big(e^{-tq(x,\eta)}-e^{-tq_1(\eta)}\big)\,\mathrm{d}x\Big|\\
&\qquad\qquad \leq \frac{1}{t^k}\,\|\Psi_{N\beta,k}\|_{L^1}\Big(\frac{1}{1+a^2(\xi)}\Big)^k
\end{aligned}$$

for any $N\in\mathbb{N}_0$ and $|\beta|=1$. The assertion follows by summation over these indices. □

Using Corollary 5.5 we can study some mapping properties of the operators $e^{-tq(x,D)}$ and $e^{-tq(x,D)}-e^{-tq_1(D)}$.

Theorem 5.6. *Suppose that the* General Assumption, (2.6), *and* (2.10.m) *hold for some* $m>n-2$. *Then* $e^{-tq(x,D)}-e^{-tq_1(D)}$ *is for any fixed* t *a continuous operator from* $L^2(\mathbb{R}^n)$ *into* $H^{a^2,k}(\mathbb{R}^n)$, $-1\leq k<m-n/2$. *Moreover, we have for* $0<t\leq 1$ *the estimate*

$$\|\big(e^{-tq(x,D)}-e^{-tq_1(D)}\big)u\|_{a^2,k}\leq \widetilde{C}(N,n,k,\Phi_\bullet,\phi_\bullet)\,\frac{1}{t^k}\,\|u\|_0. \tag{5.12}$$

(A similar estimate with a larger constant holds for any $0<t\leq$ const.*)*

Proof. It is clearly sufficient to prove (5.12). Choose $u\in L^2(\mathbb{R}^n)$ and $v\in C_0^\infty(\mathbb{R}^n)$ and put

$$\widehat{q}_t(\eta,\xi)=\mathbf{F}_{x\mapsto\eta}\left(e^{-tq(x,\xi)}-e^{-tq_1(\xi)}\right)(\eta,\xi).$$

Then Corollary 5.5 gives with $C=C(m,n,k,\Phi_\bullet,\phi_\bullet)$

$$\begin{aligned}
&\big|\big([e^{-tq(x,D)}-e^{-tq_1(D)}]u,v\big)_0\big|\\
&\qquad = (2\pi)^{-n}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widehat{q}_t(\xi-\eta,\eta)\widehat{u}(\eta)\overline{\widehat{v}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi\Big|\\
&\qquad \leq (2\pi)^{-n}C\,\frac{1}{t^k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-m/2}\\
&\qquad\qquad \times\Big(\frac{1}{1+a^2(\eta)}\Big)^k|\widehat{u}(\eta)|\,|\widehat{v}(\xi)|\,\mathrm{d}\eta\,\mathrm{d}\xi =
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-n} C \, \frac{1}{t^k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-m/2} \\
&\qquad \times \Big(\frac{1 + a^2(\xi)}{1 + a^2(\eta)} \Big)^k |\widehat{u}(\eta)| \Big(\frac{1}{1 + a^2(\xi)} \Big)^k |\widehat{v}(\xi)| \; \mathrm{d}\eta \; \mathrm{d}\xi \\
&\leq 2^k \, (2\pi)^{-n} C \, \frac{1}{t^k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{(2k-m)/2} \, |\widehat{u}(\eta)| \\
&\qquad \times \Big(\frac{1}{1 + a^2(\xi)} \Big)^k |\widehat{v}(\xi)| \; \mathrm{d}\eta \; \mathrm{d}\xi \\
&\leq 2^k \, (2\pi)^{-n} C \, \frac{1}{t^k} \Big(\int_{\mathbb{R}^n} (1 + |\tau|^2)^{(2k-m)/2} \; \mathrm{d}\tau \Big) \|u\|_0 \, \|v\|_{a^2,-k}.
\end{aligned}
$$

For $-1 \leq k < \frac{m-n}{2}$, we obtain

$$
\frac{\big| \big([e^{-tq(x,D)} - e^{-tq_1(D)}] u, v \big)_0 \big|}{\|v\|_{a^2,-k}} \leq 2^k \, (2\pi)^{-n} C \, \frac{1}{t^k} \Big(\int_{\mathbb{R}^n} (1 + |\tau|^2)^{(2k-m)/2} \; \mathrm{d}\tau \Big) \|u\|_0,
$$

and the theorem follows. $\square$

Since $e^{-tq_1(D)} \colon L^2(\mathbb{R}^n) \to H^{a^2,k}(\mathbb{R}^n)$ is continuous, too, the next corollary easily follows from the triangular inequality.

Corollary 5.7. *In the situation of Theorem* 5.6 *both* $e^{-tq(x,D)} - e^{-tq_1(D)}$ *and* $e^{-tq(x,D)}$ *are for any fixed* t *continuous operators from* $L^2(\mathbb{R}^n)$ *into* $H^{a^2,k}(\mathbb{R}^n)$, $-1 \leq k < \frac{m-n}{2}$.

From our *General Assumption*, in particular from (2.1), we know that $H^{a^2,k}(\mathbb{R}^n)$ is continuously embedded into $H^{kr_0}(\mathbb{R}^n)$. By Sobolev's embedding theorem we conclude for $k > (n - \ell)/(2r_0)$, $\ell \in \mathbb{N}_0$, that $H^{a^2,k} \subset C_b^{\ell}(\mathbb{R}^n)$, thus $\bigcap_{k \in \mathbb{N}} H^{a^2,k}(\mathbb{R}^n) \subset C_b^{\infty}(\mathbb{R}^n)$. This already proves the following Corollary.

Corollary 5.8. *Suppose that the assumptions of Theorem* 5.6 *hold with* $k > (n - \ell)/(2 r_0)$, $\ell \in \mathbb{N}_0$. *Then the operators* $e^{-tq(x,D)} - e^{-tq_1(D)}$ *and* $e^{-tq(x,D)}$ *map* $L^2(\mathbb{R}^n)$ *continuously into* $C_b^{\ell}(\mathbb{R}^n)$ *for any fixed* $t > 0$. *In case* $m = \infty$, *the operators map* $L^2(\mathbb{R}^n)$ *continuously into* $C_b^{\infty}(\mathbb{R}^n)$.

Note that Corollary 5.8 *does not require* any smallness condition on $\sum_{|\alpha| \leq m} \|\phi_\alpha\|_{L^1}$.

We will now proceed with the comparison of $\{T_t\}_{t \geq 0}$ and $\{e^{-tq(x,D)}\}_{t \geq 0}$.

Lemma 5.9. *Let $\{T_t\}_{t\geq 0}$ be a Fellerian semigroup generated by the pseudodifferential operator $-q(x,D)$. We assume the General Assumption, (2.10.0), and (4.1)–(4.4). Then we have for all $u \in C_0^\infty(\mathbb{R}^n)$, $t \geq 0$, and every $x \in \mathbb{R}^n$ the estimate*

$$\left|T_t u(x) - e^{-tq(x,D)}u(x)\right| \leq 2\,t\left(\gamma_0 + \|\phi_0\|_{L^\infty}\right)\|(1+a^2)\,\widehat{u}\|_{L^1}. \tag{5.13}$$

Proof. Observe that for every $x \in \mathbb{R}^n$ and $t \geq 0$ the operator $e^{-tq(x,D)}$ is well-defined for those u with $\widehat{u} \in L^1(\mathbb{R}^n)$ which is clearly the case if $u \in C_0^\infty(\mathbb{R}^n)$. This follows from the fact that $\xi \mapsto q(x,\xi)$ is negative definite, i.e., $q(x,\xi) \geq 0$, and therefore $e^{-tq(x,\xi)} \leq 1$. Now, with the notation of Section 4,

$$\begin{aligned}\left|T_t u(x) - e^{-tq(x,D)}u(x)\right| &= \left|\int_{\mathbb{R}^n} e^{ix\xi}\big(\lambda_t(x,\xi) - e^{-tq(x,\xi)}\big)\widehat{u}(\xi)\,\mathrm{d}\xi\right| \\ &\leq \int_{\mathbb{R}^n}\left|\lambda_t(x,\xi) - e^{-tq(x,\xi)}\right|\left|\widehat{u}(\xi)\right|\,\mathrm{d}\xi \\ &\leq 2\,t\,(\gamma_0 + \|\phi_{0\,L^\infty}\|)\,\|(1+a^2)\,\widehat{u}\|_{L^1}.\end{aligned}$$

□

As in Section 4 we have:

Corollary 5.10. *Under the assumptions of Lemma 5.9 with $m > n+1$ we have for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\left|T_t u(x) - e^{-tq(x,D)}u(x)\right| \leq t\,\gamma'\,\|u\|_{n+1}, \tag{5.14}$$

where γ' is given by

$$\gamma' = 2\left(\gamma_0 + \|\phi_0\|_{L^\infty}\right)\kappa_1\Big(\int_{\mathbb{R}^n}(1+|\tau|^2)^{-(n+1)/2}\,\mathrm{d}\tau\Big)^{1/2}. \tag{5.15}$$

The operator $e^{-tq(x,D)}$ is therefore, at least for small $t > 0$, a reasonable approximation of T_t. If one wants to compare $e^{-tq(x,D)}u$ and $T_t u$ in L^2-norm, it is necessary to have some estimates for the partial derivatives of $\lambda_t(x,\xi)$ w.r.t. the variable x. Such estimates are difficult to obtain from (4.1). Another approach using the resolvent of $-q(x,D)$ rather than the semigroup seems to be possible. This will be considered in a forthcoming paper.

References

[1] C. Berg and G. Forst, *Potential theory on locally compact Abelian groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, II. Ser., vol. 87, Springer, Berlin, 1975.

[2] Ph. Courrège, *Générateur infinitésimal d'un semi-groupe de convolution sur R^n, et formule de Lévy-Khinchine*, Bull. Sci. Math. 2^e sér. **88** (1964), 3–30.

[3] Ph. Courrège, *Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principe du maximum*, Sém. Théorie du Potentiel (1965/66), 38.

[4] Ph. Courrège, *Sur la forme intégro-différentielle au génerateur infinitesimal d'un semi-groupe de Feller sur une variété*, Sém. Théorie du Potentiel (1965/66), 48.

[5] I. Gradshteyn and I. Ryzhik, *Tables of integrals, series, and products. Corrected and enlarged edition*, Academic Press, San Diego, CA, 1992, (7th ed.).

[6] W. Hoh, *Pseudodifferential operators with negative definite symbols and the martingale problem*, Stochastics Stochastics Rep. **55** (1995), 225–252.

[7] W. Hoh, *A symbolic calculus for pseudodifferential operators generating Feller semigroups*, Preprint (1995).

[8] W. Hoh and N. Jacob, *On the Dirichlet problem for pseudodifferential operators generating Feller semigroups*, J. Funct. Anal. (to appear).

[9] L. Hörmander, *The analysis of linear partial differential operators III*, Grundlehren Math. Wiss., vol. 274, Springer, Berlin, 1985.

[10] N. Jacob, *A class of Feller semigroups generated by pseudodifferential operators*, Math. Z. **215** (1994), 151–166.

[11] N. Jacob, *Non-local (semi-)Dirichlet forms generated by pseudodifferential operators*, Dirichlet forms and stochastic processes (Z. M. Ma, M. Röckner and J. A. Yan, eds.), Proc. Intern. Conf. Dirichlet Forms and Stochastic Processes, Beijing 1993, de Gruyter, Berlin, 1995, pp. 223–233.

[12] N. Jacob, *Lineare partielle Differentialgleichungen*, Akademie Verlag, Berlin, 1995.

[13] N. Jacob, *Characteristic functions and symbols in the theory of Feller processes*, Potential Anal. (to appear).

[14] N. Jacob and H.-G. Leopold, *Pseudodifferential operators with variable order of differentiation generating Feller semigroups*, Integral Equations Operator Theory **17** (1993), 544–553.

[15] H. Kumano-go, *Pseudo-differential operators*, MIT Press, Cambridge, MA, 1981.

[16] V. P. Maslov and I. A. Shishmareyev, *On the behavior of one-parameter operator semigroups e^{tA} as $t \to \infty$*, Uspekhi Mat. Nauk **29** (1973), 177. (Russian)

[17] A. Negoro, *Stable-like processes: construction of the transition density and the behavior of sample paths near $t = 0$*, Osaka J. Math. **31** (1994), 189–214.

[18] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, Berlin, 1987.

[19] M. Tsuchyia, *Lévy measure with generalized polar decomposition and the associated SDE with jumps*, Stochastics Stochastics Rep. **38** (1992), 95–117.

[20] K. Yosida, *Functional analysis*, Grundlehren Math. Wiss., vol. 123, Springer, Berlin, 1980, (6th ed.).

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