ON AN ISOMORPHISM BETWEEN GENERALIZED HÖLDER SPACES IMPLEMENTED BY THE FRACTIONAL INTEGRATION OPERATOR

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1. Introduction

Let $I_+^\alpha \varphi$ denote the Riemann-Liouville fractional integral operator of order $\alpha(>0)$:

$$(1) \qquad \big(I_+^\alpha \varphi\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, \mathrm{d}t, \quad 0 < x < 1.$$

Here we consider an isomorphism problem for the generalized Hölder spaces $H_p^\omega[0,1]$. We establish this isomorphism by fractional integration operator acting between the spaces $H_p^\omega[0,1]$ and $H_p^{\omega_\alpha}[0,1]$ with a characteristic ω_α. A classical result of Hardy and Littlewood is well known and concerns the case $p=\infty$ and the usual Hölder space $H_0^\mu[0,1]$. It asserts that the fractional integral operator $I_+^\alpha \varphi$ is a one-to-one mapping between the spaces $H_0^\mu[0,1]$ and $H_0^{\mu+\alpha}[0,1]$ if $0<\mu$, $\alpha<1$, $\mu+\alpha<1$. Here $H_0^\mu[0,1]$ is a subspace of $H^\mu[0,1]$ consisting of Hölder continuous functions $\varphi(x)$ such that $\varphi(0)=0$. This assertion was generalized in various directions. B. S. Rubin extended this result to the case of the weighted Hölder spaces $H_0^\mu([0,1],\rho)$ with the power weight

$$\rho(x) = \prod_{k=1}^n |x-x_k|^{\mu_k}, \quad 0 \le x_1 < x_2 < \cdots < x_n \le 1.$$

He showed that $I_+^\alpha \varphi$ implements an isomorphism between $H_0^\mu([0,1],\rho)$ and $H_0^{\mu+\alpha}([0,1],\rho)$. S. G. Samko and Kh. M. Murdaev extended this result to the case of the generalized Hölder spaces $H_0^\omega[0,1]$ (and weighted spaces

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$H_0^\omega([0,1],\rho)$ with a power weight). They showed that $I_+^\alpha\varphi$ establishes an isomorphism between the spaces $H_0^\omega[0,1]$ and $H_0^{\omega_\alpha}[0,1]$, where $\omega_\alpha(t) = t^\alpha\omega(t)$ and $\omega(t)$ satisfies certain conditions of Bari-Stechkin type. They also considered the so called Weyl fractional integrals in periodic classes $H^\omega[0,1]$. Besides, B. S. Rubin, V. G. Vakulov and S. G. Samko obtained some multidimensional results for potentials in $\mathbb{R}^n$ or on a sphere in $\mathbb{R}^n$. These and many other results as well as historical notes can be found in the encyclopaedic monograph by S. G. Samko, A. A. Kilbas and O. I. Marichev [1], published on Russian in 1987 and translated into English in 1993 (Gordon & Breach).

Here we consider an isomorphism problem for spaces $H_p^\omega[0,1]$ in the case $p<\infty$. The main problem is to find a natural replacement for the condition $\varphi(0)=0$ in the case $p=\infty$. It should be observed that the main role in investigations concerning the case $p=\infty$ was connected with the Zygmund type estimates for the fractional integral $I_+^\alpha\varphi$ and the fractional derivative $D_+^\alpha f$. The investigation of the case $p<\infty$ is based on two approaches: a) using Zygmund type estimates [2]; b) using natural extension [3], [4] of a function in $H_p^\omega[0,1]$ by zero for $x>1$. We describe here both these approaches.

2. Zygmund type estimates

Let $\omega_p(\varphi,\sigma)$ denote the integral modulus of continuity:

$$\omega_p(\varphi,\sigma) = \sup_{0<h<\sigma} \|\varphi(x-h)-\varphi(x)\|_p,$$

where $\|\cdot\|_p$ is a norm in $L_p(0,1)$ and $\varphi(x-h)=0$ if $x-h\notin[0,1]$. Further, let $D_+^\alpha f$ be a fractional derivative in the Marchaud form:

$$(D_+^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{f(x)}{x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{f(x)-f(y)}{(x-y)^{1+\alpha}}\,\mathrm{d}y.$$

Lemma 1. *The following Zygmund estimates are valid:*

$$\omega_p(I_+^\alpha\varphi,\sigma) \le c\left\{\sigma\int_\sigma^1 \frac{\omega_p(\varphi,t)}{t^{2-\alpha}}\,\mathrm{d}t + \sigma^\alpha\left(\int_0^\sigma |\varphi(y)|^p\,\mathrm{d}y\right)^{1/p} + \sup_{h<\sigma} h\left(\int_h^1\left|\frac{\varphi(y)}{y^{1-\alpha}}\right|^p\mathrm{d}y\right)^{1/p}\right\},$$

$$\omega_p(D_+^\alpha f,\sigma) \le c\left\{\int_0^\sigma \frac{\omega_p(f,t)}{t^{1+\alpha}}\,\mathrm{d}t + \left(\int_0^{2\sigma} \frac{1}{t^{1+\alpha}}\,\mathrm{d}t \int_0^t |f(y)|^p\,\mathrm{d}y\right)^{1/p} + \sup_{h<\sigma} h\left(\int_h^1 \left|\frac{f(y)}{y^{1+\alpha}}\right|^p \mathrm{d}y\right)^{1/p}\right\}.$$

Let W denote the class of characteristics, consisting of continuous functions $\omega(t)$ such that $\omega(0)=0$, $\omega(t)>0$, $t>0$, and $\omega(t)$ almost increases. By $H_p^\omega[0,1]$ we denote the generalized Hölder space that consists of functions $\varphi(x)$ from $L_p(0,1)$, satisfying the condition

$$\omega_p(\varphi,t) \le c\,\omega(t),$$

where $\omega(t)$ is called a characteristic of the class $H_p^\omega[0,1]$.

We need the following

Definition 1. By $H_p^{1,\omega}[0,1]$ we denote a subspace in $H_p^\omega[0,1]$ of functions which satisfy the conditions

$$\text{(2)}\qquad \left(\int_0^t |\varphi(y)|^p\,\mathrm{d}y\right)^{1/p} \le c\,\omega(t), \qquad \left(\int_t^1 \left|\frac{\varphi(y)}{y^{1-\alpha}}\right| \mathrm{d}y\right)^{1/p} \le c\,\frac{\omega(t)}{t^{1-\alpha}},$$

and by $H_p^{2,\omega_\alpha}[0,1]$ an analogous space with ω in (2) replaced by ω_α and α by $-\alpha$.

It should be observed that the conditions in (2) are just the replacement of the assumption $\varphi(0)=0$ in the case $p<\infty$.

The main result implies the following theorem proved in [2].

Theorem 2. *Let $0<\alpha<1$ and let the characteristic $\omega(t)$ satisfy the Bari-Stechkin type conditions:*

$$\int_\sigma^1 \frac{\omega(t)}{t^{2-\alpha}}\,\mathrm{d}t \le c\,\frac{\omega(\sigma)}{\sigma^{1-\alpha}}, \qquad \int_0^\sigma \frac{\omega(t)}{t}\,\mathrm{d}t \le c\,\omega(\sigma).$$

Then the fractional integral $I_+^\alpha\varphi$ establishes an isomorphism between the spaces $H_p^{1,\omega}[0,1]$ and $H_p^{2,\omega_\alpha}[0,1]$.

It should be observed that this result may be extended to the weighted case for generalized Hölder spaces with a power weight.

3. Spaces $\overline{H}_p^{\omega,m,k}(0,1)$

Here we consider another approach, which allows to consider also the case $\alpha > 0$ (see [4]) and which was suggested in [3] for $0 < \alpha < 1$. We assume that the characteristic belongs to the class A_μ^λ of functions $\omega(x)$ in W such that $\omega(x)x^{-\mu}$ almost decreases and $\omega(x)x^{-\lambda}$ almost increases.

Definition 2. Let $m \geq 0$ and $k > 0$ be integers, $p \geq 1$ and $\omega(t) \in W$. We say that $f \in H_p^{\omega,m,k}(\mathbb{R}_+^1)$ if $f \in L_p(\mathbb{R}_+^1)$ and f has generalized derivatives up to the order m such that

$$\left\| \Delta_h^k f^{(j)} \right\|_{L_p} \leq ch^{-m}\omega(h),$$

where Δ_h^k is a non-centered finite difference of order k with the step h. Let also $\overline{H}_p^{\omega,m,k}(0,1)$ denote the class of functions which belong to the space $H_p^{\omega,m,k}(\mathbb{R}_+^1)$ when extended by zero for $x > 1$.

It may be shown that the spaces $H_p^{\omega,m,k}(\mathbb{R}_+^1)$ coincide if their parameters m and k satisfy the following so called accordance condition:

$$m \leq \lambda < \mu \leq m + k.$$

It should be observed that the accordance condition can be also formulated in terms of the so called index numbers defined for functions in W.

Theorem 3. *Let $\omega \in A_\mu^\lambda$, $0 < \lambda < \mu$, and let the parameters λ, μ, m and k satisfy the accordance condition with $\mu + \alpha < m + k$. Further, let $\alpha > 0$, $\alpha \neq N$. Then the operator $I_+^\alpha \varphi$ establishes an isomorphism between the spaces $\overline{H}_p^{\omega,m,k}(0,1)$ and $\overline{H}_p^{\omega_\alpha,m,k}(0,1)$, where $1 \leq p < \infty$ and $\omega_\alpha(t) = t^\alpha \omega(t)$.*

The proof of Theorem 3 is based on the following steps:

1) the operator $I_+^\alpha \varphi$ maps $\overline{H}_p^{\omega,m,k}(0,1)$ into $\overline{H}_p^{\omega_\alpha,m,k}(0,1)$;

2) the operator $D_+^\alpha f$ maps $\overline{H}_p^{\omega_\alpha,m,k}(0,1)$ into $\overline{H}_p^{\omega,m,k}(0,1)$;

3) a function f in $\overline{H}_p^{\omega_\alpha,m,k}(0,1)$ can be represented as $f = I_+^\alpha \varphi$ where $\varphi \in \overline{H}_p^{\omega,m,k}(0,1)$.

An analogous theorem can be obtained for an integer α under the condition $\mu + [\alpha] + 1 < m + k$.

4. Concluding remarks

In contrast to Theorem 1, in Theorem 2 we unfortunately need a finite difference of higher order. The question of coincidence of the two approaches presented in Theorems 1 and 2 is open for all $0 < \alpha < 1$. The positive answer to this question was given S. G. Samko and Z. U. Mussalaeva in terms of index numbers for some interval $0 < \alpha < \alpha_0 \leq 1$, where $\alpha_0 = 1$ in the case when the upper and lower index numbers coincide.

The question, however, remains open for $\alpha \geq 1$.

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