NEW ASPECTS IN THE WEIGHT THEORY AND APPLICATIONS

Vakhtang Kokilashvili

Introduction

The purpose of this paper is to describe some of the recent developments in the weight theory of integral operators arising in the harmonic analysis, boundary value problems for analytic functions, and the partial differential equations.

The paper is organized as follows.

Section 1 is auxiliary. It contains the basic notion of homogeneous type spaces (HTS) and several examples.

In Sections 2 and 3 we derive solutions of two weight problems for integral transforms with positive kernels. For the weak type problems these transforms are assumed to be defined on general spaces with a measure and with a given quasi-metric, while a strong type problem is solved in the case of a HTS. Similar problems were investigated previously for some particular cases [1]–[17]. Our main idea, which underlies the solutions of the above mentioned problems, consists in developing the method and conditions proposed in [1] and [5], where the weak type problems were solved in the Euclidean spaces using the well known Besicovitch covering lemma, which, as known (see, e.g., [11]), does not hold in a general HTS. Nevertheless, for the general case we have succeeded in finding criteria for two weight inequalities which are as simple and elegant as those given in [5] for the Euclidean spaces. We observe that the results similar to ours were previously obtained in [11], [12] (see also [16]) only for the spaces having a group structure.

Section 4 gives a complete description of couples of weight functions, guaranteeing the validity of two weight inequalities of strong type for fractional maximal functions defined on a general HTS. Similar results were obtained in [20] for the HTS with a group structure.

Criteria for two weight inequalities for the Riemann-Liouville and the Weyl operators as well as for one-sided fractional integrals are stated in Section 5.

Section 6 is devoted to the applications of these results to two weight imbedding theorems. An improvement of imbedding theorems in the scale of weighted Lizorkin-Triebel spaces of "different metrics" is presented.

In Section 7 we discuss the two weight problem for singular integrals defined on homogeneous groups. This problem remains open even for classical singular integrals. We derive conditions, particularly useful for applications, for the validity of two weight inequalities for singular integrals.

The final Section 8 contains an application of the above-mentioned results to the Dirichlet problem in domains with non-smooth boundaries. The complete picture of solvability in certain setting of this problem is given, and explicit formulae for the solutions are found.

The present exposition is a synthesis of various papers by the author in collaboration with Ioseb Genebashvili, Amiran Gogatishvili, Petr Lizorkin, Aleko Meskhi and Vakhtang Paatashvili. It is a survey of results, comments and some examples. In order to keep a reasonable length we omit all proofs, and the reader is referred to the corresponding original papers [22], [23], [24], [27], [28], [31], [34].

1. Homogeneous type spaces

Let (X, d, μ) be a topological space with a complete measure μ and a given quasi-metric d, that is, with a non-negative function $d\colon X \times X \to \mathbb{R}^1$, satisfying the following conditions:

1) $d(x, x) = 0$ for any x from X;

2) $d(x, y) > 0$ for any x and y from X;

3) there exists a constant a_0 such that $d(x, y) \le a_0 d(y, x)$ for any x and y from X;

4) there exists a constant a_1 such that $d(x, y) \le a_1\big(d(x, z) + d(z, y)\big)$ for any x, y and z from X;

5) the class of continuous functions with a compact support is dense in $L^1(X, \mu)$;

6) we assume that each ball $B(x, r)$ is measurable and that there exists a constant $b > 0$ such that $\mu B(x, 2r) \le b\mu B(x, r)$ for an arbitrary ball $B(x, r)$, that is, the measure μ satisfies the doubling condition.

Spaces satisfying the conditions 1)–6) are called homogeneous type spaces (HTS) (see [18], [19]). Several examples follow:

1° Let $X = \mathbb{R}^n$, and let d be the Euclidean distance. Put $\mathrm{d}\mu(x) = |x|^\alpha\, \mathrm{d}x$,

$\alpha \geq 0$. We see that several Borel measures can be associated to a quasi-distance in order to get a HTS.

2° Let $X = \mathbb{R}^n$, $d(x, y) = \sum_{j=1}^n |x_j - y_j|^{\alpha_j}$ with positive $\alpha_1, \dots, \alpha_n$, and let μ be the Lebesgue measure. In general, with α_j's not all equal, the last formula defines an anisotropic distance.

3° $X = (0, 1)$, and $d(x, y)$ is the length of the smallest dyadic interval containing x and y.

4° $X = [-1, 1]$, d is the usual distance, and $\mathrm{d}\mu(x) = (1-x)^\alpha (1+x)^\beta \,\mathrm{d}x$ the measure, where α, $\beta > -1$.

5° $X = [0, \infty)$, $d(x, y) = |x^2 - y^2|$, the measure $\mathrm{d}\mu = x \,\mathrm{d}x$.

6° Let now $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be an arc-length measure on Γ. By definition Γ is regular if

$$\nu\big(\Gamma \cap B(z, r)\big) \leq cr$$

for any $z \in \mathbb{C}$ and $r > 0$.

For r less than a half of the diameter of Γ the reverse inequality

$$\nu\big(\Gamma \cap B(z, r)\big) \geq r$$

holds for all $z \in \Gamma$. When equipped with ν and the Euclidean distance, a regular curve becomes a HTS.

7° Piecewise smooth or Lipschitzian manifold X in $\mathbb{R}^n$ with the induced Euclidean metric and the Hausdorff measure of the respective dimension.

8° The boundary of a bounded Lipschitzian domain in $\mathbb{R}^n$ with the corresponding harmonic measure.

9° Any C^∞ compact Riemannian manifold with the Riemannian metric and volume.

10° On an arbitrary closed set $F \subset \mathbb{R}^n$ there exists a measure satisfying the doubling condition [21]. Consequently, any such set with the Euclidean metric and the mentioned measure becomes a HTS.

11° The boundary of a strongly pseudo-convex domain in C^n with the Lebesgue measure and anisotropic distance represents a HTS.

In what follows a locally integrable and almost everywhere positive function $w \colon X \to \mathbb{R}^1$ is called a weight function (shortly a weight). For a μ-measurable set E we put

$$wE = \int_E w(x) \,\mathrm{d}\mu.$$

2. A criterion for two weight inequality of weak type for integral transforms with a positive kernel

For a time being, in our discussion of weak type problems we assume that the space (X, d, μ) is a general measure space with a quasi-metric d, i.e., we shall not automatically assume that μ satisfies the doubling condition. We derive a necessary and sufficient condition for the validity of the two weight inequality of weak type for the operator

$$T(f)(x) = \sup_{t\geq 0}\Big|\int\limits_X k(x,y,t)f(y)\,\mathrm{d}\mu\Big|,$$

where $k\colon X\times X\times[0,\infty)\to\mathbb{R}^1$ is a positive measurable kernel. For the general case we have the following sufficient condition for the inequality of weak type.

Theorem 2.1 ([23])**.** *Let $1<p<q<\infty$, let v and μ be arbitrary locally finite measures on X such that $\mu\{x\}=0$ for any $x\in X$. If*

$$c_0 = \sup\big(vB(x,2N_0r)\big)^{1/q}\times \times\Big(\int\limits_{X\setminus B(x,r)} k^{p'}(x,y,t)w^{1-p'}(y)\,\mathrm{d}\mu\Big)^{1/p'}<\infty, \tag{2.1}$$

where the supremum is taken over all $t\geq 0$ and balls $B(x,r)$, such that $\mu B(x,r)>0$, $N_0=a_1(1+2a_0)$, and the constants a_0 and a_1 are from the definition of the quasi-metric, then there exists a constant $c>0$ such that

$$v\{x\in X: T(f)(x)>\lambda\}\leq c\lambda^{-q}\Big(\int\limits_X |f(x)|^p w(x)\,\mathrm{d}\mu\Big)^{q/p} \tag{2.2}$$

for every μ-measurable non-negative function $f\colon X\to\mathbb{R}^1$ and arbitrary $\lambda>0$.

Under some additional condition imposed on the kernel k we have a criterion for the validity of (2.2).

Theorem 2.2 ([23])**.** *Let $1<p<q<\infty$, $\mu\{x\}=0$ for all $x\in X$. If there exists a constant $c_1>0$ such that*

$$k(a,y,t)\leq c_1k(x,y,t) \tag{2.3}$$

for every $t\geq 0$, $a\in X$, $y\in X\setminus B(a,r)$, $r>0$ and $x\in B(a,2N_0r)$, then the conditions (2.1) and (2.2) are equivalent.

Remark 2.1. If $X=\mathbb{R}^n$ and d is the Euclidean distance, then the constant N_0 in the above theorems can be taken equal to 1. This can be done because the Besicovitch covering lemma can be applied to the Euclidean spaces. If, in addition, the measure v is such that $vB(x,r)$ is continuous with respect to r, we can replace $vB(x,2N_0r)$ in (1.1) by $vB(x,r)$ and, accordingly, the condition (1.3) can be weakened as follows: there exists a constant $c_1>0$ such that

$$k(a,y,t)\le c_1k(x,y,t)$$

for all $t\ge 0$, a, x and y from X, satisfying $d(a,x)\le d(a,y)$.

Next we will consider the case where $k(x,y,t)\equiv k(x,y)$. Again, we shall assume that μ is an arbitrary locally finite measure. Let

$$K(f)(x)=T(f)(x)=\int_X k(x,y)f(y)\,\mathrm{d}\mu,$$

$$K^*(f)(x)=\int_X k(y,x)f(y)\,\mathrm{d}\mu.$$

Definition 2.1. A positive kernel $k\colon X\times X\to\mathbb{R}^1$ is said to satisfy the condition (V) $(k\in V)$ if there exists a constant $c>0$ such that

$$k(x,y)<ck(x',y) \tag{2.4}$$

for arbitrary x, y and x' from X such that $d(x,x')<Nd(x,y)$, where $N=2N_0$.

Theorem 2.3 ([23]). *Let $1<p<q<\infty$, let μ and v be locally finite measures, $\mu\{x\}=0$, and let $k\in V$. Then the following conditions are equivalent:*

(i) *there exists a constant $c_1>0$ such that*

$$v\{x\in X: K(f)(x)>\lambda\}\le c_1\lambda^{-q}\Big(\int_X |f(x)|^p\,\mathrm{d}\mu\Big)^{q/p} \tag{2.5}$$

for all $\lambda>0$ and every non-negative measurable f;

(ii) *there exists a constant $c_2>0$ such that*

$$\Big(\int_X \big(K^*(\chi_B\,\mathrm{d}v)(x)\big)^{p'}\,\mathrm{d}\mu\Big)^{1/p'}\le c_2(vB)^{1/q'} \tag{2.6}$$

for any ball B in X;

(iii)

$$\sup\big(vB(x,2N_0r)\big)^{1/q}\Big(\int_{X\setminus B(x,r)} k^{p'}(x,y)\,\mathrm{d}\mu(y)\Big)^{1/p'}<\infty. \tag{2.7}$$

3. A solution of two weight problem of strong type for integral transforms with positive kernels

In this section X is assumed to be a space satisfying the conditions 1)–6) from Section 1 and, in addition, we suppose that $B(x,2r)\setminus B(x,r)\neq\emptyset$ for all r and R, $0<r<R<\infty$.

For a positive measurable kernel $k\colon X\to\mathbb{R}^1$ we put $k^*(x,y)=k(y,x)$.

Definition 3.1. A kernel k satisfies the condition (V') ($k\in V'$) if there exist positive constants $c_1>1$ and $c_2>1$ such that

$$k(x,y)\leq c_1k(x',y)$$

for every x, x' and y from X such that $d(x',y)\leq c_2d(x,y)$ (see [11]).

Theorem 3.1 ([23])**.** *The conditions (V) and (V') are equivalent. Let $1<p<q<\infty$, and assume that k and k^* satisfy condition (V). Then for the inequality*

$$\Big(\int\limits_X|K(f)(x)|^qv(x)\,\mathrm{d}\mu\Big)^{1/q}\leq c_3\Big(\int\limits_X|f(x)|^pw(x)\,\mathrm{d}\mu\Big)^{1/p}\tag{3.1}$$

to hold with a constant c_3 independent on f it is necessary and sufficient that the following two conditions be fulfilled simultaneously:

$$\sup_{\substack{x\in X\\ r>0}}\big(vB(x,2N_0r)\big)^{1/q}\Big(\int\limits_{X\setminus B(x,r)}k^{p'}(x,y)w^{1-p'}(y)\,\mathrm{d}\mu\Big)^{1/p'}<\infty,\tag{3.2}$$

and

$$\sup_{\substack{x\in X\\ r>0}}\big(w^{1-p'}B(x,2N_0r)\big)^{1/p'}\Big(\int\limits_{X\setminus B(x,r)}k^{q}(x,y)v(y)\,\mathrm{d}\mu\Big)^{1/q}<\infty.\tag{3.3}$$

In some particular cases the criteria for the validity of two weight inequalities of strong type have a simpler form.

By definition, the measure ν satisfies the reverse doubling condition if there exist constants $\beta_1>1$ and $\beta_2<1$ such that

$$\nu B(x,r)\leq\beta_2\nu B(x,\beta_1r)\tag{3.4}$$

for all $x\in X$ and $r>0$.

We put $h(B)=\sup\{k(x,y)\colon x,y\in B, d(x,y)>cr(B)\}$, where $r(B)$ is the radius of the ball B and c is a sufficiently small positive constant depending on a_1.

Theorem 3.2 ([23]). *Let $1 < p < q < \infty$ and let the measures v and $w^{1-p'}$ satisfy the reverse doubling condition. Then for the inequality* (3.1) *to hold it is necessary and sufficient that*

$$\sup_B h(B)\Big(\int_B v(x)\,\mathrm{d}\mu\Big)^{1/q}\Big(\int_B w^{1-p'}(x)\,\mathrm{d}\mu\Big)^{1/p'} < \infty. \tag{3.5}$$

Theorem 3.3 ([23]). *Let $1 < p < q < \infty$ and assume that there exists an $r > 1$ such that*

$$\sup_B h(B)(\mu B)^{1/q+1/p'}\Big(\frac{1}{\mu B}\int_B v^q(y)\,\mathrm{d}\mu\Big)^{1/q}\times \times\Big(\frac{1}{\mu B}\int_B w^{(1-p')r}(y)\,\mathrm{d}\mu\Big)^{1/(1-p')} < \infty. \tag{3.6}$$

Then (3.1) *holds.*

4. Criteria for two-weight inequalities of strong type for the fractional maximal function

For a locally integrable function $f\colon X \to \mathbb{R}^1$ the fractional maximal function is defined by

$$M\gamma(f)(x) = \sup(\mu B)^{\gamma-1}\int_B |f(y)|\,\mathrm{d}\mu, \quad 0 < \gamma < 1, \tag{4.1}$$

where the supremum is taken over all balls B of positive measure that contain x.

Theorem 4.1 ([24]). *Let $1 < p < q < \infty$, $0 < \gamma < 1$, let v and w be weight functions. For the inequality*

$$\Big(\int_X \big(M\gamma(f)(x)\big)^q v(x)\,\mathrm{d}\mu\Big)^{1/q} \le c\Big(\int_X |f(x)|^p w(x)\,\mathrm{d}\mu\Big)^{1/p} \tag{4.2}$$

to hold with a constant $c > 0$ it is necessary and sufficient that

$$\sup\big(w^{1-p'}B(x,2N_0r)\big)^{1/p'}\times \times\Big(\int_{X\setminus B(x,r)} v(y)\big(\mu B\big(x,d(x,y)\big)\big)^{(\gamma-1)q}\,\mathrm{d}\mu\Big)^{1/q} < \infty. \tag{4.3}$$

If the measure $w^{1-p'}$ satisfies the reverse doubling condition, then (4.3) takes a simpler form.

Theorem 4.2 ([24]). *Let $1 < p < q < \infty$, $0 < \gamma < 1$, and let the measure $w^{1-p'}$ satisfy the reverse doubling condition. Then* (4.2) *holds if and only if*

$$\sup_{\substack{x\in X\\ r>0}} \big(\mu B(x,r)\big)^{\gamma-1}\big(w^{1-p'}B(x,r)\big)^{1/p'}\big(vB(x,r)\big)^{1/q} < \infty.$$

The above theorems remain valid also for the fractional maximal function

$$M\gamma(f\,\mathrm{d}\sigma)(x) = \sup_{B\ni x}(\mu B)^{\gamma-1}\int_B |f(y)|\,\mathrm{d}\sigma(y), \tag{4.4}$$

where $0 < \gamma < 1$, σ is a Borel measure and the supremum is taken over all balls $B \subset X$ of positive measure that contain x.

Theorem 4.3 ([24]). *Let X be an arbitrary HTS, $1 < p < q < \infty$, $0 < \gamma < 1$, and let σ and ω be positive Borel measures. For the inequality*

$$\Big(\int_X \big(M\gamma(f\,\mathrm{d}\sigma)(x)\big)^q\,\mathrm{d}\omega(x)\Big)^{1/q} \le c\Big(\int_X |f(x)|^p\,\mathrm{d}\omega(x)\Big)^{1/p} \tag{4.5}$$

to hold with a constant $c > 0$ independent of f it is necessary and sufficient that

$$\sup_{\substack{x\in X\\ r>0}} \big(\sigma B(x,2N_0)\big)^{1/p'} \times \\ \times \Big(\int_{X\setminus B(x,r)} \big(\mu B\big(x,d(x,y)\big)\big)^{(\gamma-1)q}\,\mathrm{d}\omega(y)\Big)^{1/q} < \infty. \tag{4.6}$$

Theorem 4.4 ([24]). *Let $1 < p < q < \infty$, $0 < \gamma < 1$, and assume that the measure σ satisfies the reverse doubling condition. Then* (4.5) *holds if and only if*

$$\sup_{B,\,\mu B\neq 0} (\mu B)^{\gamma-1}(\sigma B)^{1/p'}(\omega B)^{1/q} < \infty.$$

Theorems 4.1 and 4.3 were proved earlier in [20] under the additional assumption that the space X possesses a group structure, while Theorem 4.4 deals with the case of homogeneous type spaces under a stronger restriction imposed on the measure σ.

Remark 4.1. Extensions of these results as well as of those in the preceding sections to a half-space can be found in [24].

5. Solutions of weight problems for one-sided potentials and maximal functions

In this section at first we present solutions of two weight problems of weak and strong type for the Riemann-Liouville transform

$$R_\alpha(f)(x) = \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t,$$

and the Weyl integral

$$W_\alpha(f)(x) = \int_x^\infty (t-x)^{\alpha-1} f(t)\,\mathrm{d}t,$$

where $0<\alpha<1$, $x>0$.

As a consequence of Theorems 2.1 and 2.2 we conclude that the following statements are valid.

Theorem 5.1 ([24])**.** *Let $1<p<q<\infty$, $0<\alpha<1$. Then the inequality*

$$v\{x\in(0,\infty)\colon |R_\alpha(f)(x)|>\lambda\} \le c\lambda^{-q}\Big(\int_0^\infty |f(x)|^p w(x)\,\mathrm{d}x\Big)^{q/p}$$

holds with a constant $c>0$ independent of f if and only if

$$\sup_{\substack{a,h\\ 0<h<a}} \Big(\int_{a-h}^{a+h} v(y)\,\mathrm{d}y\Big)^{1/q}\Big(\int_0^{a-h} \frac{w^{1-p'}(y)}{(a-y)^{(1-\alpha)p'}}\,\mathrm{d}y\Big)^{1/p'} < \infty. \tag{5.1}$$

Theorem 5.2 ([24])**.** *Let $1<p<q<\infty$ and $0<\alpha<1$. Then the following two conditions are equivalent:*

(i) *there exists a constant $c_1>0$ such that*

$$v\{x\in(0,\infty)\colon |W_\alpha(f)(x)|>\lambda\} \le c_1\lambda^{-q}\Big(\int_0^\infty |f(x)|^p w(x)\,\mathrm{d}x\Big)^{q/p}$$

holds for every $\lambda>0$ and every measurable f.

(ii)

$$\sup_{\substack{a,h\\ 0<h<a}} \Big(\int_{a-h}^{a+h} v(y)\,\mathrm{d}y\Big)^{1/q}\Big(\int_{a+h}^{\infty} \frac{w^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}\,\mathrm{d}y\Big)^{1/p'} < \infty. \tag{5.2}$$

In particular, Theorem 2.2 yields a solution of the two weight problem of weak type for more general operators, e.g., for

$$N^{+}_{\alpha,\beta}(f)(x) = \sup_{c>x} \frac{1}{(c-x)^{\beta}} \int_{x}^{c} \frac{|f(s)|}{(s-x)^{1-\alpha}}\,\mathrm{d}s,$$

$$M^{+}_{\alpha,\beta}(f)(x) = \sup_{c>x} \frac{1}{(c-x)^{\beta}} \int_{x}^{c} \frac{|f(s)|}{(c-s)^{1-\alpha}}\,\mathrm{d}s,$$

and their left-side analogs.

These operators were investigated in [17].

For the case $\alpha > 1$ the two weight problem for R_α and W_α was solved earlier by many authors, while for $0 < \alpha < 1$ it remained open. We now give solutions for the latter case as well.

Theorem 5.3 ([24]). *Let $1 < p < q < \infty$ and $0 < \alpha < 1$. For the inequality*

$$\Big(\int_{0}^{\infty} |R_\alpha(f)(x)|^{q} v(x)\,\mathrm{d}x\Big)^{1/q} \le c_1 \Big(\int_{0}^{\infty} |f(x)|^{p} w(x)\,\mathrm{d}x\Big)^{1/p}$$

to hold with a constant $c_1 > 0$ independent of f it is necessary and sufficient that (5.1) *is satisfied and*

$$\sup_{\substack{a,h\\ 0<h<a}} \Big(\int_{a-h}^{a+h} w^{1-p'}(y)\,\mathrm{d}y\Big)^{1/p'} \Big(\int_{0}^{a-h} \frac{v(y)}{(a-y)^{(1-\alpha)q}}\,\mathrm{d}y\Big)^{1/q} < \infty.$$

Theorem 5.4 ([24]). *Let $1 < p < q < \infty$ and $0 < \alpha < 1$. For the inequality*

$$\Big(\int_{0}^{\infty} |W_\alpha(f)(x)|^{q} v(x)\,\mathrm{d}x\Big)^{1/q} \le c_2 \Big(\int_{0}^{\infty} |f(x)|^{p} w(x)\,\mathrm{d}x\Big)^{1/p}$$

to hold with a constant independent of f it is necessary and sufficient that (5.1) *is satisfied and*

$$\sup_{\substack{a,h\\ 0<h<\alpha}} \Big(\int_{a-h}^{a+h} w^{1-p'}(y)\,\mathrm{d}y\Big)^{1/p'} \Big(\int_{a+h}^{\infty} \frac{v(y)}{(y-a)^{(1-\alpha)q}}\,\mathrm{d}y\Big)^{1/q} < \infty.$$

6. Imbedding theorems for weighted Lizorkin-Triebel spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and denote by $x = (x_1, \dots, x_n)$ a point in this space. The weighted Lebesgue space $L^p_\sigma(\mathbb{R}^n)$ $(1 \le p < \infty)$ is defined as the space of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}^1$ with the finite norm

$$\|f\|_{L^p_\sigma} = \Big(\int\limits_{\mathbb{R}^n} |f(x)|^p \sigma^p(x)\, \mathrm{d}x \Big)^{1/p}.$$

The symbol $\hookrightarrow$ will stand for a continuous imbedding.

In [28] we investigated the weighted Lizorkin-Triebel spaces $L^{p,\theta}_w$.

Definition 6.1. We say that $f \in L^{p,\theta,r}_w$ $(r > 0,\ 1 < p < \infty)$, if there exists $g \in L^{p,\theta}_w$ such that

$$f(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} G_r(x-y) g(y)\, \mathrm{d}y,$$

where G_r is the Bessel-Macdonald kernel. By definition,

$$\|f \mid L^{p,\theta,r}_w\| = \|g \mid L^{p,\theta}_w\|.$$

We now introduce a special class of couples of weight functions.

Definition 6.2. Let $1 < \alpha < \beta < \infty$, $\mu > 0$. By $V^{\alpha,\beta}_\mu(\mathbb{R}^n)$ we denote the set of couples of weight functions (ϱ, σ) satisfying

$$\sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \Big(\int\limits_{B(x,r)} \varrho^\alpha(y)\, \mathrm{d}y \Big)^{1/\alpha} \Big(\int\limits_{\mathbb{R}^n \setminus B(x,r)} \frac{\sigma^{-\beta'}(y)}{|x-y|^{\mu\beta'}}\, \mathrm{d}y \Big)^{1/\beta'} < \infty. \tag{6.1}$$

Here $B(x,r)$ is a ball in $\mathbb{R}^n$, and, as usual $\beta' = \beta/(\beta-1)$.

The condition (6.1) is a special case of (3.2) when $X = \mathbb{R}^n$, d is the Euclidean distance, μ is the Lebesgue measure, and $k(x,y) = |x-y|^{-\mu}$.

Using Theorem 3.1 we arrive at the following two weight imbedding theorems for “different metrics” and “different dimensions”.

Theorem 6.1 ([27]). *Let $1 < p < q < \infty$, $1 < \theta < \infty$, and $0 < \gamma < r \le n$. Let (ϱ, σ) be a weight couple such that $(\varrho, \sigma) \in V^{q,p}_{n-\gamma}$ and $(1/\sigma, 1/\varrho) \in V^{p',q'}_{n-\gamma}$. Then*

$$L^{p,\theta,r}_\sigma \hookrightarrow L^{q,\theta,s}_\varrho,$$

where $s = r - \gamma$.

Theorem 6.2. *Suppose that $1 < p < q < \infty$, $1 < \theta < \infty$, $1 \le m \le n$, and $0 < \gamma < m$. If $(\varrho, \sigma) \in V^{q,p}_{m-\gamma}(\mathbb{R}^m)$, $(\sigma^{-1}, \varrho^{-1}) \in V^{p',q'}_{m-\gamma}(\mathbb{R}^m)$, and $f(x_1, \dots, x_n) \in L^{p,\theta,r}_\sigma(\mathbb{R}^n)$, then for any fixed $(x^0_{m+1}, \dots, x^0_n) \in \mathbb{R}^{n-m}$ the function $f(x_1, \dots, x_m, x^0_{m+1}, x^0_n)$ belongs to $L^{q,\theta,s}_\varrho(\mathbb{R}^m)$ with $s = r - \gamma - (n-m)/p$, and*

$$\|f(x, x^0_{m+1}, \dots, x^0_n) \mid L^{q,\theta,s}_\varrho(\mathbb{R}^m)\| \le c\|f \mid L^{p,\theta,r}_\sigma(\mathbb{R}^n)\|,$$

where the constant c is independent of f.

These theorems improve the classical imbedding theorems. It should be observed that the one weight case $v = w$ of Theorem 6.1 was proved by the author earlier in [28]. The two weight imbedding theorems for mixed normed spaces of anisotropic Bessel potentials when the weight is in fact the product of one-dimensional weight functions were proved in [29].

We now pass to the consideration of the imbedding theorems for the weighted anisotropic Lizorkin-Triebel spaces. These spaces are defined for $\vec{r} = (r_1, \dots, r_n)$, $r_j > 0$, in the following way: In Definition 6.1 we replace the usual kernel by the anisotropic Bessel-Macdonald kernel. As above, we assume that

$$\|f \mid L^{p,\theta,\vec{r}}_w\| = \|g \mid L^{p,\theta}_w\|, \quad g \in L^{p,\theta}_w.$$

We need also certain new classes $\widetilde{V}^{\alpha,\beta}_\mu$ of couples of weight functions. Their definitions are analogous to those of the classes V_μ with the corresponding replacement of the usual Euclidean distance and n-dimensional balls by the so called a-distance

$$|x-y|_a = \Big(\sum_{j=1}^n |x_j - y_j|^{-2/a_j}\Big)^{1/2}, \quad \frac{1}{a_j} = \frac{r_j}{n}\sum_{j=1}^n \frac{1}{r_j}, \quad j = 1, \dots, n,$$

and the "anisotropic" balls generated by the a-distance, respectively.

The following statement is true.

Theorem 6.3. *Let us suppose that $1 < p < q < \infty$, $1 < \theta < \infty$, $(\varrho, \sigma) \in \widetilde{V}^{q,p}_{n-\gamma}(\mathbb{R}^n)$, $\left(\frac{1}{\sigma}, \frac{1}{\varrho}\right) \in V^{p',q'}_{n-\gamma}(\mathbb{R}^n)$, and $0 < \gamma < n$. If*

$$\kappa = 1 - \frac{\gamma}{n}\sum_{j=1}^n \frac{1}{r_j} \ge 0,$$

then

$$L^{p,\theta,\vec{r}}_\sigma \hookrightarrow L^{q,\theta,\vec{s}}_\varrho$$

with $\vec{s} = \kappa\vec{r}$.

7. Two-weight inequalities for singular integrals on homogeneous groups

In this section we present sufficient conditions for the validity of two weight inequalities of weak and strong type for singular integrals. In general, the problem of a complete description of couples of weights ensuring the validity of two weight inequalities has not been fully solved even for classical singular integrals. Nevertheless, we find some sufficient conditions, which are useful for applications, and which are in certain cases also necessary.

A homogeneous group (HG) is a simply connected nilpotent Lie group G on the Lie algebra g with a given one-parametric group of dilations $\delta_t = \exp(A \ln t)$, where A is a diagonalizable linear operator on g with positive eigenvalues ([30, p. 5]).

For a homogeneous group G the map $\exp \circ \delta_t \circ \exp$, $t > 0$, is an automorphism on G, which will be denoted by the same symbol δ_t. The number $Q = \operatorname{tr} A$ is called a dimension of homogeneity of G.

Homogeneous groups are particular cases of the HTS. The examples of a HG are: the n-dimensional Euclidean space with a fixed diagonal matrix $A = (a_{ij})$ $(i, j = 1, 2, \dots)$, the Heisenberg group, etc.

On a homogeneous group G, a homogeneous norm is introduced, that is, a continuous function $r\colon G \to [0, \infty)$, smooth on $G \setminus \{e\}$ and satisfying

(i) $r(x) = r(x^{-1})$,
(ii) $r(\delta_t x) = \operatorname{tr}(x)$,
(iii) $r(x) = 0 \Leftrightarrow x = e$,
(iv) there exists a constant $c_0 > 0$ such that

$$r(xy) = c_0\big(r(x) + r(y)\big)$$

for all x and y from G.

For $x \in G$ and $\varrho > 0$ we put $B(x, \varrho) = \{y \in G\colon r(xy^{-1}) < \varrho\}$. Let $S(x, \varrho) = \{y \in G\colon r(xy^{-1}) = \varrho\}$. Observe that $\delta_\varrho B(e, 1) = B(e, \varrho)$.

We fix a normalized Haar measure on G such that the measure of the unit ball $B(e, 1)$ is equal to 1. The Haar measure of a measurable set $E \subset G$ will be denoted by $|E|$ and the integral with respect to this measure by $\int_E f(x)\,\mathrm{d}x$.

It is evident that

$$|\delta_t(E)| = t^Q |E|, \quad \mathrm{d}(\delta_t x) = t^Q\,\mathrm{d}x.$$

In particular, $|B(x, r)| = r^Q$ for any $x \in G$ and $r > 0$.

In the sequel, we denote by $L^p_w(G)$ $(1 \le p < \infty)$ the set of all measurable functions $f\colon G \to \mathbb{R}^1$ for which

$$\|f\|_{L^p_w(G)} = \Big(\int\limits_G |f(x)|^p w(x)\,\mathrm{d}x\Big)^{1/p} < \infty.$$

This notation differs slightly from that used in the preceding section.

Definition 7.1. A weight function w belongs to $A_p(G)$ $(1 < p < \infty)$ if

$$\sup \Big(\frac{1}{|B|}\int\limits_B w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int\limits_B w^{1-p'}(x)\,\mathrm{d}x\Big)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset G$.

By definition, $w \in A_1(G)$ if there exists $c_1 > 0$ such that

$$\frac{1}{|B|}\int\limits_B w(x)\,\mathrm{d}x \le c_1 \operatorname*{ess\,inf}_{x\in B} w(x)$$

for every ball $B \subset G$.

For an integrable function $f\colon G \to \mathbb{R}^1$ we consider the Calderón-Zygmund singular integral

$$T(f)(x) = \int\limits_G k(xy^{-1})f(y)\,\mathrm{d}y \overset{\text{def}}{=} \lim_{r\to 0} \int\limits_{G\setminus B(x,r)} k(xy^{-1})f(y)\,\mathrm{d}y,$$

where the measurable function k satisfies the following conditions:

(i) $|k(x)| \le c|x|^{-1}$, $x \in G$,

(ii) k is homogeneous of order $-Q$, that is,

$$k(\delta_t x) = t^{-Q}k(x)$$

for $x \in G$ and $t > 0$,

(iii) the modulus of continuity of k, $\omega(k,\delta) = \sup\{|k(x) - k(y)|\colon x, y \in S_G, r(xy^{-1}) < \delta\}$, satisfies

$$\int\limits_0^1 \frac{\omega(t,k)}{t}\,\mathrm{d}t < \infty.$$

It is known (see for example [33]) that if $w \in A_p(G)$, $1 < p < \infty$, then T is continuous in $L^p_w(G)$.

Definition 7.2. The weight function ϱ is called radial if there exists a function $\beta\colon (0,\infty) \to (0,\infty)$ such that $\varrho(x) = \beta\big(r(x)\big)$.

In the sequel we use the symbol ϱ instead of β.

Theorem 7.1 ([31])**.** *Let $1 < p < \infty$, let ϱ be radial, $\varrho \in A_p(G)$, and let u and σ be positive increasing functions on $(0,\infty)$. Put $v = \sigma\varrho$, $w = u\varrho$. If*

$$\sup_{t>0} \Big(\int_t^\infty v(\tau)\tau^{Q-Qp-1}\,\mathrm{d}\tau \Big)\Big(\int_0^{t/2} w^{1-p'}(\tau)\tau^{Q-1}\,\mathrm{d}\tau \Big)^{p-1} < \infty, \tag{7.1}$$

then there exists a constant $c_2 > 0$ such that for every $f \in L^p_w(G)$ we have

$$\int_G |T(f)(x)|^p V\big(r(x)\big)\,\mathrm{d}x \le c_2 \int_G |f(x)|^p w\big(r(x)\big)\,\mathrm{d}x. \tag{7.2}$$

Theorem 7.2 [31]**.** *Let $1 < p < \infty$, let $\varrho \in A_p(G)$ be a radial function and let u, v be positive and decreasing on $(0,\infty)$. Put $v = \varrho\sigma$, $w = \varrho u$. If*

$$\sup_{t>0} \Big(\int_0^{t/2} v(\tau)\tau^{Q-1}\,\mathrm{d}\tau \Big)\Big(\int_t^\infty w^{1-p'}(\tau)\tau^{-1-Q/(p-1)}\,\mathrm{d}\tau \Big)^{p-1} < \infty, \tag{7.3}$$

then (7.1) *holds.*

We now give some conditions for two weight inequalities of weak type.

Theorem 7.3 [31]**.** *Let $\varrho \in A_1(G)$ be a radial function and let σ and ϱ be positive increasing functions on $(0,\infty)$, $v = \varrho u$, $w = \varrho u$. If*

$$\sup_{t>0} \Big(\int_t^\infty \frac{v(\tau)}{\tau}\,\mathrm{d}\tau \Big) \operatorname*{ess\,sup}_{\tau\in(0,t)} \frac{1}{w(\tau)} < \infty, \tag{7.4}$$

then there exists a constant $c_3 > 0$ such that

$$\int_{\{x\in G\colon |T(f)(x)|>\lambda\}} v\big(r(x)\big)\,\mathrm{d}x \le c_3\lambda^{-1} \int_G |f(x)| w\big(\tau(x)\big)\,\mathrm{d}x \tag{7.5}$$

for every $f \in L^1_w(G)$ *and* $\lambda > 0$.

If the functions σ and u are increasing on $(0,\infty)$ and if we replace (7.4) by

$$\sup_{t>0} \Big(\int_0^t v(\tau)\tau^{Q-1}\, \mathrm{d}\tau \Big) \operatorname*{ess\,sup}_{\tau\in(t,\infty)} \frac{1}{\tau^Q w(\tau)} < \infty, \tag{7.6}$$

then (7.5) holds, too.

We now present two theorems for the conjugate function

$$\widetilde{f}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)}{2\,\mathrm{tg}(t/2)}\, \mathrm{d}t.$$

The following statements with particular couples of weights are true:

Theorem 7.4. *If* $1 < p < \infty$, *then*

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x|^{p-1}\, \mathrm{d}x \le c \int_{-\pi}^{\pi} |f(x)|^p |x|^{p-1} \ln \frac{\pi}{|x|}\, \mathrm{d}x.$$

Moreover, the power p on the right hand side with the logarithm is sharp, that is, it cannot be replaced by any $\gamma < p$.

Theorem 7.5. *If* $1 < p < \infty$, *then*

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p \frac{\mathrm{d}x}{|x| \ln^p \frac{\pi}{|x|}} \le c \int_{-\pi}^{\pi} |f(x)|^p \frac{\mathrm{d}x}{|x|}.$$

The particular cases of Theorems 7.1 and 7.2 were previously considered by Guseinov, Guliev, and by Edmunds and the author [33].

8. Applications to the Dirichlet problem in domains with non-smooth boundaries

Let D be a simply connected bounded domain in the complex plane $\mathbb{C}$. Let $z = z(w)$ be a conformal mapping of $U = \{w : |w| < 1\}$ onto D such that $z(0) = 0$, $z'(0) > 0$.

We shall assume that the boundary Γ of D can be represented in the form $\Gamma = \bigcup_k \Gamma_k$, where Γ_k are non-intersecting Lyapunov arcs meeting at the points C_k $(k = 1, \dots, n)$, forming with respect to the domain (inner with respect to D) angles with values $\nu_k \pi$, $0 < \nu_k \le 2$.

We denote by $E^p(D)$ $(1 < p < \infty)$ the class of analytic functions f in D for which

$$\sup_{0<r<1} \int_{\Gamma_r} |f(z)|\,|\mathrm{d}z| < \infty,$$

where Γ_r is the image of $|w| = r$ by $z = z(w)$.

This class of analytic functions is known as the Smirnov class.

Let us consider the Dirichlet problem

$$\begin{aligned}\Delta u = 0, \quad u \in \operatorname{Re}\phi, \quad \phi \in E^p(D),\\ u^+(t) = f(t), \quad t \in \Gamma.\end{aligned} \tag{8.1}$$

For simplicity, only the case of one angle point will be considered.

Case I: If $0 < \nu < p$, then (8.1) is solvable for an arbitrary $f \in L^p(\Gamma)$. The homogeneous problem has the trivial solution only and in general we have the following formula for the solution

$$u(z) = \operatorname{Re}\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(z(\xi))}{\xi - w(z)}\,\mathrm{d}t + \frac{w(z)}{2\pi i}\int_\gamma \frac{\overline{f(z(\xi))}}{\xi - w(z)}\,\mathrm{d}t\Big), \tag{8.2}$$

where γ denotes the unite circle $|w| = 1$.

Case II: If $p < \nu < 2$, then (8.1) is solvable for any $f \in L^p(\Gamma)$.

The homogeneous problem has infinitely many solutions:

$$u_0(z) = M\frac{w(C) + w(z)}{w(C) - w(z)}, \tag{8.3}$$

where M is an arbitrary real constant.

The solution of the non-homogeneous problem can be written in the following form:

$$u(z) = u_0(z) + \widetilde{u}(z),$$

where

$$\begin{aligned}\widetilde{u}(z) = \operatorname{Re}\Big[\frac{1}{w(z) - w(C)}\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(z(\xi))(\xi - w(C))}{\xi - w(z)}\,\mathrm{d}\xi\\ - \frac{w(C)w^2(z)}{2\pi i}\int_\gamma \frac{\overline{f(z)(\xi))}\,\overline{(\xi - w(C))}}{\xi - w(z)}\frac{\mathrm{d}\xi}{\xi}\Big)\Big].\end{aligned} \tag{8.4}$$

Case III: If $\nu = p$, then the homogeneous problem has only the trivial solution. Moreover, there exists $f \in L^p(\Gamma)$ for which (8.1) has no solution.

Nevertheless, using Theorem 7.4 we can prove that if

$$f(t)\big|\ln|t-C|\,\big| \in L^p(\Gamma), \tag{8.5}$$

then (8.1) is solvable and the solution is given by (8.2).

Case IV: Let $\nu = 2$. If $1 < p < 2$, then (8.5) is a sufficient condition for the solvability of (8.1).

The case $p > 2$ was in fact discussed above; more precisely, (8.1) is solvable for any $f \in L^p(\Gamma)$ and the solution is given by (8.4).

In [34] we obtained a complete solution of the Riemann-Hilbert problem in the domains with above considered boundaries. This problem is more general than (8.1); it consists of finding a function $\phi \in E^p(D)$, with the angular boundary values satisfying the condition

$$\mathrm{Re}\left[\big(a(t)+ib(t)\big)\phi^+(t)\right] = f(t), \quad t \in \Gamma,$$

where a and b are certain bounded measurable functions defined on Γ and $f \in L^p(\Gamma)$.

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