COMPACT OPERATORS AND NORMAL FAMILIES OF SUBHARMONIC FUNCTIONS

Andriy A. Kondratyuk and Svyatoslav I. Tarasyuk

The properties of potential type compact operators are applied to the proof of the following theorem concerning normal families of subharmonic functions in the sense of Montel.

Let S^{m-1} be the unit sphere in $\mathbb{R}^m$, $|S^{m-1}|$ its area. If $m = 2$ we assume $1 + \frac{1}{m-2} = +\infty$.

Theorem 1. *Let $\mathcal{F}$ be a family of subharmonic in $\mathbb{R}^m$, $m \geq 2$, functions u such that*

$$\sup_{u \in \mathcal{F}} \int_{S^{m-1}} |u(rx)| \, \mathrm{d}S(x) < \infty$$

for every $r \geq 0$. Then for every sequence $\{u_n\}$ from $\mathcal{F}$ there exist a subharmonic in $\mathbb{R}^m$ function v and a subsequence $\{u_{n_j}\}$ such that for every $r > 0$ and every q, $1 \leq q < 1 + \frac{1}{m-2}$, we have

$$\lim_{j \to \infty} \int_{S^{m-1}} |u_{n_j}(rx) - v(rx)|^q \, \mathrm{d}S(x) = 0. \tag{1}$$

For $m = 2$, $q = 1$ this result was obtained in [1].

To prove Theorem 1 we need some preliminary results.

Let $K = \overline{K}_R$ be a closed ball of radius R. The sequence $\{\mu_n\}$ of positive measures on K is called weakly convergent to measure μ, if for every function g from $C(K)$ we have

$$\lim_{n \to \infty} \int_K g \, \mathrm{d}\mu_n = \int_K g \, \mathrm{d}\mu.$$

By standard arguments we obtain the following lemma.

Lemma. *Let X_j be the Banach spaces, $X_1 \xrightarrow{T} X_2 \xrightarrow{A_n} X_3$, where T is a linear compact operator, $\{A_n\}$ is a sequence of linear bounded operators convergent pointwise to zero and let $F_n = A_n T$. Then the sequence of norms $\|F_n\|$ tends to zero.*

We shall denote the Newton potential of measure μ_n by

$$P_n(y) = \int_K \frac{\mathrm{d}\mu_n(\eta)}{|y-\eta|^{m-2}}, \quad m \geq 3.$$

The potential of measure μ is denoted by $P(y)$.

Theorem 2. *Let $\mu_n \to \mu$ weakly on $\overline{K}_R$. Then*

$$\lim_{n\to\infty} \int_{S^{m-1}} |P_n(rx) - P(rx)|^q \,\mathrm{d}S(x) = 0 \tag{2}$$

for $r > 0, 1 \leq q < 1 + \frac{1}{m-2}$.

Proof of Theorem 2. For r fixed, $0 < r < \infty$, we define the linear operator T as follows

$$(Tf)(\eta) = \int_{S^{m-1}} \frac{f(x)}{|rx-\eta|^{m-2}} \,\mathrm{d}S(x).$$

The q-degree of Newton kernel is integrable for $q \in \left[1, 1 + \frac{1}{m-2}\right)$. So that by Sobolev-Kantorovich theorem [2], [3, p. 428] the operator T is a compact operator from $L^p(S^{m-1})$ into $C(K)$ if $1/q + 1/p = 1$. Using the Fubini theorem we have

$$\begin{aligned}
&\int_{S^{m-1}} [P_n(rx) - P(rx)] f(x) \,\mathrm{d}S(x) \\
&= \int_K \left\{ \int_{S^{m-1}} \frac{f(x)}{|rx-\eta|^{m-2}} \,\mathrm{d}S(x) \right\} [\mathrm{d}\mu_n(\eta) - \mathrm{d}\mu(\eta)] \\
&= \int_K (Tf)(\eta)[\mathrm{d}\mu_n(\eta) - \mathrm{d}\mu(\eta)] = A_n T f = F_n f.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{S^{m-1}} |P_n(rx) - P(rx)|^q \,\mathrm{d}S(x) \\
&= \sup_{\|f\|_{L_p} \leq 1} \int_{S^{m-1}} [P_n(rx) - P(rx)] f(x) \,\mathrm{d}S(x) = \|F_n\|.
\end{aligned}$$

Applying Lemma we obtain (2). $\square$

Proof of Theorem 1. Put

$$V_1(r) = \sup_{u\in\mathcal{F}} \frac{1}{|S^{m-1}|}\int_{S^{m-1}} |u(rx)|\,\mathrm{d}S(x), \quad 0\le r<\infty,$$

and consider a continuous majorant V of V_1, for example,

$$V(r) = \int_r^{r+1} \sup\{V_1(\tau)\colon \tau\le t\}\,\mathrm{d}t.$$

Let m=2. We may take u as a function of complex variable. Put

$$c_k(r,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}u(re^{i\theta})\,\mathrm{d}\theta, \quad k\in\mathbb{Z}.$$

By the Rubel-Taylor theorem [4], [5, p. 16], [6]

$$|c_k(r,u)| \le \frac{V(3r)}{|k|+1}, \quad u\in\mathcal{F},\ r>0,\ k\in\mathbb{Z}.$$

By the Anderson-Bearnstein theorem [1] for every sequences $\{u_n\}$ from $\mathcal{F}$ exist a subsequence $\{u_{n_j}\}$ and a subharmonic in $\mathbb{C}$ function v such that

$$c_k(r,u_{n_j}) \to c_k(r,v) \tag{3}$$

when $j\to\infty$, $r>0$, $k\in\mathbb{Z}$.

In so doing, evidently,

$$|c_k(r,v)| \le \frac{V(3r)}{|k|+1}, \quad r>0,\ k\in\mathbb{Z}. \tag{4}$$

By the Hausdorff-Young theorem for $q\ge 2$, $1/p+1/q=1$, we have

$$\begin{aligned} m_q(r,u_n-v) &= \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |u_{n_j}(re^{i\theta})-v(re^{i\theta})|^q\,\mathrm{d}\theta\right\}^{1/q} \\ &\le \left\{\sum_{k\in\mathbb{Z}} |c_k(r,u_{n_j})-c_k(r,v)|^p\right\}^{1/p}. \end{aligned}$$

The relation (1) for $q\ge 2$ follows from this, (3) and (4). For $q\in[1,2)$ it follows in the same way if, moreover, we take into account that $m_q(r,u)$ is increasing as a function of q.

Now suppose $m \geq 3$. Let

$$N(r,u) = \frac{1}{|S^{m-1}|} \int_{S^{m-1}} u(rx)\ \mathrm{d}S(x).$$

If μ is the Riesz measure of the subharmonic function u then by classical Jensen's formula,

$$\begin{aligned} N(2r,u) - N(r,u) &= (m-2) \int_r^{2r} \mu(\overline{K}_t) \frac{\mathrm{d}t}{t^{m-1}} \\ &\geq \mu(\overline{K}_r) r^{m-2} (1 - 2^{2-m}). \end{aligned}$$

Hence, for every sequence $\{\mu_n\}$ of Riesz measures associated with $u_n \in \mathcal{F}$ we have

$$\mu_n(\overline{K}_r) \leq \frac{r^{m-2}}{1-2^{2-m}} \big(V(2r) + V(r)\big).$$

The sequence $\{\mu_n\}$ is, consequently, bounded on compact sets from $\mathbb{R}^m$.

Using the Banach-Alaoglu theorem and diagonal process we find a measure μ and a subsequence $\{\mu_{n_j}\}$ weakly converging to μ in every ball $\overline{K}_l$, $l \in \mathbb{N}$. We denote $\{\mu_{n_j}\}$ by $\{\mu_n\}$.

Let l be fixed. The function $h_n(y) = u_n(y) - P_n(y)$, where $P_n(y)$ denotes the Newton potential of μ_n in $\overline{K}_l$, is harmonic in K_l. Hence, for $0 < r < l$ we have

$$N(r, |h_n|) \leq N(r, |u_n|) + N(r, |P_n|).$$

By virtue of (2) and condition $N(r, |u_n|) \leq V(r)$ we can write

$$\sup_n N(r, |h_n|) < \infty$$

for every $r \in (0, l)$. It follows from representation of h_n by Poisson integral that $\{h_n\}$ is bounded sequence on compact subsets of $\overline{K}_l$. Using the compact principle we find a harmonic function h such that $\{h_n\}$ converges uniformly to h on compact subsets of K_l. Denote $v = h + P$, where P is the Newton potential of μ. Using theorem 2 we get (1) for $0 < r < l$.

If $\{u_n^{(1)}\}$ is the subsequence for K_1 and $v^{(1)}$ is the corresponding limit function, $\{u_n^2\}, v^{(2)}$ for K_2, then $v^{(1)}$ coincides with $v^{(2)}$ in K_1 almost everywhere. Since $v^{(1)}$ and $v^{(2)}$ are subharmonic, they coincide everywhere in K_1. By recurrence, using diagonal process, we obtain (1) which is what had to be proved. $\square$

References

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