INTEGRABLE MAJORANT OF THE PARTIAL SUMS FOR MULTIPLE HAAR SERIES

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Let $[a, b]$ be an interval in $\mathbb{R}$, let $\{g_n\}$ be some orthonormal system of real functions defined on $[a, b]$ and let

$$\sum_{n=1}^{\infty} c_n g_n(x)$$

be a Fourier series of some function $f \in L^2(a, b)$ with respect to the system $\{g_n\}$, i.e., $c_n = (f, g_n)$ is the n-th Fourier coefficient. In problems concerning the almost everywhere convergence of orthogonal Fourier series the majorant of the partial sums

$$(Pf)(x) = \sup_m \Big| \sum_{n=1}^{m} c_n g_n(x) \Big|$$

is frequently used.

Numerous propositions concerning the convergence of Fourier series are expressed in the form of estimates for Pf. For example the classical Menshov-Rademacher theorem (see e.g. [1]) states the inequality

$$\|Pf\|_{L^2}^2 \leq K \sum_{n=1}^{\infty} c_n^2 \log^2(n+1).$$

Also in the theory of convergence systems of Fourier series the majorant Pf plays an important role.

Definition 1. Let $\{g_n\}$ be an orthonormal system in $L^2(a, b)$. We call it a convergence system if the condition

$$\sum_{n=1}^{\infty} c_n^2 < \infty \tag{A}$$

implies

$$\sum_{n=1}^{\infty} c_n g_n(x) < \infty \tag{B}$$

almost everywhere on $[a,b]$.

Definition 2. Let $\{g_n\}$ be a convergence system defined on $[a,b]$. If the validity of (B) on some subinterval E of $[a,b]$ implies (A), then we call this system a strong convergence system.

Theorem (see e.g. [2]). *An orthonormal system $\{g_n\}$ defined on $[a,b]$ is the convergence system if and only if Pf is finite a.e. for any $f \in L^2(a,b)$.*

Remark 1. P. L. Ulyanov studied convergence problems of Fourier series and he stated in [3] that under certain conditions on $\{c_n\}$ a complete system becomes a strong convergence system.

In this paper we deal with the majorant Pf for a Haar orthonormal system in $\mathbb{R}^N$. Namely, we shall deal with conditions on f ensuring $Pf \in L^1$. Let us define the Haar system on $[0,1]$.

Definition 3. Given a natural number n we can write n uniquely as $n = 2^k + i$ with $k = 0,1,2,\dots$ and $i = 0,1,\dots,2^k - 1$. Denote $I = [0,1]$ and $I_n = (i2^{-k}, (i+1)2^{-k})$. We introduce the Haar system $\{h_n\}$ in I in the form:

$$h_0(t) = 1 \quad \text{for } t \in I,$$

$$h_n(t) = \begin{cases} 2^{k/2} & \text{for } t \in I_n^+ = (i2^{-k}, (2i+1)2^{-k-1}) \\ -2^{k/2} & \text{for } t \in I_n^- = ((2i+1)2^{-k-1}, (i+1)2^{-k}) \\ 0 & \text{for } t \in I \setminus \overline{I_n^+ \cup I_n^-} \\ h_n(0+) & \text{for } t = 0 \\ h_n(1-) & \text{for } t = 1 \\ \frac{1}{2}\big(h_n(t-) + h_n(t+)\big) & \text{for all other } t \in I, \end{cases}$$

where $h_n(t-) = \lim\limits_{s \to t-} h_n(s)$ and $h_n(t+) = \lim\limits_{s \to t+} h_n(s)$.

Let N be a natural number and let $\mathbb{N}$ be the set of all non-negative integers. We shall use the following notation: $\mathbb{N}^N = \mathbb{N} \times \dots \times \mathbb{N}$, $I^N = I \times \dots \times I$, $x = (x_1, \dots, x_N) \in I^N$, $\boldsymbol{m} = (m_1, \dots, m_N) \in \mathbb{N}^N$, $|\boldsymbol{m}| = m_1 + \dots + m_N$, $\mathbf{0} = (0, \dots, 0)$, $\boldsymbol{\infty} = (\infty, \dots, \infty)$, $I_{\boldsymbol{m}} = I_{m_1} \times \dots \times I_{m_N}$, $D_{\boldsymbol{m}} =$

$(0,2^{-m_1})\times\cdots\times(0,2^{-m_N})$, $J_{\boldsymbol{m}}=(2^{-m_1-1},2^{-m_1})\times\cdots\times(2^{-m_N-1},2^{-m_N})$ and $2^{\boldsymbol{m}}=(2^{m_1},\ldots,2^{m_N})$.

We shall say that the series

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_N=1}^{\infty}b_{k_1\ldots k_N}h_{k_1}(x_1)\ldots h_{k_N}(x_N)=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{\infty}}b_{\boldsymbol{k}}h_{k_1}(x_1)\ldots h_{k_N}(x_N)\tag{1}$$

is a multiple Haar series of some measurable function $f(x)=f(x_1,\ldots,x_N)$ defined on I^N, i.e.,

$$b_{\boldsymbol{k}}=\int\limits_{I^N}f(x)h_{k_1}(x_1)\ldots h_{k_N}(x_N)\,\mathrm{d}x,$$

and that the sum

$$\begin{aligned}S_{\boldsymbol{m}}(f;x)&=S_{m_1\ldots m_N}(f;x_1,\ldots,x_N)\\&=\sum_{k_1=0}^{m_1}\cdots\sum_{k_N=0}^{m_N}b_{k_1\ldots k_N}h_{k_1}(x_1)\ldots h_{k_N}(x_N)\\&=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{m}}b_{\boldsymbol{k}}h_{k_1}(x_1)\ldots h_{k_N}(x_N)\end{aligned}\tag{2}$$

is a partial sum of the series (1) of order $|\boldsymbol{m}|$. We introduce the function

$$(Pf)(x)=\sup_{\boldsymbol{m}}|S_{\boldsymbol{m}}(f;x)|$$

defined on I^N. We shall call this function a majorant of the partial sums for the multiple Haar series (1).

Using the Haar kernel on $[0,1]$ (see e.g. [1]),

$$K_n(u,v)=\sum_{l=0}^{n}h_l(u)h_l(v)=\begin{cases}(\operatorname{meas}I_n)^{-1} & \text{for } (u,v)\in I_n\times I_n,\\ 0 & \text{for other } (u,v),\end{cases}\tag{3}$$

we obtain the following representation for the sum (2):

$$\begin{aligned}S_{\boldsymbol{m}}(f;x)&=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{m}}\int\limits_{I^N}f(t)h_{k_1}(t_1)\ldots h_{k_N}(t_N)\,\mathrm{d}t\cdot h_{k_1}(x_1)\ldots h_{k_N}(x_N)\\&=\int\limits_{I^N}f(t)K_{m_1}(t_1,x_1)\ldots K_{m_N}(t_N,x_N)\,\mathrm{d}t,\end{aligned}$$

i.e.,

$$S_{\boldsymbol{m}}(f;x)=\begin{cases}(\operatorname{meas} I_{\boldsymbol{m}})^{-1}\int\limits_{I_{\boldsymbol{m}}} f(t)\ \mathrm{d}t & \text{for } x\in I_{\boldsymbol{m}},\\ 0 & \text{otherwise.}\end{cases}$$

Theorem A. *Let $\Phi(u)$ be a non-negative function increasing on $[0,\infty)$ such that*

$$\Phi(u)=o\big(\log^N(u+1)\big) \qquad \textit{for } u\to+\infty.$$

Then there exists a function f_0 in the Orlicz class $L\Phi(L)$ of measurable functions defined on I^N such that the majorant Pf_0 of the partial sums of its multiple Haar series is not integrable (in the Lebesgue sense) over I^N.

Proof. Choose $u=2^{Nn}$ ($n\in\mathbb{N}$). Due to the properties of Φ we obtain

$$\Phi(2^{Nn})=o\big(\log^N(2^{Nn}+1)\big),$$

and thus

$$\Phi(2^{Nn})=o\big((Nn+1)^N\big).$$

Hence there exists a sequence $\{n_l\}$ of natural numbers such that

$$\sum_{l=1}^{\infty}\frac{\Phi(2^{Nn_l})}{(Nn_l+1)^N}<+\infty,$$

and due to the monotonicity of Φ we get

$$\sum_{l=1}^{\infty}\frac{\Phi\big(2^{Nn_l}(Nn_l+1)^{-N}\big)}{(Nn_l+1)^N}<+\infty. \tag{4}$$

For $\boldsymbol{m}$, $|\boldsymbol{m}|=0,1,\dots$, we take a step function

$$f_0(x)=a_{\boldsymbol{m}}\geq 0 \quad \text{for } x\in J_{\boldsymbol{m}}.$$

Then

$$S_{2^{\boldsymbol{m}}}(f_0;x)=2^{|\boldsymbol{m}|}\int\limits_{D_{\boldsymbol{m}}} f_0(t)\ \mathrm{d}t \qquad \text{for } x\in D_{\boldsymbol{m}}$$

which yields (for $x\in D_{\boldsymbol{m}}$) that

$$S_{2^{\boldsymbol{m}}}(f_0;x)=2^{|\boldsymbol{m}|}\sum_{\boldsymbol{i}=\boldsymbol{m}}^{\infty}\int\limits_{J_{\boldsymbol{i}}} f_0(t)\ \mathrm{d}t=2^{|\boldsymbol{m}|}\sum_{\boldsymbol{i}=\boldsymbol{m}}^{\infty}a_{\boldsymbol{i}}2^{-|\boldsymbol{i}|-N}$$

and

$$(Pf_0)(x) \geq |S_{2^{\boldsymbol{m}}}(f_0;x)| = 2^{|\boldsymbol{m}|-N} \sum_{\boldsymbol{i}=\boldsymbol{m}}^{\infty} a_{\boldsymbol{i}} 2^{-|\boldsymbol{i}|}.$$

Therefore, we obtain

$$\begin{aligned}
\int_{I^N} (Pf_0)(x)\ \mathrm{d}x &= \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\infty} \int_{J_{\boldsymbol{k}}} (Pf_0)(x)\ \mathrm{d}x \\
&\geq \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\infty} \int_{J_{\boldsymbol{k}}} 2^{|\boldsymbol{k}|-N} \sum_{\boldsymbol{i}=\boldsymbol{k}}^{\infty} a_{\boldsymbol{i}} 2^{-|\boldsymbol{i}|}\ \mathrm{d}x \\
&= 2^{-2N} \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\infty} \sum_{\boldsymbol{i}=\boldsymbol{k}}^{\infty} a_{\boldsymbol{i}} 2^{-|\boldsymbol{i}|} \\
&= 2^{-2N} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{i}} a_{\boldsymbol{i}} 2^{-|\boldsymbol{i}|} \\
&= 2^{-2N} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} a_{\boldsymbol{i}} 2^{-|\boldsymbol{i}|} \prod_{j=1}^{N} (i_j+1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{I^N} |f_0(x)|\Phi(|f_0(x)|)\ \mathrm{d}x &= \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} \int_{J_{\boldsymbol{i}}} |f_0(x)|\Phi(|f_0(x)|)\ \mathrm{d}x \\
&= \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} a_{\boldsymbol{i}} \Phi(a_{\boldsymbol{i}}) 2^{-|\boldsymbol{i}|-N}.
\end{aligned}$$

According to the selection of the sequence $\{n_l\}$ above and to the property (4) we choose

$$a_{\boldsymbol{i}} = \begin{cases} 2^{Nn_l}(Nn_l+1)^{-N} & \text{for } i_1 = \cdots = i_N = n_l,\ l = 1, 2, \ldots \\ 0 & \text{for other } \boldsymbol{i}. \end{cases}$$

Then we conclude that

$$\int_{I^N} (Pf_0)(x)\ \mathrm{d}x \geq 2^{-2N} \sum_{l=1}^{\infty} \frac{2^{Nn_l}}{(Nn_l+1)^N} 2^{-Nn_l}(n_l+1)^N = +\infty$$

and

$$\int\limits_{I^N} |f_0(x)|\Phi(|f_0(x)|)\,\mathrm{d}x = 2^{-N}\sum_{l=1}^{\infty}\frac{2^{Nn_l}}{(Nn_l+1)^N}\Phi\Big(\frac{2^{Nn_l}}{(Nn_l+1)^N}\Big)2^{-Nn_l}$$
$$= 2^{-N}\sum_{l=1}^{\infty}\frac{1}{(Nn_l+1)^N}\Phi\Big(\frac{2^{Nn_l}}{(Nn_l+1)^N}\Big) < +\infty,$$

which completes the proof of Theorem A. □

Remark 2. Using the properties of maximal function (see e.g. [4]) we obtain the following theorem.

Theorem B. *Let* $\Phi(u) = \log^N(u+1)$. *Then* $Pf \in L^1(I^N)$ *for every function* $f \in L\Phi(L)$ *on* I^N.

References

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