REGULARITY PROPERTIES OF THE SOLUTION TO THE WAVE EQUATION IN BESOV SPACES WITH $p < 1$

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We consider the usual homogeneous wave equation in $\mathbb{R}^n$

$$\Box u(x,t) \equiv 0, \quad u(x,0)=0, \quad \frac{\partial u}{\partial t}(x,0)=f(x), \tag{1}$$

where, of course, $\Box \equiv \frac{\partial^2}{\partial t^2} - \triangle$ is the d'Alembert operator in $\mathbb{R}^{n+1}$.

Given a function $f = f(x)$ on $\mathbb{R}^n$ we denote by $f(x,t)$ the (unique) solution of the Cauchy problem (1). For a fixed $t > 0$ we denote the map $f(x) \mapsto f(x,t)$ by A_t, i.e., $f(x,t) = (A_t f)(x)$. Of course, A_t is linear.

Function spaces

We recall the definitions of the function spaces we will use. Let $S(\mathbb{R}^n)$ be the Schwartz space and $S'(\mathbb{R}^n)$ its dual, the space of tempered distributions. We denote by $\mathcal{F}$, $\mathcal{F}^{-1}$ the Fourier transform and its inverse, respectively.

Definition 1. Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Then

$$H_p^s(\mathbb{R}^n) = \big\{ f \in S'(\mathbb{R}^n) \colon \|f \mid H_p^s\| = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f \mid L_p\| < \infty \big\}. \tag{2}$$

Given $\phi \in S(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subseteq \{y \colon |y| < 2\}$ and $\phi(x) = 1$ if $|x| \le 1$, we set: $\phi_j(x) = \phi(2^{-j}x) - \phi(2^{-j+1}x)$, $j > 0$, and $\phi_0 = \phi$.

Definition 2. Let $0 < p \le \infty$, $0 < q \le \infty$, and $-\infty < s < \infty$.

$$B_{pq}^s(\mathbb{R}^n) = \Big\{ f \colon f \in S'(\mathbb{R}^n), \\ \|f \mid B_{pq}^s\| = \Big(\sum_{j=0}^{\infty} 2^{sjq} \|\mathcal{F}^{-1}(\phi_j \mathcal{F} f) \mid L_p\|^q \Big)^{1/q} < \infty \Big\} \tag{3}$$

(usual modification if $q = \infty$).

Problems

- If $f \in H_p^s$, we ask whether $(A_t f)$ belongs to the same space H_p^σ for a suitable σ.
- The same question for $f \in B_{pq}^s$.

The answer to the first Problem is known, and it reads as follows, cf. [2]: Let $1 < p < \infty$, $s \in \mathbb{R}$, and $\alpha_p = (n-1)|1/p - 1/2|$. Then

$$A_t \colon H_p^s \longrightarrow H_p^{s+1-\alpha_p}, \tag{4}$$

which is sharp in the following sense: $\forall \varepsilon > 0\ \exists f_\varepsilon \in H_p^s$ such that $(A_t f_\varepsilon) \notin H_p^{s+1-\alpha_p+\varepsilon}$.

Unfortunately Peral's proof does not apply to the spaces with $p < 1$. Of course the result (4) can be extended to Besov spaces with $\min(p,q) > 1$ simply by interpolation. Up to now nothing is known about B_{pq}^s with $p < 1$.

Result

Theorem 1. *Let $2 \nmid n$, $n \geq 3$, and $0 < q \leq p$. Let $s > \sigma_p = n(1/p - 1)_+$, $s - (n-1)/2 > \delta - n$, $\delta > n$ if $p > 1$, and $\delta > (n-1)/p$ if $p \leq 1$.*

If $f \in B_{pq}^s(\mathbb{R}^n)$, then

$$f(\cdot, t) \in B_{pq}^{s+(n+1)/2-\delta}(\mathbb{R}^n) \quad \forall t > 1. \tag{5}$$

Remark 1. If $p > 1$, then $\delta > n$. Set $\delta = n + \varepsilon$. Then for $s > 1/2(n-1)$ we have

$$f(\cdot, t) \in B_{pq}^{s-(n-1)/2-\varepsilon}(\mathbb{R}^n) \quad \forall \varepsilon > 0. \tag{6}$$

Unfortunately this is worse than (4).

Remark 2. If $p \leq 1$, then $\delta > (n-1)/p$. Set $\delta = (n-1)/p + \varepsilon$. Let $\alpha_p = (n-1)(1/p - 1/2)$. Then for $s > \max(\alpha_p, \sigma_p)$ we have

$$f(\cdot, t) \in B_{pq}^{s-\alpha_p+1-\varepsilon}(\mathbb{R}^n) \quad \forall \varepsilon > 0. \tag{7}$$

About the proof

The main tool of the proof is the atomic decomposition of the spaces involved. So we start with the definition of atoms and recall the decomposition theorem.

Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the cube with center $m_\nu = 2^{-\nu} m$ and side length $2^{-\nu}$. Let $c \in \mathbb{R}$, $c > 0$, and let Q be an arbitrary cube with side length a. Then cQ denotes the cube with the same center as Q but side length ca.

Definition 3. (i) Let $K \in \mathbb{N}_0$. A function $a(x)$ is said to be an 1_K-*atom* if

$$\operatorname{supp} a \subseteq 5Q_{0m},$$

and

$$|D^\alpha a(x)| \leq 1, \quad \text{if } |\alpha| \leq K.$$

(ii) Let $L+1$, $K \in \mathbb{N}_0$, $s \in \mathbb{R}$ and $0 < p \leq \infty$. A function $a(x)$ is said to be an $(s,p)_{K,L}$-*atom* if

$$\operatorname{supp} a \subseteq 5Q_{\nu m} \quad \text{for suitable } \nu \in \mathbb{N}_0 \text{ and } m \in \mathbb{Z}^n$$
$$|D^\alpha a(x)| \leq 2^{-\nu(s-n/p)+\nu|\alpha|}, \quad \text{if } |\alpha| \leq K,$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\, \mathrm{d}x = 0 \quad \text{for } |\beta| \leq L.$$

Theorem 2 (Atomic decompositions). *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\sigma_p = n(1/p - 1)_+$, $K \geq (\lceil s \rceil + 1)_+$, and $L \geq \max(\lceil \sigma_p - s \rceil, -1)$, where $\lceil x \rceil$ denotes the integer part of x. Then we have*

$$f \in B^s_{pq}(\mathbb{R}^n) \;\Leftrightarrow\; f = \sum_{m \in \mathbb{Z}^n} \lambda_m a_m + \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \tag{8}$$

where a_m is a 1_K-atom at Q_{0m}, $a_{\nu m}$ is an $(s,p)_{K,L}$-atom at $Q_{\nu m}$, and

$$\Big(\sum_{m \in \mathbb{Z}^n} |\lambda_m|^p \Big)^{1/p} + \Big(\sum_{\nu=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

Furthermore, we have the equivalence

$$\|f \mid B^s_{pq}\| \sim \inf \Big[\Big(\sum_{m \in \mathbb{Z}^n} |\lambda_m|^p \Big)^{1/p} + \Big(\sum_{\nu=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \Big)^{q/p} \Big)^{1/q} \Big],$$

where the infimum is taken over all admissible decompositions (8).

Proof may be found in [FJ1, Theorem 7.1], cf. also [3, 1.9.2]. □

Since n is odd, we know that if the initial data have compact support, say $\operatorname{supp} f(x) \subseteq Q$, Q compact, then for a fixed $t > 0$ one has $\operatorname{supp} f(x,t) \subseteq Q \oplus S_t(0)$, where $\oplus$ denotes the Minkowski sum and $S_t(x_0) = \{x \in \mathbb{R}^n : |x - x_0| = t\}$.

We now consider this in the opposite way, i.e., given a compact set Q, we ask, which area in $\mathbb{R}^n$ will "affect" Q at time t. The answer is obvious. We take $Q = 5Q_{\mu k}$. Then using the notations

$$R^{\mu kt} = 5Q_{\mu k} \oplus S_t(0)$$

and

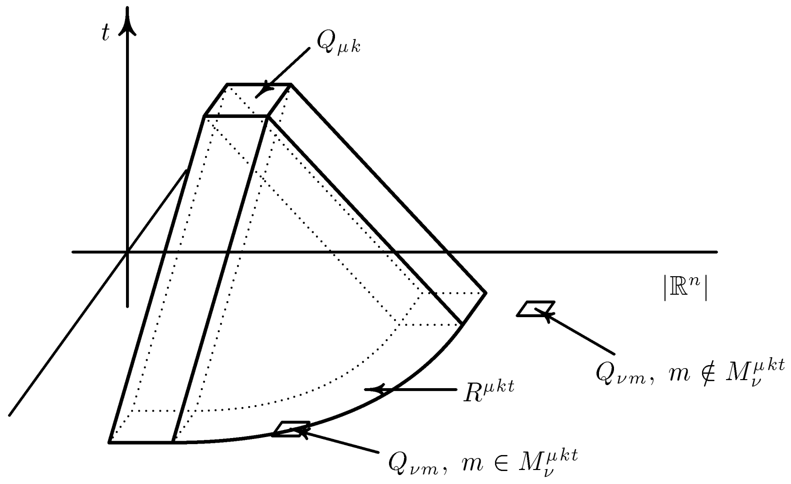
$$M_\nu^{\mu kt} = \{m \in \mathbb{Z}^n : 5Q_{\nu m} \cap R^{\mu kt} \neq \emptyset\},$$

we have the following observations.

(i) Let $a_{\nu m}$ be an atom in $Q_{\nu m}$. Then $\forall m \notin M_\nu^{\mu kt}$ we have $a_{\nu m}(x,t) = 0$ if $x \in 5Q_{\mu k}$, i.e., at the time t the atom $a_{\nu m}(x)$ does not affect the domain $5Q_{\mu k}$.

(ii) Furthermore, there exists a constant $c > 0$, independent of μ, ν and k such that

$$\operatorname{card} M_\nu^{\mu kt} \leq c 2^{\nu n - \min(\mu,\nu)} (t+1)^{n-1}. \tag{9}$$



Now take $f \in B^s_{pq}$. Since $s > \sigma_p$, it follows from Theorem 2 that

$$f(x) \in B^s_{pq} \;\Leftrightarrow\; f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x).$$

Because $(A_t f)(x) = f(x,t)$ is linear, we get

$$(A_t f)(x) = f(x,t) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x,t). \tag{10}$$

Our aim is to redistribute this decomposition in such a way that we get an atomic decomposition again. We will describe here only how to redistribute the decomposition, and we omit the proof, which we end up with an atomic decomposition.

To do this we need a suitable resolution of unity. Let $J \in \mathbb{N}$ and let Φ^J be the collection of all families $\{\phi_{\mu k}(x)\}_{\mu \in \mathbb{N}, k \in \mathbb{Z}^n}$ with the following properties:

(i) $\phi_{\mu k} \in C_0^\infty(\mathbb{R}^n)$, $0 \le \phi_{\mu k} \le 1$,
(ii) $\operatorname{supp} \phi_{\mu k} \subseteq 5Q_{\mu k}$,
(iii) $|D^\gamma \phi_{\mu k}(x)| \le 2^{-\mu J} 2^{\mu|\gamma|}(1 - 2^{-J})$, $\forall |\gamma| \le J' \le J$,
(iv) $\sum_{k \in \mathbb{Z}^n} \phi_{\mu k}(x) \equiv 2^{-\mu J}(1 - 2^{-J})$,
(v) $\left| \sum_{\mu=0}^{\infty} \phi_{\mu k}(x) \right| \le 1$.

Obviously, $\sum_{\mu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \phi_{\mu k}(x) \equiv 1$ and $\Phi^J \neq \emptyset$.

We now fix t. Starting with (10) we get:

$$\begin{aligned} f(x,t) &= \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x,t) \\ &= \sum_{\mu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \phi_{\mu k}(x) \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x,t) \\ &= \sum_{\mu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \sum_{\nu=0}^{\infty} \phi_{\mu k}(x) \sum_{m \in M_\nu^{\mu k t}} \lambda_{\nu m} a_{\nu m}(x,t) \\ &= \sum_{\mu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \Big(\sum_{\nu=0}^{\mu-1} \phi_{\mu k}(x) \sum_{m \in M_\nu^{\mu k t}} \lambda_{\nu m} a_{\nu m}(x,t) \\ &\quad + \sum_{\nu=\mu}^{\infty} \phi_{\mu k}(x) \sum_{m \in M_\nu^{\mu k t}} \lambda_{\nu m} a_{\nu m}(x,t) \Big) = \end{aligned} \tag{11}$$

$$= \sum_{k\in\mathbb{Z}^n} \Big[\Big(\sum_{\mu=0}^{\infty} \phi_{\mu k}(x) \sum_{\nu=0}^{\mu-1} \sum_{m\in M_\nu^{\mu k t}} \lambda_{\nu m} a_{\nu m}(x,t) \Big)$$

$$+ \sum_{\mu=0}^{\infty} \Big(\phi_{\mu k}(x) \sum_{\nu=\mu}^{\infty} \sum_{m\in M_\nu^{\mu k t}} \lambda_{\nu m} a_{\nu m}(x,t) \Big) \Big]$$

$$= \sum_{k\in\mathbb{Z}^n} \Big(\eta_k\, b_k(x,t) + \sum_{\mu=0}^{\infty} \eta_{\mu k}\, b_{\mu k}(x,t) \Big).$$

We choose the coefficients as follows. Let $\delta > 0$ and put

$$\eta_{\mu k} = c_o\, t^{-(n-1)/2} 2^{-\mu n/p} \sum_{\nu=\mu}^{\infty} \sum_{m\in M_\nu^{\mu k t}} 2^{-\nu(\delta - n/p)} |\lambda_{\nu m}|. \tag{12}$$

Let $N \in \mathbb{N}$, $\theta > 0$, and

$$\eta_k = g_t 2^N \sum_{\mu=0}^{\infty} 2^{-\mu\theta} c_{\mu k} \begin{cases} \mu, & p > 1 \\ 1, & p \le 1, \end{cases} \tag{13}$$

where

$$c_{\mu k} = \Big(\sum_{\nu=0}^{\mu-1} \Big(\sum_{m\in M_\nu^{\mu k t}} |\lambda_{\nu m}|^p \Big)^{q/p} \Big)^{1/q} \tag{14}$$

and g_t is a suitable constant depending only on t.

References

[1] M. Frazier and B. Jawerth, *Decomposition of Besov-spaces*, Indiana Univ. Math. J. **34** (1985), 777–799.

[2] J. Peral, *L_p estimates for the wave equation*, J. Funct. Anal. **36** (1980), 114–145.

[3] H. Triebel, *Theory of function spaces. II*, Birkhäuser-Verlag, Basel, Boston, Stuttgart, 1992.