THE HARDY-LITTLEWOOD MAXIMAL FUNCTION AND SOBOLEV SPACES ON A METRIC SPACE

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1. Introduction

The first order Sobolev space $W^{1,p}(\Omega)$, Ω open in $\mathbb{R}^n$ and $1 \le p \le \infty$, is based on the space of differentiable functions: The space $W^{1,p}(\Omega)$ is the closure of $C^1(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} = \big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\big)^{1/p}.$$

P. Hajłasz [H] introduced the Sobolev spaces $W^{1,p}(X)$ on an arbitrary metric space X with a measure μ. His approach rests on a pointwise inequality

$$|u(x) - u(y)| \le C|x-y|\big(M(|\nabla u|)(x) + M(|\nabla u|)(y)\big), \tag{1.1}$$

where $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, $C = C(n)$, and $M(f)$ stands for the Hardy-Littlewood maximal function of f. In this paper we show that this approach allows many variations, which can be used to generate useful first order Sobolev spaces on metric spaces.

In Section 2 we consider the Hardy-Littlewood maximal function $M(u)$. We show that $M(u) \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, provided that $u \in W^{1,p}(\mathbb{R}^n)$. Section 3 is devoted to the pointwise inequality (1.1). Section 4 contains a general approach to the Sobolev space $W^{1,p}(X)$ on an arbitrary metric space (X, d) with a measure μ. The starting point is a subspace of $L^p(X, \mu)$ with admissible "derivatives", which take the role of the maximal function $M(|\nabla u|)$ in (1.1). In Section 5 a capacity on $W^{1,p}(X)$ is considered.

2. The Hardy-Littlewood maximal function

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The (Hardy-Littlewood) maximal function of f is defined as

$$M(u)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |u| \, \mathrm{d}m. \tag{2.1}$$

We have the following well-known result, see [S, p. 5]:

2.2. Theorem. (a) $|u(x)| \le M(u)(x)$ a.e.

(b) $u \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty \Rightarrow M(u)(x) < \infty$ a.e.

(c) $u \in L^1(\mathbb{R}^n) \Rightarrow m\{x\colon M(u)(x) > \alpha\} \le \frac{A}{\alpha} \int_{\mathbb{R}^n} |u| \, \mathrm{d}m$, $A = A(n)$.

(d) $u \in L^p(\mathbb{R}^n)$, $1 < p \le \infty \Rightarrow \|M(u)\|_{L^p} \le A\|u\|_{L^p}$, $A = A(n,p)$.

Next we are going to show that $M(u) \in W^{1,p}(\mathbb{R}^n)$ provided that $u \in W^{1,p}(\mathbb{R}^n)$. We need a pair of real-analytic lemmas. The first is an application of the Mazur lemma.

2.3. Lemma. *Suppose that (u_i) is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, and $u_i \to u$ a.e. Then $u \in W^{1,p}(\mathbb{R}^n)$ and $u_i \to u$, $D_k u_i \to D_k u$ weakly in $L^p(\mathbb{R}^n)$.*

2.4. Lemma. *Let $u \in W^{1,p}(\mathbb{R}^n)$ and let $f \in L^\infty(\mathbb{R}^n)$ have a compact support. Then $u * f \in W^{1,p}(\mathbb{R}^n)$,*

$$(u * f)(x) = \int_{\mathbb{R}^n} u(x-y) f(y) \, \mathrm{d}m(y),$$

and $D_k(u * f) = D_k u * f$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. By the Fubini theorem

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(x)(D_k u * f)(x) \, \mathrm{d}m(x) &= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} D_k u(x-y)\varphi(x) \, \mathrm{d}m(x) \, \mathrm{d}m(y) \\ &= -\int_{\mathbb{R}^n} (u * f) D_k \varphi \, \mathrm{d}x, \end{aligned}$$

and since $u*f$, $D_k u*f \in L^p(\mathbb{R}^n)$, this shows that $D_k u*f$ is the distributional derivative of $u * f$. □

The following result is due to J. Kinnunen [K].

2.5. Theorem. *Let $1 < p < \infty$. If $u \in W^{1,p}(\mathbb{R}^n)$, then $M(u) \in W^{1,p}(\mathbb{R}^n)$. Moreover,*

$$|D_k M(u)| \le M(D_k|u|) \text{ a.e., } k = 1, \ldots, n. \tag{2.6}$$

Proof. For $r > 0$ let $\chi_r = \dfrac{\chi_{B(0,r)}}{m(B(0,r))}$. With this notation,

$$\fint_{B(x,r)} |u| \, \mathrm{d}m = \frac{1}{m(B(x,r))} \int_{B(x,r)} |u| \, \mathrm{d}m = \frac{1}{m(B(0,r))} \int_{B(0,r)} |u(x-y)| \, \mathrm{d}m(y) = (|u| * \chi_r)(x), \tag{2.7}$$

and write

$$u_r(x) = (|u| * \chi_r)(x).$$

Now $|u| \in W^{1,p}(\mathbb{R}^n)$ and by Lemma 2.4

$$D_k u_r = D_k|u| * \chi_r, \tag{2.8}$$

and by (2.7)

$$M(u)(x) = \sup_j u_{r_j}(x),$$

where the supremum is taken over all rational positive radii r_j. Write $u_j = u_{r_j}$ and define

$$v_i(x) = \max\{u_j(x) \colon 1 \le j \le i\}, \quad i = 1, 2, \ldots .$$

Then $v_i \le M(u)$ a.e. and

$$\begin{aligned} |D_k v_i| &\le \max_{1 \le j \le i} |D_k u_j| = \max_{1 \le j \le i} \big|\chi_{r_j} * D_k|u|\big| \\ &\le \max_{1 \le j \le i} \chi_{r_j} * \big|D_k|u|\big| \le \sup_j \chi_{r_j} * \big|D_k|u|\big| \\ &= M(D_k|u|) \end{aligned} \tag{2.9}$$

holds a.e. On the other hand, Theorem 2.2(d) and (2.9) yield

$$\begin{aligned} \|v_i\|^p_{W^{1,p}} = \|v_i\|^p_{L^p} + \|\nabla v_i\|^p_{L^p} &\le \|M(u)\|^p_{L^p} + \Big\| \Big(\sum_{k=1}^n M(D_k|u|)^2 \Big)^{1/2} \Big\|^p_{L^p} \\ &\le A^p \big(\|u\|^p_{L^p} + \|\nabla u\|^p_{L^p} \big) = A^p \|u\|^p_{W^{1,p}}. \end{aligned}$$

Thus (v_i) is a bounded sequence in $W^{1,p}$ and $v_i \to M(u)$ a.e. Lemma 2.3 gives $M(u) \in W^{1,p}(\mathbb{R}^n)$ and (2.6) follows from (2.9). $\square$

2.10. Comments. (a) By Theorem 2.2 we have $\|M(u)\|_{W^{1,p}} \le c\|u\|_{W^{1,p}}$, $c = c(n,p)$, $1 < p < \infty$.

(b) The inequality in (a) can be used to bypass many applications of the maximal function in potential theory.

3. Sobolev spaces and the maximal function

We are first going to prove an inequality due to Hedberg. We need an inequality for C^1-functions.

3.1. Lemma ([GT, p. 155]). *Let $B(x_0, r) \subset \mathbb{R}^n$ and $u \in C^1(\mathbb{R}^n)$. Then*

$$|u(x) - u_{B(x_0,r)}| \le C \int_{B(x_0,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y), \tag{3.2}$$

$$u_{B(x_0,r)} = \fint_{B(x_0,r)} u \, \mathrm{d}m,$$

for all $x \in B(x_0, r)$. Here $C = C(n) < \infty$.

3.3. Lemma. (Hedberg's inequality). *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\int_{B(x_0,r)} \frac{|u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) \le CrM(u)(x) \tag{3.4}$$

for all $x \in B(x_0, r)$. Here $C = C(n)$.

Proof. Fix $x \in B(x_0, r)$ and let

$$A_i = \big(B(x, 2r/2^i) \setminus B(x, 2r/2^{i+1})\big) \cap B(x_0, r), \quad i = 0, 1, \dots .$$

Now

$$\begin{aligned}
\int_{B(x_0,r)} \frac{|u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) &= \sum_{i=0}^{\infty} \int_{A_i} \frac{|u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) \\
&\le \sum_{i=0}^{\infty} \left(\frac{r}{2^i}\right)^{1-n} \int_{B(x,2^{1-i}r)} |u(y)| \, \mathrm{d}m(y) \\
&= \sum_{i=0}^{\infty} 2^{i(n-1)} \frac{m(B(0,1)) r 2^{(1-i)n}}{m(B(x, 2^{1-i}r))} \int_{B(x,2^{1-i}r)} |u(y)| \, \mathrm{d}m(y) \\
&= 2^n m(B(0,1)) \sum_{i=0}^{\infty} 2^{-i} \fint_{B(x,2^{1-i}r)} |u(y)| \, \mathrm{d}m(y) \le CrM(u)(x).
\end{aligned}$$

$\square$

3.5. Remark. A slightly better inequality than (3.4) is possible because for (3.4) there is no need to compute $M(u)(x)$ at large distances: $M(u)(x)$ can be replaced by

$$M_{2r}(u)(x) = \sup_{0<t\leq 2r} \fint_{B(x,t)} |u| \, \mathrm{d}m.$$

3.6. Corollary. *If $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, then (3.2) holds for a.e. $x \in B(x_0, r)$.*

Proof. There is a sequence $u_i \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $u_i \to u$ in L^p and $\nabla u_i \to \nabla u$ in L^p. Passing to a subsequence we may assume that $\nabla u_i \to \nabla u$ a.e. Since

$$\begin{aligned}\Big| \int_{B(x_0,r)} & \frac{|\nabla u_i(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) - \int_{B(x_0,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) \Big| \\ &\leq \int_{B(x_0,r)} \frac{|\nabla u_i(y) - \nabla u(y)|}{|x-y|^{n-1}} \, \mathrm{d}m(y) \leq CrM(|\nabla u_i - \nabla u|)(x) \to 0\end{aligned}$$

for a.e. $x \in B(x_0, r)$, the claim follows from (3.2). $\square$

3.7. Theorem (Hajłasz [H]). *Suppose that $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Then for a.e. $x, y \in \mathbb{R}^n$,*

$$|u(x) - u(y)| \leq C|x-y|\big(M(\nabla u)(x) + M(\nabla u)(y)\big), \tag{3.8}$$

where $C = C(n)$.

Proof. For a.e. $x, y \in \mathbb{R}^n$ we have by Corollary 3.6 and by (3.4) with $B = B(\frac{x+y}{2}, |x-y|)$

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_B| + |u(y) - u_B| \\ &\leq C\Big[\int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, \mathrm{d}m(z) + \int_B \frac{|\nabla u(z)|}{|y-z|^{n-1}} \, \mathrm{d}m(z)\Big] \\ &\leq C|x-y|\big(M(\nabla u)(x) + M(\nabla u)(y)\big).\end{aligned}$$

$\square$

3.9. Remark. A slightly better "local" result is true in view of Remark 3.5.

3.10. Remark. It is not difficult to show, see [H] or [HM], that a function $u \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, which satisfies

$$|u(x) - u(y)| \leq |x - y|\big(g(x) + g(y)\big)$$

for a.e. $x, y \in \mathbb{R}^n$ and some $g \in L^p(\mathbb{R}^n)$, belongs to $W^{1,p}(\mathbb{R}^n)$. Thus the inequality (3.8) is both necessary and sufficient for a function u to belong to the class $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Indeed, this is the starting point to develop the theory of Sobolev spaces on a metric space.

4. The space $\mathcal{F}^p$, $1 \leq p \leq \infty$

Let (X, d) be a metric space and let μ be a Borel measure on X. Suppose that $\mathcal{F}$ is a linear space of μ-measurable functions $u\colon X \to [-\infty, \infty]$ and that $\mathcal{K}$ is a positive cone of μ-measurable functions on X. A positive cone means that each $g \in \mathcal{K}$ satisfies $g \geq 0$ μ-a.e. and $\lambda_1 g_1 + \lambda_2 g_2 \in \mathcal{K}$ whenever $g_1, g_2 \in \mathcal{K}$ and $\lambda_1, \lambda_2 \geq 0$. For a measurable function u we let $D(u) = D(u, \mathcal{K})$ denote the set of all $g \in \mathcal{K}$ such that

$$|u(x) - u(y)| \leq d(x, y)\big(g(x) + g(y)\big) \tag{4.1}$$

holds for μ-a.e. $x, y \in X$.

If $1 \leq p \leq \infty$, then we let $\mathcal{F}_p$ be the set of all $u \in \mathcal{F}$ such that

$$\|u\|_{\mathcal{F}_p} = \big(\|u\|_{L^p}^p + \inf_{g \in D(u,\mathcal{K})} \|g\|_{L^p}^p\big)^{1/p} < \infty. \tag{4.2}$$

Here we use the abbreviation $L^p = L^p(X) = L^p(X, \mu)$. If $D(u, \mathcal{K}) = \emptyset$, then we put $\|u\|_{\mathcal{F}_p} = \infty$. Now it is easy to see that $\mathcal{F}_p$ is a linear subspace of L^p and (4.2) defines a norm in $\mathcal{F}_p$. We complete the space $\mathcal{F}_p$ with respect to the norm (4.2) and obtain the space $\mathcal{F}^p$, which consists of equivalence classes of Cauchy sequences of functions in $\mathcal{F}_p$. Since L^p is complete, every equivalence class is represented by a unique function $u \in L^p$, and it is not difficult to see that the induced norm on $\mathcal{F}^p$ is

$$\|u\|_{\mathcal{F}^p} = \big(\|u\|_{L^p}^p + \lim_{\varepsilon \to 0} \inf_{g \in D_\varepsilon(u)} \|g\|_{L^p}^p\big)^{1/p} \tag{4.3}$$

where for $\varepsilon > 0$

$$\begin{aligned} D_\varepsilon(u) &= D_\varepsilon(u, \mathcal{K}) \\ &= \{g \in \mathcal{K}\colon g \in D(v, \mathcal{K}) \text{ for some } v \in \mathcal{F} \text{ with } \|u - v\|_{L^p} < \varepsilon\}. \end{aligned}$$

For the construction of $\mathcal{F}^p$ only the space $\mathcal{F} \cap L^p$ and the cone $\mathcal{K} \cap L^{p+}$ are needed; here L^{p+} is the positive cone of non-negative functions in L^p.

Now $\mathcal{F}^p$ is a linear space and, by construction, complete, hence $\mathcal{F}^p$ is a Banach space. From the definition of $\mathcal{F}^p$ we obtain the following characterization.

4.4. Lemma. *A function $u \in L^p$ belongs to $\mathcal{F}^p$ if and only if there are sequences of functions $u_i \in \mathcal{F}$ and $g_i \in D(u_i, \mathcal{K})$ such that $u_i \to u$ in L^p and*

$$\liminf_{i \to \infty} \|g_i\|_{L^p} < \infty.$$

In general, it is difficult to compute $\|\cdot\|_{\mathcal{F}^p}$ from (4.3). The following lemma makes it sometimes easier to estimate $\|\cdot\|_{\mathcal{F}^p}$.

4.5. Lemma. *Let $1 < p < \infty$ and $u \in \mathcal{F}^p$. Then there is a unique function $g \in D(u, L^{p+})$ such that*

$$\|u\|_{\mathcal{F}^p} = \left(\|u\|_{L^p}^p + \|g\|_{L^p}^p\right)^{1/p}$$

and $g_i \to g$ in L^p for some sequence $g_i \in \mathcal{K}$.

Proof. By the definition of the norm in $\mathcal{F}^p$, there are sequences of functions $\widetilde{v}_i \in \mathcal{F}$ and $\widetilde{g}_i \in D(\widetilde{v}_i, \mathcal{K})$ such that $\widetilde{v}_i \to u$ in L^p and

$$\left(\|\widetilde{v}_i\|_{L^p}^p + \|\widetilde{g}_i\|_{L^p}^p\right)^{1/p} \to \|u\|_{\mathcal{F}^p}.$$

Passing to a subsequence we may assume that $\widetilde{v}_i \to u$ μ-a.e. Since $1 < p < \infty$ and the sequence $\|\widetilde{g}_i\|_{L^p}$ is bounded, we may assume, passing again to a subsequence, that $\widetilde{g}_i \to g$ weakly in L^p. By the Banach-Saks theorem, passing to a subsequence, the sequence,

$$g_j = \frac{1}{j} \sum_{i=1}^{j} \widetilde{g}_i$$

converges to g in L^p and hence, passing again to a subsequence if necessary, we may assume that $g_j \to g$ μ-a.e. By the cone property of $\mathcal{K}$ we have $g_j \in D(v_j, \mathcal{K})$ where

$$v_j = \frac{1}{j} \sum_{i=1}^{j} \widetilde{v}_j.$$

Since $v_j \to u$ μ-a.e. and

$$|v_j(x) - v_j(y)| \le d(x,y)\big(g_j(x) + g_j(y)\big)$$

holds for μ-a.e. in X, we obtain as $j \to \infty$

$$|u(x) - u(y)| \le d(x,y)\big(g(x) + g(y)\big)$$

μ-a.e. Thus $g \in D(u, L^{p+})$ and the uniqueness of g follows from the uniform convexity of the L^p-norm, $1 < p < \infty$. □

4.6. Example. Let $1 < p < \infty$. If we take $\mathcal{F} = L^p(X)$ and $\mathcal{K} = L^p(X)^+$, then the norm of $\mathcal{F}^p$, given by (4.3), is the same as

$$\|u\| = \big(\|u\|_{L^p}^p + \inf_{g \in D(u, L^{p+})} \|g\|_{L^p}^p\big)^{1/p}; \tag{4.7}$$

this can be easily seen by using the method of the proof of Lemma 4.5. The space $\mathcal{F}^p$ is now the space $W^{1,p}(X)$ introduced by P. Hajłasz [H].

5. The capacity associated with the space $\mathcal{F}^p$

In this section we show that there is a natural variational capacity based on the norm (4.3). In [KM] we introduced this capacity in the space $W^{1,p}(X)$ of Example 4.6.

Let (X, d), μ, $\mathcal{F}$, and $\mathcal{K}$ be as in Section 4. Furthermore, we assume that both $\mathcal{F}$ and $\mathcal{K}$ have the following lattice property: If $u_i \in \mathcal{F}$ (resp. $\mathcal{K}$), $i = 1$, 2, then $\max(u_1, u_2) \in \mathcal{F}$ (resp. $\mathcal{K}$). Moreover, we assume that if $c \ge 0$, then $\min(u, c) \in \mathcal{F}$ for each $u \in \mathcal{F}$. The last condition means that functions in $\mathcal{F}$ can be truncated.

The following lemma is easy to prove.

5.1. Lemma. *Suppose that $u_i \in \mathcal{F}$ and $g_i \in D(u_i, \mathcal{K})$, $i = 1, 2$. Then $\max(g_1, g_2) \in D(\max(u_1, u_2), \mathcal{K})$, and $\max(g_1, g_2) \in D(\min(u_1, u_2), \mathcal{K})$. Also for $c \ge 0$ we have $g_1 \in D(\min(u_1, c), \mathcal{K})$.*

Lemma 5.1 implies that if $g \in D(u, \mathcal{K})$, then $g \in D(|u|, \mathcal{K})$.

For $E \subset X$ and $1 < p < \infty$ we introduce the p-capacity of E as

$$C_p(E) = C_p(E; \mathcal{F}, \mathcal{K}) = \inf_{u \in \mathcal{A}} \|u\|_{\mathcal{F}^p}^p, \tag{5.2}$$

where $\mathcal{A} = \mathcal{A}(E)$ is the set of admissible functions for E, i.e., $\mathcal{A}$ consists of functions $u \in L^p(X)$ such that $u \ge 1$ μ-a.e. in a neighbourhood of E. If $\mathcal{A} = \emptyset$, then we set $C_p(E) = \infty$.

5.3. Remark. As the admissible set of functions in (5.2) the set $\mathcal{A}' = \{u \in \mathcal{A} \colon 0 \le u \le 1\}$ can be used.

The following theorem shows that the capacity C_p has the same basic measure theoretic property as the classical capacities.

5.4. Theorem. *The capacity C_p is an outer measure.*

Proof. Since $C_p(\emptyset) = 0$ and since C_p is clearly monotone, it remains to check the subadditivity. For this let $E_i \subset X$, $i = 1, 2, \ldots$, and $\varepsilon > 0$. We may assume that

$$\sum_{i=1}^{\infty} C_p(E_i) < \infty. \tag{5.5}$$

Fix $t > 0$ and choose $u_i \in \mathcal{A}(E_i)$ and $v_i \in \mathcal{F}$ with $g_i \in D(v_i, \mathcal{K})$ such that

$$\|u_i - v_i\|_{L^p} < t/2^{i+2} \tag{5.6}$$

and

$$\|u_i\|_{L^p}^p + \|g_i\|_{L^p}^p \le C_p(E_i) + \varepsilon/2^i. \tag{5.7}$$

Next consider $u = \sup u_i$; clearly $u \ge 1$ in a neighbourhood of $E = \bigcup E_i$. Choose k so large that the function $\widetilde{v}_k = \max\limits_{1\le i\le k} v_i$ satisfies

$$\|\widetilde{v}_k - u\|_{L^p} < t.$$

Indeed, since $\widetilde{v}_k \to v = \sup v_i$ in L^p and $\|v - u\|_{L^p} \le t/2$ by (5.6), this is possible. Hence

$$\inf_{\widetilde{g}\in D_t(u)} \|\widetilde{g}\|_{L^p}^p \le \|\widetilde{g}_k\|_{L^p}^p,$$

where $\widetilde{g}_k = \max\limits_{1\le i\le k} g_i$; note that $\widetilde{g}_k \in D(\widetilde{v}_k, \mathcal{K})$ by Lemma 5.1.

Thus we have

$$\inf_{\widetilde{g}\in D_t(u)} \|\widetilde{g}\|_{L^p}^p \le \|\widetilde{g}_k\|_{L^p}^p \le \|g\|_{L^p}^p,$$

where $g = \sup g_i$. Letting $t \to 0$ we obtain

$$\lim_{t\to 0} \inf_{\widetilde{g}\in D_t(u)} \|\widetilde{g}\|_{L^p}^p \le \|g\|_{L^p}^p.$$

On the other hand,

$$\|g\|_{L^p}^p \le \sum_{i=1}^{\infty} \|g_i\|_{L^p}^p,$$

and

$$\|u\|_{L^p}^p \leq \sum_{i=1}^{\infty} \|u_i\|_{L^p}^p,$$

and hence

$$\begin{aligned} C_p(E) &\leq \|u\|_{L^p}^p + \lim_{t\to 0} \inf_{\widetilde{g}\in D_t(u)} \|\widetilde{g}\|_{L^p}^p \\ &\leq \sum_{i=1}^{\infty} \left(\|u_i\|_{L^p}^p + \|g_i\|_{L^p}^p\right) \\ &\leq \sum_{i=1}^{\infty} C_p(E_i) + \varepsilon, \end{aligned}$$

and the proof is complete since $\varepsilon > 0$ was arbitrary. □

5.8. Remark. It is not very difficult to show the following properties of C_p.

If $C_1 \supset C_2 \supset \dots$ are compact sets in X and $C = \bigcap C_i$, then

(5.9) $$C_p(C) = \lim_{i\to\infty} C_p(C_i).$$

If $O_1 \subset O_2 \subset \dots$ are open sets in X and $O = \bigcup O_i$, then

(5.10) $$C_p(O) = \lim_{i\to\infty} C_p(O_i).$$

We close this section by studying the capacity properties of functions in $W^{1,p}(X)$. A property is said to hold p-quasieverywhere (p-q.e.) if it holds except for a set of p-capacity zero.

5.11. Theorem. *For each Cauchy sequence (u_i) of functions $u_i \in \mathcal{F}^p \cap C(X)$ there exists a subsequence which converges p-q.e. to a function $u \in \mathcal{F}^p$. Moreover, the convergence is uniform outside a set of arbitrary small p-capacity.*

Proof. Choose a subsequence, denoted again by (u_i), such that

$$\sum_{i=1}^{\infty} 2^{ip} \|u_i - u_{i+1}\|_{\mathcal{F}^p}^p < \infty.$$

For $i = 1, 2, \dots,$ let

$$E_i = \{x \in X : |u_i(x) - u_{i+1}(x)| > 2^{-i}\},$$

and $F_j = \bigcup_{i=j}^{\infty} E_i$. By continuity the function $2^i|u_i - u_{i+1}|$ is admissible for E_i, which implies

$$C_p(E_i) \le 2^{ip}\|u_i - u_{i+1}\|^p_{\mathcal{F}^p}.$$

By subadditivity we obtain

$$C_p(F_j) \le \sum_{i=j}^{\infty} C_p(E_i) \le \sum_{i=j}^{\infty} 2^{ip}\|u_i - u_{i+1}\|^p_{\mathcal{F}^p}.$$

Next let $F = \bigcap F_j$. Then

$$C_p(F) \le \lim_{j\to\infty} C_p(F_j) = 0.$$

Now $u_i(x)$ converges at each point $x \in X \setminus F$. Also

$$\begin{aligned} |u_j(x) - u_k(x)| &\le \sum_{i=j}^{k-1} |u_i(x) - u_{i+1}(x)| \\ &\le \sum_{i=j}^{k-1} 2^{-i} \le 2^{1-j} \end{aligned}$$

for $x \in X \setminus F_j$, $k > j$, which shows that the convergence is uniform in $X \setminus F_j$. $\square$

We say that a function $u\colon X \to [-\infty, \infty]$ is p-quasicontinuous, if for every $\varepsilon > 0$ there is $E \subset X$ such that $C_p(E) < \varepsilon$ and $u|_{X\setminus E}$ is continuous.

5.12. Corollary. *Suppose that $C(X)$ is dense in $\mathcal{F}^p$. Then for each $u \in \mathcal{F}^p$ there is a p-quasicontinuous function $v \in \mathcal{F}^p$ such that $u = v$ μ-a.e.*

5.13. Remark. Corollary 5.12 holds, in particular, in $\mathcal{F}^p$, if each $u \in \mathcal{F}$ is continuous.

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