BANACH FUNCTION SPACES AND THE TWO-WEIGHT PROBLEM FOR MAXIMAL FUNCTIONS

Carlos Pérez

1. Motivation and description of main results

The purpose of this paper is to give an exposition of our recent work on the two-weight problem for maximal functions [P1], [P2]. In particular, we wish to describe the relevance of certain boundedness properties of a variant of the Hardy-Littlewood maximal function, which we denote by M_X, where X is a Banach function space (BFS). Then, we shall outline some applications to higher singular operators as obtained in [P3], [P4], [P5], [P6].

Let $0 \leq \alpha < n$ be the Marcinkiewicz (or fractional maximal) operator defined for locally integrable functions f by

$$M_\alpha f(x) = \sup_{x \in Q} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q |f(y)| \, \mathrm{d}y,$$

where Q denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes. We are interested in the problem of obtaining estimates of the form

$$\int_{\mathbb{R}^n} \big(w(x) M_\alpha f(x)\big)^p \, \mathrm{d}x \leq c \int_{\mathbb{R}^n} \big(v(x)|f(x)|\big)^p \, \mathrm{d}x, \tag{1}$$

where $1 < p < \infty$, c is a positive constant independent of f, and w, v are weight functions, that is, measurable and a.e. nonnegative functions.

1.1. The Hardy-Littlewood maximal function.

The starting point of the modern theory of weighted norm inequalities goes back to the paper of B. Muckenhoupt [M] concerning the Hardy-Littlewood maximal function which corresponds to the case $\alpha = 0$. In that paper the inequality (1) with $\alpha = 0$ and with $w = v$ was characterized in terms of the surprisingly simple geometric condition:

There exists a finite constant c such that for all cubes Q

$$\left(\frac{1}{|Q|} \int_Q w(y)^p \, \mathrm{d}y\right)^{1/p} \left(\frac{1}{|Q|} \int_Q w(y)^{-p'} \, \mathrm{d}y\right)^{1/p'} \leq c.$$

A characterization in the case of two weights was obtained by E. Sawyer [S1], [S2] by means of the condition:

There exists a finite constant c such that for all cubes Q

$$S_p \qquad \int_Q \big(w(x)M(v^{-p'}\chi_Q)(x)\big)^p \,\mathrm{d}x \le c \int_Q v(x)^{-p'} \,\mathrm{d}x.$$

This condition is difficult to handle in practice since it involves the operator M itself. On the other hand, if we consider the weak type inequality, namely,

$$\sup_{\lambda>0} \lambda^p \, w^p(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \le C \int_{\mathbb{R}^n} \big(v(x)|f(x)|\big)^p \,\mathrm{d}x, \tag{2}$$

it is well known [GCRdF] that it holds if and only if the weights satisfy the two-weight version of the A_p condition:

There exists a finite constant c such that for all cubes Q

$$A_p \qquad \left(\frac{1}{|Q|}\int_Q w(x)^p \,\mathrm{d}x\right)^{1/p} \left(\frac{1}{|Q|}\int_Q v(x)^{-p'} \,\mathrm{d}x\right)^{1/p'} \le c,$$

It is also well known that the A_p condition is not sufficient for the strong type inequality (1), namely we have the strict inclusion

$$S_p \subset A_p.$$

It is therefore natural to study the "gap" between the two conditions. In particular, since the condition A_p is simple, it would be interesting to obtain sufficient conditions "close" in form to A_p.

A first result in this direction was obtained by C. Neugebauer in [N]. He observed that if (w, v) is a couple of weights such that for some $r > 1$

$$\left(\frac{1}{|Q|}\int_Q w(x)^{pr} \,\mathrm{d}x\right)^{1/(pr)} \left(\frac{1}{|Q|}\int_Q v(x)^{-p'r} \,\mathrm{d}x\right)^{1/(p'r)} \le c \tag{3}$$

for all cubes Q, then

$$\int_{\mathbb{R}^n} \big(w(x)Mf(x)\big)^p \,\mathrm{d}x \le c \int_{\mathbb{R}^n} \big(v(x)f(x)\big)^p \,\mathrm{d}x \tag{4}$$

for all nonnegative functions f.

The method in [N] is interesting because it shows that (3) is equivalent to "inserting" an A_p weight between the given weights. However, it is not sharp enough to obtain better conditions for (4).

We showed in [P1] that (4) holds assuming much weaker conditions on the weights. We shall see that it is enough to replace the average norm associated to the weight v^{-1} in A_p by a "stronger" norm defined in terms of any BFS whose associated space satisfies certain mapping property.

To be more precise we give the following definition.

Definition 1.1. Let X be a BFS over $\mathbb{R}^n$ with respect to the Lebesgue measure. Given a measurable function f and any cube Q, we define the X-average of f over Q by

$$\|f\|_{X,Q} = \|\tau_{\ell(Q)}(f\chi_Q)\|_X, \tag{5}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$, χ_E is the characteristic function of E and $\ell(Q)$ is the sidelength of the cube Q.

If $X = L^B$ is the Orlicz space defined by a Young function B [KR], [KJF], then the average is defined by

$$\|f\|_{X,Q} = \|f\|_{B,Q} = \inf\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)\,\mathrm{d}y \le 1\right\}. \tag{6}$$

If X is the Lorentz space $X = L^{s,q}$, then the average is given by

$$\|f\|_{X,Q} = \|f\|_{s,q,Q} = \frac{1}{|Q|^{1/s}}\|f\chi_Q\|_{s,q}.$$

We now define a natural maximal operator associated to the space X the interest of which will be shown in Theorem 1.3.

Definition 1.2. For each locally integrable function f the maximal operator M_X is defined by

$$M_X f(x) = \sup_{x\in Q}\|f\|_{X,Q},$$

where the supremum is taken over all the cubes containing x.

1.2. The fractional maximal function.

Given a BFS X, X' will denote the space associate to X, which is again BFS (cf. next section).

Theorem 1.3. *Let $1 < p < \infty$ and let $0 \le \alpha < n$. Let X be a BFS such that the maximal operator corresponding to its associate space X' is bounded on $L^p(\mathbb{R}^n)$, namely*

$$M_{X'}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n). \tag{7}$$

Suppose that (w, v) is a couple of weights satisfying the condition: there exists a positive constant K such that for all cubes Q

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q w(y)^p\,\mathrm{d}y\right)^{1/p}\|v^{-1}\|_{X,Q} \le K. \tag{8}$$

Then there exists a positive constant c such that

$$\int_{\mathbb{R}^n} \big(w(x)M_\alpha f(x)\big)^p \,\mathrm{d}x \le c \int_{\mathbb{R}^n} \big(v(x)f(x)\big)^p \,\mathrm{d}x \tag{9}$$

for all nonnegative functions f.

We remark that the spaces X that we are interested in must satisfy the following inequalities for all functions f and for all cubes Q:

$$\|f\|_{L^{p'},Q} \le \|f\|_{X,Q} \le \|f\|_{L^{p'r},Q},$$

where $r > 1$. The first one follows from the condition imposed on X in Theorem 1.3: $M_{X'}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, as can be seen directly [P1]. The second inequality follows from the fact that the associate space to $X = L^{p'r}$ is $X' = L^{(p'r)'}$ whose corresponding maximal operator

$$M_{X'}f(x) = M_{(p'r)'}f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |f(y)|^{(p'r)'}\,\mathrm{d}y\right)^{1/(p'r)'},$$

is bounded on $L^p(\mathbb{R}^n)$. Therefore we can deduce the following.

Corollary 1.4. *Let $1 < p < \infty$ and let $0 \le \alpha < n$. Suppose that (w, v) is a couple of weights such that for some $r > 1$ there is a positive constant K for which for all cubes Q*

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q w(x)^p\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'r}\,\mathrm{d}x\right)^{1/(p'r)} \le K. \tag{10}$$

Then

$$\int_{\mathbb{R}^n} \big(w(x)M_\alpha f(x)\big)^p \,\mathrm{d}x \le c \int_{\mathbb{R}^n} \big(v(x)|f(x)|\big)^p \,\mathrm{d}x \tag{11}$$

for all functions f.

We refer the reader to the survey paper [W] by R. Wheeden for more information about this condition in another context.

Observe that when $r = 1$ the associate space to $X = L^{p'}$ is $X' = L^p$ for which the corresponding maximal function M_p is not bounded on $L^p(\mathbb{R}^n)$ since M itself fails to be bounded on $L^1(\mathbb{R}^n)$.

We remark that the case of the Hardy-Littlewood maximal function has a different flavour since it can be deduced by interpolation using a self-improving property of (10) valid only when $\alpha = 0$ (see Section 3).

A particular instance which still is not interesting enough is given by the Lorentz space

$$X = L^{p'r,\infty}.$$

Indeed the associate space is given by $X' = L^{(p'r)',1}$ and it is not hard to show that $M_{X'}$ is bounded on $L^p(\mathbb{R}^n)$ [P1]. Therefore if (w, v) satisfies for some positive K and for all cubes Q,

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q w(x)^p\,\mathrm{d}x\right)^{1/p}\frac{1}{|Q|^{1/(rp')}}\|\chi_Q v^{-1}\|_{rp',\infty} \le K, \tag{12}$$

then (11) holds.

More interesting examples are provided by the theory of Orlicz spaces such as

$$X = L^{p'}(\log L)^{p'-1+\delta}$$

or the larger one

$$X = L^{p'}(\log L)^{p'-1}\big(\log(\log L)\big)^{p'-1+\delta}$$

where $\delta > 0$, and for which the case $\delta = 0$ is **not** allowed. This follows from:

Corollary 1.5. *Let $1 < p < \infty$, and let B be a doubling Young function such that*

$$\int_c^\infty \left(\frac{t^{p'}}{B(t)}\right)^{p-1}\frac{\mathrm{d}t}{t} < \infty, \tag{13}$$

for some positive constant c. Let (w, v) be a couple of weights such that there is a positive constant K for which

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q w(y)^p\,\mathrm{d}y\right)^{1/p}\|v^{-1}\|_{B,Q} \le K \tag{14}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n}\big(w(x)M_\alpha f(x)\big)^p\,\mathrm{d}x \le c\int_{\mathbb{R}^n}\big(v(x)f(x)\big)^p\,\mathrm{d}x \tag{15}$$

for all nonnegative functions f.

The Corollary follows from Theorem 1.3 once we show that the tail condition of B (13) implies that the maximal function corresponding to the associate space to L^B, namely $M_{(L^B)'} \approx M_{L^{\overline{B}}}$ is bounded on $L^p(\mathbb{R}^n)$. In fact we have the following characterization.

Theorem 1.6. *Let $1 < p < \infty$. Suppose that B is a Young function. Then the following statements are equivalent.*

(i)

$$\int_c^\infty \frac{B(t)}{t^p}\,\frac{\mathrm{d}t}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\overline{B}(t)}\right)^{p-1} \frac{\mathrm{d}t}{t} < \infty. \tag{16}$$

(ii) *There is a constant c such that*

$$\int_{\mathbb{R}^n} \big(M_B f(y)\big)^p\,\mathrm{d}y \le c \int_{\mathbb{R}^n} f(y)^p\,\mathrm{d}y \tag{17}$$

for all nonnegative functions f.

(iii) *There is a constant c such that*

$$\int_{\mathbb{R}^n} \big(M_B f(y)\big)^p\, w(y)\ \mathrm{d}y \le c \int_{\mathbb{R}^n} f(y)^p\, Mw(y)\ \mathrm{d}y \tag{18}$$

for all nonnegative functions f and w.

(iv) *There is a constant c such that*

$$\int_{\mathbb{R}^n} \big(Mf(y)\big)^p\, \frac{w(y)}{\big[M_{\overline{B}}(u^{1/p})(y)\big]^p}\ \mathrm{d}y \le c \int_{\mathbb{R}^n} f(y)^p\, \frac{Mw(y)}{u(y)}\ \mathrm{d}y, \tag{19}$$

for all nonnegative functions f, w and u.

(v) *Let $1 < s \le \infty$. Then*

$$M_B : L^p_{\ell^s}(\mathbb{R}^n) \to L^p_{\ell^s}(\mathbb{R}^n).$$

The proof of this Theorem can be found in [P1] except for the last equivalence which is needed in [P5].

Inequality (19) looks intricate; however, it has found to be useful in obtaining sharp weighted estimates for the following important nonlinear commutator of Rochberg and Weiss:

$$N = [T, \Omega] = T \circ \Omega - \Omega \circ T,$$

where $\Omega f = f \log|f|$ [P6]. In particular it is used in the following form

$$\int_{\mathbb{R}^n} \big(Mf(x)\big)^{p'}\, \frac{u(x)}{\big[M^{[p]+1} w(x)\big]^{p'-1}}\ \mathrm{d}x \le c \int_{\mathbb{R}^n} |f(x)|^{p'}\, \frac{Mu(x)}{w(x)^{p'-1}}\ \mathrm{d}x. \tag{20}$$

1.3. The vector-valued maximal function.

We shall denote by $\overline{M}_q$ the following celebrated vector-valued generalization of the Hardy-Littlewood maximal function which was introduced by C. Fefferman and E. M. Stein in [FS]:

$$\overline{M}_q f(x) = \left(\sum_{i=1}^{\infty} \left(M f_i(x) \right)^q \right)^{1/q}.$$

In [P5] we prove the following weighted estimate which reflects the higher singularity of $\overline{M}_q$ when compared with the scalar case.

Theorem 1.7. *Let $1 < q < p < \infty$.*

(a) *There exists a constant C such that*

$$\int_{\mathbb{R}^n} \left(\overline{M}_q f(x) \right)^p w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|_q^p\, M^{[p/q]+1} w(x)\ \mathrm{d}x, \tag{21}$$

for all locally integrable functions f and $w \ge 0$.

(b) *The condition* (a) *is sharp since there exists* **no** *constant C such that*

$$\int_{\mathbb{R}^n} \left(\overline{M}_q f(x) \right)^p w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|_q^p\, M^{[p/q]} w(x)\ \mathrm{d}x, \tag{22}$$

for all locally integrable functions f and $w \ge 0$. Likewise, the corresponding weak type (p,p) estimate is false.

In this paper we are going to present a different proof based on the equivalences i) and ii) of Theorem 1.6. In fact the method is very general and it uses some of the basic principles of the theory of extrapolation [GCRdF]. We shall give another application in Section 1.4.

Remark 1.8. It follows from part b) of Theorem 1.7 that the vector-valued analogue of the classical Fefferman-Stein inequality

$$\int_{\mathbb{R}^n} \left(\overline{M}_q f(x) \right)^p w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|_q^p\, M w(x)\ \mathrm{d}x. \tag{23}$$

is **false** in general when $p > q$.

Remark 1.9. If we look at the proof of (21) we see that we can refine such inequality by replacing $M^{[p/q]+1}$ by $M_{L(\log L)^{p/q-1+\delta}}$, $\delta > 0$, or by M_A where A satisfies

$$\int_c^{\infty} \left(\frac{t}{A(t)} \right)^{(p/q)'-1} \frac{\mathrm{d}t}{t} < \infty.$$

Proof of Theorem 1.7. Let $r = p/q > 1$ and let I be defined as

$$I = \left\| \left(\sum_{i=0}^{\infty} (Mf_i)^q \right)^{1/q} \right\|_{L^p(w)}^q = \left\| \sum_{i=0}^{\infty} (w^{1/p}\, Mf_i)^q \right\|_{L^r(\mathbb{R}^n)}$$

Then there exists $g \in L^{r'}(\mathbb{R}^n)$ with unit norm such that

$$I = \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} \big(M(f_i)(x)\big)^q\, w(x)^{1/r} g(x)\ \mathrm{d}x.$$

Now, by using the classical Fefferman-Stein inequality (28) (with $\alpha = 0$) together with the generalized Hölder inequality (36) in the context of Orlicz space we have

$$\begin{aligned} I &\le C \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} |f_i(x)|^q\, M(w^{1/r} g)(x)\ \mathrm{d}x \\ &\le C \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} |f_i(x)|^q\ M_B(w^{1/r})(x) M_{\overline{B}}(g)(x)\ \mathrm{d}x. \end{aligned}$$

On letting $v = \big(M_B(w^{1/r})\big)^r$ we can continue with the usual Hölder inequality to obtain

$$I \le \left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^p(v)}^q \| M_{\overline{B}}(g) \|_{L^{r'}(\mathbb{R}^n)}$$

All we have to do to conclude the proof is to choose B such that $\overline{B}$ satisfies the tail condition (16) with $p = r'$, namely that

$$\int_c^{\infty} \left(\frac{t^r}{B(t)} \right)^{r'-1} \frac{\mathrm{d}t}{t} < \infty,$$

which would imply by Theorem 1.6 that $M_{\overline{B}}\colon L^{r'}(\mathbb{R}^n) \to L^{r'}(\mathbb{R}^n)$. This is done with $B(t) \approx t^r (\log t)^{[r]}$. Finally, we are left with showing that

$$v = \big(M_B(w^{1/r})\big)^r \le C\, M^{[r]+1}(w).$$

Indeed, since $B(t^{1/r}) \approx t(\log t)^{[r]}$ and $\big(M_B(w^{1/r})\big)^r = M_{L(\log L)^{[r]}}(w)$. Then the inequality follows from the following reasoning:

Let $k = 1, 2, 3, \ldots$, then there exists a constant $C = C_n$ such that for all bounded functions f with support contained in Q

$$\|f\|_{L(\log L)^k, Q} \le \frac{C}{|Q|} \int_Q M^k f(y)\, \mathrm{d}y, \tag{24}$$

where $M^k = M \circ \cdots \circ M$, $k = 1, 2, \ldots$, denotes the Hardy-Littlewood maximal operator M iterated k times. Indeed, by homogeneity we can assume that the right hand side is one. Then by the definition of the Luxemburg norm it is enough to prove

$$\frac{1}{|Q|} \int_Q f(y) \big(1 + \log^+(f(y))\big)^k \, \mathrm{d}y \le C,$$

which is a consequence of iterating the following well known inequality of E. M. Stein [St]:

$$\int_Q f(y)\big(1+\log^+(f(y))\big)^k \, \mathrm{d}y \le C \int_Q Mf(y)\big(1+\log^+(Mf(y))\big)^{k-1} \, \mathrm{d}y, \tag{25}$$

with $k = 1, 2, 3, \ldots$. □

1.4. Littlewood-Paley square functions.

The basic steps of the proof we just gave can be used to derive sharp weighted estimates for other operators for large values of p once there is a good starting point (this is related to extrapolation theory). Due to its central role in Harmonic Analysis let us consider the classical Area function S_φ. This nonlinear operator is defined by the integral

$$S_\varphi(f)(x) = \left(\int_{B_t(x)} |f * \varphi_t(y)|^2 \, \frac{\mathrm{d}t \, \mathrm{d}y}{t^{n+1}} \right)^{1/2},$$

where $\varphi \in C_0^\infty$ with $\int \varphi = 0$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$, $t > 0$. Then the Area function satisfies the following weighted inequality due to A. Chang, M. Wilson, and T. Wolff in [CWW]:

There exists a constant C such that

$$\int_{\mathbb{R}^n} \big(S_\varphi(f)(x)\big)^2 \, w(x) \, \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^2 \, Mw(x) \, \mathrm{d}x. \tag{26}$$

Besides of its intrinsic interest, this inequality is related to the problem of estimating the eigenvalues of Schrödinger operators in terms of the potential as shown in the celebrated paper by C. Fefferman [F] and the subsequent papers [CWW], [CW1]. The estimate for small values of p, namely $1 < p < 2$, follows from the result due to S. Chanillo and R. Wheeden in [CW2]. However, it is also shown in this paper that the estimate is false for $p > 2$. See also the work by M. Wilson [Wil1]–[Wil4] which motivate our research.

We state now the following best possible result at least in the scale we are looking at.

Theorem 1.10. *Let $p > 2$, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \big(S_\varphi(f)(x)\big)^p\, w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^p\, M_{L(\log L)^{p/2-1+\delta}} w(x)\ \mathrm{d}x, \tag{27}$$

where $\delta > 0$. In fact we could replace the maximal function by M_B, where B satisfies

$$\int_c^\infty \left(\frac{t}{B(t)}\right)^{(p/2)'-1} \frac{\mathrm{d}t}{t} < \infty.$$

Remark 1.11. As above we can select $0 < \delta \le [\frac{p}{2}] - \frac{p}{2} + 1 > 0$ such that

$$M_{L(\log L)^{p/2-1+\delta}} \le C\, M^{[\frac{p}{2}]+1}.$$

This gives a better result than the one obtained by A. Uchiyama in [U] where $M^{[p/2]+1}$ is replaced by $M^{[p/2]+2}$.

Proof. Let $r = p/2 > 1$ and let I be defined as

$$\begin{aligned} I &= \|S_\varphi(f)\|^2_{L^p(w)} = \|\big(S_\varphi(f)\big)^2\, w^{1/r}\|_{L^r(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \big(S_\varphi(f)(x)\big)^2\, w(x)^{1/r} g(x)\ \mathrm{d}x \end{aligned}$$

for some $g \in L^{r'}(\mathbb{R}^n)$ with unit norm. Now, combining (26) together with the generalized Hölder inequality (36) we have that

$$\begin{aligned} I &\le C \int_{\mathbb{R}^n} |f(x)|^2\, M(w^{1/r} g)(x)\ \mathrm{d}x \\ &\le C \int_{\mathbb{R}^n} |f(x)|^2\, M_B(w^{1/r})(x) M_{\overline{B}}(g)(x)\ \mathrm{d}x \\ &\le \|f\|^2_{L^p(v)} \|M_{\overline{B}}(g)\|_{L^{r'}(\mathbb{R}^n)} \end{aligned}$$

where $v = \big(M_B(w^{1/r})\big)^r$. As for the vector-valued maximal function, all we have to do is to choose B such that $\overline{B}$ satisfies the tail condition (16) with $p = r'$, namely that

$$\int_c^\infty \left(\frac{t^r}{B(t)}\right)^{r'-1} \frac{\mathrm{d}t}{t} < \infty,$$

which implies that $M_{\overline{B}}\colon L^{r'}(\mathbb{R}^n) \to L^{r'}(\mathbb{R}^n)$, concluding the proof. $\square$

1.5. A sharp dual inequality.

To understand the next result we recall the following generalization of the classical Fefferman-Stein inequality due to E. Sawyer:

$$\int_{\mathbb{R}^n} \left(M_\alpha f(x)\right)^p w(x)\, \mathrm{d}x \le c \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p} w(x)\, \mathrm{d}x. \tag{28}$$

Now, if M_α were a linear (self-adjoint) operator this inequality would imply

$$\int_{\mathbb{R}^n} \left(M_\alpha f(x)\right)^{p'} \left(M_{\alpha p} w(x)\right)^{1-p'} \mathrm{d}y \le c \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'}\, \mathrm{d}x.$$

However, this inequality is **false** in general and the following can be seen as a sharp replacement.

Corollary 1.12. *Let $1 < p < \infty$ and let $0 \le \alpha < n$. Then, there exists a constant C such that*

$$\int_{\mathbb{R}^n} \left(M_\alpha f(x)\right)^p \left(M_{\alpha p'}(M^{[p']}w)(x)\right)^{1-p} \mathrm{d}x \le c \int_{\mathbb{R}^n} f(x)^p\, w(x)^{1-p}\, \mathrm{d}x \tag{29}$$

for all nonnegative functions f, $w \ge 0$.

Proof. Consider the doubling Young function $B(t) \approx t^{p'}(\log t)^{p'-1+\delta}$ with $\delta = [p'] - p' + 1 > 0$, namely

$$B(t) \approx t^{p'}(\log t)^{[p']}.$$

Then if we denote by $A(t) = B(t^{1/p'}) \approx t(\log t)^{[p']}$ we have

$$\begin{aligned}
&|Q|^{\alpha/n} \left(\frac{1}{|Q|}\int_Q \left(M_{\alpha p'}(M^{[p']}w)(y)\right)^{1-p} \mathrm{d}y\right)^{1/p} \|w^{1/p'}\|_{B,Q} \\
&\quad = |Q|^{\alpha/n} \left(\frac{1}{|Q|}\int_Q \left(M_{\alpha p'}(M^{[p']}w)(y)\right)^{1-p} \mathrm{d}y\right)^{1/p} \|w\|_{A,Q}^{1/p'} \\
&\quad \le \left(\frac{1}{|Q|}\int_Q M^{[p']}w(y)\, \mathrm{d}y\right)^{-1/p'} \|w\|_{A,Q}^{1/p'}
\end{aligned}$$

Therefore, the inequality will follow from (24). □

1.6. Fractional integrals.

A first consequence of Corollary 1.12 in the case $\alpha \ne 0$ concerns classical fractional integrals or Riesz potentials defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, \mathrm{d}y.$$

Theorem 1.13. *Let $1 < p < \infty$ and let $0 < \alpha < n$. Then, we can find a constant C such that*

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^p\, w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^p\, M_{\alpha p}(M^{[p]}w)(x)\ \mathrm{d}x, \tag{30}$$

for all functions f and $w \ge 0$.

The proof we sketch uses duality combined with Corollary 1.12 after applying the following basic result relating I_α and M_α due to Muckenhoupt and Wheeden [MW]:

Let $0 < p < \infty$ and suppose that $w \in A_\infty(\mathbb{R}^n)$. Then the inequality

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^p\, w(x)\ \mathrm{d}x \le C\,[w]_{A_\infty}^p \int_{\mathbb{R}^n} \big(M_\alpha f(x)\big)^p\, w(x)\ \mathrm{d}x, \tag{31}$$

holds for every function f for which the left hand side is finite.

We remark that the fact that $(M_\alpha g)^{1-p'} \in A_{p'} \subset A_\infty$ is needed.

Remark 1.14. Standard inequalities show that (30) implies

$$\int_{\mathbb{R}^n} |f(x)|^2\, w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |\nabla f(x)|^2 M_2\big(M^2(w)\big)(x)\ \mathrm{d}x, \tag{32}$$

and in fact there is a more precise estimate given by

$$\int_{\mathbb{R}^n} |f(x)|^2\, w(x)\ \mathrm{d}x \le C \int_{\mathbb{R}^n} |\nabla f(x)|^2 M_2\big(M_{L(\log L)^{1+\varepsilon}}(w)\big)(x)\ \mathrm{d}x, \tag{33}$$

being the inequality false for $\varepsilon = 0$.

Both inequalities (32) and (33) imply conditions for potentials in the spirit of Chang, Wilson, and Wolff [CWW] (see also [CW1]) when assuming that the potential w satisfies

$$M_2\big(M_{L(\log L)^{1+\varepsilon}}(w)\big) \in L^\infty.$$

We refer the reader to [P4] for more details and for a different approach to (30).

1.7. Singular integrals.

Let T be any Calderón-Zygmund singular integral operator as defined for instance in [J]. As a consequence of a) whenever $\alpha = 0$ we have:

Theorem 1.15. *Let $1 < p < \infty$. Then, there is a constant C such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, \mathrm{d}x \leq C \int_{\mathbb{R}^n} |f(x)|^p\, M^{[p]+1} w(x)\, \mathrm{d}x, \tag{34}$$

with C independent of w and f.

As in the case of fractional integral the proof uses duality combined with the following result of Coifman [C] using that $(Mw)^{1-p'} \in A_\infty$:

Let $0 < p < \infty$ and suppose that $w \in A_\infty(\mathbb{R}^n)$. Then the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, \mathrm{d}x \leq C\, [w]_{A_\infty}^p \int_{\mathbb{R}^n} \big(Mf(x)\big)^p\, w(x)\, \mathrm{d}x, \tag{35}$$

holds for every function f for which the left hand side is finite.

In [P6] can be found versions of (35) for linear and nonlinear commutators in order to deduce estimates of the type (34).

2. The general case

X will always denote a BFS over $\mathbb{R}^n$ with respect to the Lebesgue measure. We shall refer the reader to [BS] for a complete account of the theory of Banach function spaces. X always has an associate BFS X' for which the following generalized Hölder inequality holds:

$$\int_{\mathbb{R}^n} |f(y)g(y)|\, \mathrm{d}y \leq \|f\|_X \|g\|_{X'}. \tag{36}$$

Recall that for any measurable function f and arbitrary cube Q we have defined the X-average of f over Q by

$$\|f\|_{X,Q} = \|\tau_{\ell(Q)}(f\,\chi_Q)\|_X, \tag{37}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$, χ_E is the characteristic function of E and $\ell(Q)$ is the sidelength of the cube Q.

Note that the generalized Hölder inequality yields after the change of variable $y = \ell(Q)z$

$$\frac{1}{|Q|} \int_Q f(y)g(y)\, \mathrm{d}y \leq \|f\|_{X,Q} \|g\|_{X',Q}. \tag{38}$$

Recall that we have also defined in the introduction the following maximal operator associated to the space X:

$$M_X f(x) = \sup_{x \in Q} \|f\|_{X,Q},$$

where the supremum is taken over all the cubes containing x.

We need the following lemma which is nothing but a version of the classical Calderón-Zygmund decomposition adapted to the the fractional maximal operator M_α.

Lemma 2.1. *Let $0 \le \alpha < n$ and Suppose that f is a nonnegative function. For each $t > 0$, let $E_t = \{x \in \mathbb{R}^n : M_\alpha f(x) > t\}$. Then, if E_t is not empty, we have*

$$E_t \subset \bigcup_j 3Q_j,$$

where Q_j is the family of nonoverlapping maximal dyadic cubes satisfying

$$\frac{t}{4^n} < \frac{|Q_j|^{\alpha/n}}{|Q_j|} \int_{Q_j} f(y)\, \mathrm{d}y \le \frac{t}{2^n} \tag{39}$$

for each integer j.

Furthermore we have

$$\{x \in \mathbb{R}^n : M_\alpha^d f(x) > t/4^n\} = \bigcup_j Q_j.$$

Proof of Theorem 1.3. Since the set of bounded functions with compact support is dense in $L^p(v^p)$, it is enough to show that there is a constant c such that

$$\int_{\mathbb{R}^n} \big(w(x) M_\alpha f(x)\big)^p \, \mathrm{d}x \le c \int_{\mathbb{R}^n} \big(v(x) f(x)\big)^p \, \mathrm{d}x, \tag{40}$$

for each nonnegative bounded function with compact support f. Recall that we are assuming that the weights satisfy the condition

$$|Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q w(y)^p \, \mathrm{d}y \right)^{1/p} \|v^{-1}\|_{X,Q} \le K. \tag{41}$$

For each integer k, and for any constant $a > 2^n$ we let Ω_k and D_k be the sets

$$\begin{aligned} \Omega_k &= \{x \in \mathbb{R}^n : a^k < M_\alpha f(x)\}, \\ D_k &= \{x \in \mathbb{R}^n : M_\alpha^d f(x) > a^k/4^n\}. \end{aligned}$$

By Lemma 2.1 with $t = a^k$, there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ for which $\Omega_k \subset \cup_j 3Q_{k,j}$, $D_k = \cup_j Q_{k,j}$, and

$$\frac{a^k}{4^n} < \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)\, \mathrm{d}y \le \frac{a^k}{2^n}. \tag{42}$$

We can now estimate the left hand side of (40) as follows:

(43)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(M_\alpha f(x)\right)^p w(x)^p \,\mathrm{d}x \\
&\quad= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} \left(M_\alpha f(x)\right)^p w(x)^p \,\mathrm{d}x \\
&\quad\le a^p \sum_k a^{kp} w^p(\Omega_k) \le C \sum_{k,j} a^{kp} w^p(3Q_{k,j}) \\
&\quad\le C \sum_{k,j} \left(\frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{Q_{k,j}} f(x)\,\mathrm{d}x \right)^p w^p(3Q_{k,j}) \\
&\quad= C \sum_{k,j} \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(x)v(x)\,v(x)^{-1}\,\mathrm{d}x \right)^p |Q_{k,j}|^{\alpha p/n} w^p(3Q_{k,j}).
\end{aligned}$$

Applying now the generalized Hölder inequality (38) together with the hypothesis on the weights (41) we can dominate last sum by

$$\begin{aligned}
(44)\qquad & C \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} \|v^{-1}\|^p_{X,3Q_{k,j}} |Q_{k,j}|^{\alpha p/n} w^p(3Q_{k,j}) \\
&= C \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} \|v^{-1}\|^p_{X,3Q_{k,j}} |3Q_{k,j}|^{\alpha p/n} \frac{w^p(3Q_{k,j})}{|3Q_{k,j}|} |Q_{k,j}| \\
&\le CK^p \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} |Q_{k,j}|
\end{aligned}$$

Now, for each integer k, j we let $E_{k,j} = Q_{k,j} \setminus D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets, and furthermore there is a positive constant β such that for each k, j $|Q_{k,j}| < \beta\,|E_{k,j}|$. Indeed, by standard properties of the dyadic cubes combined with (42) we have

$$\begin{aligned}
\frac{|Q_{k,j}\cap D_{k+1}|}{|Q_{k,j}|} &= \sum_i \frac{|Q_{k,j}\cap Q_{k+1,i}|}{|Q_{k,j}|} = \sum_{Q_{k+1,i}\subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} \\
&< \frac{1}{|Q_{k,j}|} \sum_{Q_{k+1,i}\subset Q_{k,j}} \left(\frac{4^n}{a^{k+1}} f(Q_{k+1,i}) \right)^{\frac{1}{1-\alpha/n}} \\
&\le \frac{1}{|Q_{k,j}|} \left(\sum_{Q_{k+1,i}\subset Q_{k,j}} \frac{4^n}{a^{k+1}} f(Q_{k+1,i}) \right)^{\frac{1}{1-\alpha/n}} \le
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{|Q_{k,j}|}\left(\frac{4^n}{a^{k+1}} f(Q_{k,j} \cap \bigcup_i Q_{k+1,i})\right)^{\frac{1}{1-\alpha/n}} \\
&\leq \frac{1}{|Q_{k,j}|}\left(\frac{4^n}{a^{k+1}} \frac{a^k}{2^n} |Q_{k,j}|^{1-\frac{\alpha}{n}}\right)^{\frac{1}{1-\alpha/n}} \\
&= \left(\frac{2^n}{a}\right)^{\frac{1}{1-\alpha/n}} < \frac{2^n}{a},
\end{aligned}
$$

since $a > 2^n$. Therefore $|Q_{k,j}| < \frac{1}{1-2^n/a}\,|E_{k,j}|$ and we can continue estimating (44) by

$$
\begin{aligned}
&\leq CK^p \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} |E_{k,j}| \\
&\leq C \sum_{k,j} \int_{E_{k,j}} \big(M_{X'}(fv)(x)\big)^p \,\mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \big(M_{X'}(fv)(x)\big)^p \,\mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \big(f(x)v(x)\big)^p \,\mathrm{d}x,
\end{aligned}
$$

since the sets $\{E_{k,j}\}$ are pairwise disjoint, and because we are assuming that $M_{X'}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. This concludes the proof of the theorem. □

3. A self-improving property of couple of weights

The purpose of this section is to show that some of the problems that we have been considering so far can be approached in a different way in the case of the Hardy-Littlewood maximal function. The interest is that this other method can be applied in different contexts [W], but the drawback is that it does not apply to the fractional case.

We consider the problem

$$
\int_{\mathbb{R}^n} \big(Mf(x)\big)^p\, w(x)\, \mathrm{d}x \leq c \int_{\mathbb{R}^n} |f(x)|^p\, v(x)\, \mathrm{d}x. \tag{45}
$$

By Corollary 1.4 we already know that a sufficient condition for some $1 < r < \infty$ is given by

$$
\frac{1}{|Q|}\int_Q w(x)\, \mathrm{d}x \left(\frac{1}{|Q|}\int_Q v(x)^{(1-p')r}\, \mathrm{d}x\right)^{(p-1)/r} \leq c. \tag{46}
$$

We now argue as follows. Observe first that we always have

$$M\colon L^\infty(v) \to L^\infty(w). \tag{47}$$

Now, if we denote the condition (46) by $A_{p,r}$ we see that $A_{p,r} \subset A_{p,1} = A_p$ and if we let q such that $q-1 = \frac{p-1}{r}$, we have

$$\frac{1}{|Q|}\int_Q w\left(\frac{1}{|Q|}\int_Q v^{(1-p')r}\right)^{(p-1)/r} = \frac{1}{|Q|}\int_Q w\left(\frac{1}{|Q|}\int_Q v^{1-q'}\right)^{q-1}.$$

Then $1 < q < p$ and

$$(w,v) \in A_{p,r} \Rightarrow (w,v) \in A_q \Leftrightarrow M\colon L^q(v) \to L^{q,\infty}(w),$$

by (2). This together with (47) implies that $M\colon L^p(v) \to L^p(w)$ by the Marcinkiewicz's interpolation theorem.

It would be interesting to give a similar argument in the fractional case $\alpha \neq 0$.

Also, it would be interesting to fit this argument within the context of Corollary 1.5 at least in the case of $\alpha = 0$. There may be perhaps some sort of nonpower self-improvement property of the condition appearing in the tail condition (13).

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