A SEMILINEAR ELLIPTIC PROBLEM IN AN UNBOUNDED DOMAIN

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Let $\Omega = D \times \mathbb{R}^m$ $(m \geq 1)$ be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$, where $D \subset \mathbb{R}^{n-m}$ is bounded, with smooth boundary $\Gamma = \partial D \times \mathbb{R}^m$. In this paper we study the problem

$$(1) \qquad Au \equiv -\sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\partial_i u) + b(x)u = f(x,u) \qquad \text{in} \quad \Omega,$$

$$(2) \qquad Bu \equiv \sum_{i,j=1}^{n} n_j(\xi)a_{ij}(\xi)\partial_i u = \varphi(\xi,u) \qquad \text{on} \quad \Gamma,$$

$$(3) \qquad u(x) \longrightarrow 0 \qquad \text{as} \quad |x| \to \infty,$$

where A is a uniformly elliptic operator and n_j $(j = 1, \dots, n)$ are the components of the outer normal vector on Γ. It is assumed that f and φ are of subcritical growth in the second variable. Problems of this kind arise for example in electromagnetic wave propagation in fibres. In [4] the existence of a nontrivial solution to this problem was shown in the special case

$$f(x,u) = P(x)|u|^{p-1}u, \quad \varphi(\xi,u) = Q(\xi)|u|^{p-1}u, \quad 1 < p < n/(n-2),$$

$(n \geq 3)$, where a Mountain Pass Lemma approach on a sequence of bounded subdomains of Ω and a comparison argument were used. In this paper we make the following assumptions.

1° The coefficients a_{ij} are in $C^1(\overline{\Omega})$, b is an L^∞-function satisfying $b(x) \geq b_0 > 0$, and A satisfies the ellipticity condition $\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq \alpha|\zeta|^2$ for every $\zeta \in \mathbb{R}^n$.

2° The functions $f(x,t), \varphi(\xi,t)$ are continuously differentiable with respect to t, $\partial f/\partial t$ and $\partial\varphi/\partial t$ are bounded and uniformly continuous on $\Omega \times [-T,T]$ (resp. $\Gamma \times [-T,T]$) for each $T > 0$, and there exists a real $\theta \in (0,1)$ such that for every $t \neq 0$

$$0 \leq t^{-1}f(x,t) \leq \theta\frac{\partial f}{\partial t}(x,t), \quad 0 \leq t^{-1}\varphi(\xi,t) \leq \theta\frac{\partial\varphi}{\partial t}(\xi,t).$$

3° We have

$$\lim_{|t|\to\infty} f(x,t)/|t|^{(n+2)/(n-2)} = \lim_{|t|\to\infty} \varphi(\xi,t)/|t|^{n/(n-2)} = 0,$$
$$\limsup_{|t|\to 0} f(x,t)/t \le 0,\ \limsup_{|t|\to 0} \varphi(\xi,t)/t \le 0,\ \text{and}\ \limsup_{|t|\to\infty} |f(x,t)| > 0$$

uniformly in x and ξ. (If $n = 2$, $(n+2)/(n-2)$ and $n/(n-2)$ may be replaced by any finite power p.)

Assumption 1° implies that the norm

$$\|u\|_A^2 = \int_\Omega \Big(\sum a_{ij}(x)\partial_i u \partial_j u + b(x)u^2\Big)\,\mathrm{d}x$$

is equivalent to the usual $W^{1,2}(\Omega)$-norm. We define

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega \Big(\sum a_{ij}(x)\partial_i u \partial_j u + b(x)u^2\Big)\,\mathrm{d}x \\ - \int_\Omega F(x,u)\,\mathrm{d}x - \int_\Gamma \Phi(\xi,u)\,\mathrm{d}\Gamma,$$

$$J(u) = \int_\Omega \Big(\sum a_{ij}(x)\partial_i u \partial_j u + b(x)u^2\Big)\,\mathrm{d}x \\ - \int_\Omega f(x,u)u\,\mathrm{d}x - \int_\Gamma \varphi(\xi,u)u\,\mathrm{d}\Gamma,$$

where F and Φ are the indefinite integrals of f and φ, respectively. We consider the following minimization problem which goes back to Nehari [2]:

$$I = \inf\{\mathcal{E}(u)\colon u \in W^{1,2}(\Omega),\ u \neq 0,\ J(u) = 0\}. \tag{4}$$

It follows from our assumptions that every $u \neq 0$ with $J(u) = 0$ satisfies $\mathcal{E}(u) \ge \delta > 0$ and $\langle J'(u), u\rangle < 0$. Therefore 0 is an isolated point in the set $\{u \in W^{1,2}(\Omega) \mid J(u) = 0\}$, and the Lagrange multiplier appearing in (4) is zero. Thus a minimum is a weak solution of (1)–(3). The main result of this paper is the existence of a weak solution in the case of asymptotically constant coefficients. In addition to 1°–3° we assume that

4° $a_{ij}(x) \to \widehat{a}_{ij}$, $b(x) \to \widehat{b} > 0$ as $|x| \to \infty$ and $f(x,t) \to \widehat{f}(t)$, $\varphi(\xi,t) \to \widehat{\varphi}(t)$ as $|x| \to \infty$, $|\xi| \to \infty$, uniformly in $t \in [-T,T]$ for every T.

We also consider the problem at infinity: Define

$$\mathcal{E}^\infty(u) = \frac{1}{2}\int_\Omega \Big(\sum \widehat{a}_{ij}\partial_i u \partial_j u + \widehat{b}u^2\Big)\,\mathrm{d}x - \int_\Omega \widehat{F}(u)\,\mathrm{d}x - \int_\Gamma \widehat{\Phi}(u)\,\mathrm{d}\Gamma,$$

$$J^\infty(u) = \int_\Omega \Big(\sum \widehat{a}_{ij}\partial_i u \partial_j u + \widehat{b}u^2\Big)\,\mathrm{d}x - \int_\Omega \widehat{f}(u)u\,\mathrm{d}x - \int_\Gamma \widehat{\varphi}(u)u\,\mathrm{d}\Gamma,$$

and

$$I^\infty = \inf\{\mathcal{E}^\infty(u) : u \in W^{1,2}(\Omega),\ u \neq 0,\ J^\infty(u) = 0\}. \tag{5}$$

We remark that the inequalities $0 < I \le I^\infty$ always hold, since $\mathcal{E}^\infty$ and J^∞ are invariant with respect to translations of the form $(0, \dots, 0, \eta_1, \dots, \eta_m)$.

Theorem 1. *Under the conditions* 1°–4°, *every minimizing sequence for* (4) *is relatively compact up to translations of the form* $(0, \dots, 0, \eta_1, \dots, \eta_m)$.

Every minimizing sequence for (4) *is relatively compact if and only if* $I < I^\infty$. *In this case there exists a minimum of* (4) *which is a nontrivial solution of* (1)–(3).

Proof. The proof is based on the concentration–compactness principle of P. L. Lions [1]. For a minimizing sequence u_k for (4) we define $\rho_k = |\nabla u_k|^2 + |u_k|^2$. We see from 3° that for every $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ such that the estimates

$$f(x,u)u \le \varepsilon|u|^2 + C_\varepsilon|u|^p, \quad \varphi(\xi,u)u \le \varepsilon|u|^2 + C_\varepsilon|u|^q$$

hold for some $2 < p < 2n/(n-2)$, $2 < q < 2(n-1)/(n-2)$. From the constraint $J(u_k) = 0$ we deduce that

$$\begin{aligned}
\|u_k\|_A^2 &= \int_\Omega f(x,u_k)u_k\,\mathrm{d}x + \int_\Gamma \varphi(\xi,u_k)u_k\,\mathrm{d}\Gamma \\
&\le \varepsilon\big(\|u_k\|_{L^2}^2 + \|u_k\|_{L^2(\Gamma)}^2\big) + C_\varepsilon\big(\|u_k\|_{L^p}^p + \|u_k\|_{L^q(\Gamma)}^q\big) \\
&\le \varepsilon C\|u_k\|_A^2 + C_\varepsilon C\big(\|u_k\|_A^p + \|u_k\|_A^q\big),
\end{aligned}$$

which shows that

$$\|u_k\|_A \ge \text{const.} > 0. \tag{6}$$

Therefore we can assume that (at least for a subsequence of ρ_k) $\lambda_k = \int_\Omega \rho_k(x)\,\mathrm{d}x \to \lambda > 0$ as $k \to \infty$. Now Lemma I.1 from [1, I] states that there is a subsequence (denoted again by ρ_k) satisfying one of the following conditions:

(i) *Vanishing.* $\lim_{k\to\infty} \sup_{y\in\Omega} \int_{y+B_R\cap\Omega} \rho_k(x)\,\mathrm{d}x = 0$ for every $R<\infty$.

(ii) *Dichotomy.* There exists an $\alpha \in (0,\lambda)$ such that for every $\varepsilon>0$ there is a $k_0 \geq 1$ and positive functions $\rho_k^1, \rho_k^2 \in L^1(\Omega)$ which satisfy for $k \geq k_0$

$$\left\|\rho_k - (\rho_k^1+\rho_k^2)\right\|_{L^1} \leq \varepsilon,\ \left|\int_\Omega \rho_k^1(x)\,\mathrm{d}x - \alpha\right| \leq \varepsilon,\ \left|\int_\Omega \rho_k^2(x)\,\mathrm{d}x - (\lambda-\alpha)\right| \leq \varepsilon$$

and $\operatorname{dist}(\operatorname{supp}\rho_k^1, \operatorname{supp}\rho_k^2) \to \infty$ as $k\to\infty$.

(iii) *Compactness.* There exists a sequence $(y_k)_k$ in Ω such that for every $\varepsilon>0$ there is an $R<\infty$ satisfying $\displaystyle\int_{y_k+B_R\cap\Omega} \rho_k(x)\,\mathrm{d}x \geq \lambda-\varepsilon$.

First we show that the case (i) cannot occur. By integration we get from 2° the estimates

$$0 \leq F(x,t) \leq \frac{\theta}{1+\theta} f(x,t)t,\ 0 \leq \Phi(\xi,t) \leq \frac{\theta}{1+\theta}\varphi(\xi,t)t, \quad t\in\mathbb{R} \tag{7}$$

and from the constraint $J(u_k)=0$ we deduce that

$$\begin{aligned} C \geq \mathcal{E}(u_k) &= \int_\Omega \frac{1}{2} f(x,u_k)u_k - F(x,u_k)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}\varphi(\xi,u_k)u_k - \Phi(\xi,u_k)\,\mathrm{d}\Gamma \\ &\geq \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right)\left(\int_\Omega f(x,u_k)u_k\,\mathrm{d}x + \int_\Gamma \varphi(\xi,u_k)u_k\,\mathrm{d}\Gamma\right) \\ &= \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right)\|u_k\|_A^2, \end{aligned}$$

which shows that u_k is bounded in $W^{1,2}(\Omega)$.

Now if $\lim_{k\to\infty}\sup_{y\in\Omega}\int_{y+B_R\cap\Omega}\rho_k(x)\,\mathrm{d}x = 0$, we conclude from Lemma I.1 in [1, II] that $u_k \to 0$ in $L^p(\Omega)$, $2<p<2n/(n-2)$. The same argument then shows that $u_k\to 0$ in $L^q(\Gamma)$, $2<q<2(n-1)/(n-2)$ (in the sense of traces). Then we obtain $\int_\Omega f(x,u_k)u_k\,\mathrm{d}x \to 0$ and $\int_\Gamma \varphi(\xi,u_k)u_k\,\mathrm{d}\Gamma \to 0$, and from $J(u_k)=0$ we finally get $\|u_k\|_A \to 0$, which contradicts (6).

Next we show that the case (ii) is also impossible. For $\alpha\in\mathbb{R}$ define

$$I_\alpha = \inf\{\mathcal{E}(u) - \tfrac{1}{2}J(u) \colon J(u)=\alpha\}$$

(and I_α^∞ correspondingly). From (7) we infer that I_α, $I_\alpha^\infty \geq 0$ for every $\alpha\in\mathbb{R}$ and $I<I_\alpha$, $I^\infty < I_\alpha^\infty$ if $\alpha<0$ (see [1, II]). From (ii) we obtain sequences u_k^1, u_k^2 satisfying $\|u_k-(u_k^1+u_k^2)\|_{W^{1,2}} \leq \varepsilon$ for large k and

$\operatorname{dist}(\operatorname{supp} u_k^1, \operatorname{supp} u_k^2) \to \infty$. Since $u \mapsto f(x,u)u$ and $u \mapsto \varphi(\xi,u)u$ define continuous Nemytskii operators from L^p to L^1 (cf. [5]), we obtain

$$\int_\Omega \left| f(x,u_k)u_k - \left(f(x,u_k^1)u_k^1 + f(x,u_k^2)u_k^2 \right) \right| \mathrm{d}x \\ + \int_\Gamma \left| \varphi(\xi,u_k)u_k - \left(\varphi(\xi,u_k^1)u_k^1 + \varphi(\xi,u_k^2)u_k^2 \right) \right| \mathrm{d}\Gamma \to 0.$$

Consequently, at least for a subsequence, we get $J(u_k^1) \to \beta_1$, $J(u_k^2) \to \beta_2$ with $\beta_1 = -\beta_2$ and

$$\begin{aligned} I &= \lim \left(\mathcal{E}(u_k) - \tfrac{1}{2} J(u_k) \right) \\ &\geq \liminf \left(\mathcal{E}(u_k^1) - \tfrac{1}{2} J(u_k^1) \right) + \liminf \left(\mathcal{E}(u_k^2) - \tfrac{1}{2} J(u_k^2) \right). \end{aligned}$$

Without loss of generality we may assume that $\operatorname{supp} u_k^2$ tends to ∞, while $\operatorname{supp} u_k^1$ remains bounded. Then we conclude that $I \geq I_{\beta_1} + I_{\beta_2}^\infty$. The following three cases should be considered:

if $\beta_1 = \beta_2 = 0$, then $I \geq I + I^\infty > I$, since $I^\infty > 0$;

if $\beta_1 < 0$, then $I < I_{\beta_1}$ and $I \geq I_{\beta_1} + I_{\beta_2}^\infty \geq I_{\beta_1}$;

if $\beta_1 > 0$, then $I \geq I_{\beta_1} + I_{\beta_2}^\infty \geq I_{\beta_2}^\infty > I^\infty$.

In all three cases we arrive at a contradiction.

This shows that the case (iii) holds. If the sequence $(y_k)_k$ is bounded, we may assume that $y_k = 0$ and then, from the embedding and trace theorem, there is a subsequence satisfying $u_k \to u$ in $L^p(\Omega)$, $2 < p < 2n/(n-2)$, and in $L^q(\Gamma)$, $2 < q < 2(n-1)/(n-2)$, respectively. Therefore we have $\int_\Omega f(x,u_k)u_k \,\mathrm{d}x \to \int_\Omega f(x,u)u \,\mathrm{d}x$, $\int_\Omega F(x,u_k)\,\mathrm{d}x \to \int_\Omega F(x,u)\,\mathrm{d}x$, $\int_\Gamma \varphi(\xi,u_k)u_k \,\mathrm{d}\Gamma \to \int_\Gamma \varphi(\xi,u)u \,\mathrm{d}\Gamma$, and $\int_\Gamma \Phi(\xi,u_k)\,\mathrm{d}\Gamma \to \int_\Gamma \Phi(\xi,u)\,\mathrm{d}\Gamma$. Since $J(u_k) = 0$, we get $J(u) \leq 0$, $\mathcal{E}(u) \leq \lim \mathcal{E}(u_k) = I$ and $\mathcal{E}(u) = I + \frac{1}{2}J(u)$. If $J(u) = \alpha < 0$, then $\mathcal{E}(u) - \frac{1}{2}J(u) \geq I_\alpha > I$, which is a contradiction. Therefore $J(u) = 0$, $\mathcal{E}(u) = I$ and $u_k \to u$ strongly in $W^{1,2}(\Omega)$.

Now suppose that $y_k \to \infty$. We may assume that $y_k = (0, \ldots, 0, \eta_{k,1}, \ldots, \eta_{k,m})$. Then for the sequence $v_k = u_k(\cdot + y_k)$ we infer as above that $v_k \to v$ in $L^p(\Omega)$, $L^q(\Gamma)$, $J^\infty(v) = 0$, $\mathcal{E}^\infty(v) = I^\infty = I$, and $v_k \to v$ strongly in $W^{1,2}(\Omega)$. If $I < I^\infty$, then the case $y_k \to \infty$ is not possible. On the other hand, if $I = I^\infty$, then we can construct a minimizing sequence which is not compact. $\square$

From Theorem 1 we get immediately the following

Corollary 2. *In the case of constant coefficients* $a_{ij}(x) = \widehat{a}_{ij}$, $b(x) = \widehat{b}$, $f(x,t) = \widehat{f}(t)$, $\varphi(\xi,t) = \widehat{\varphi}(t)$, *there exists a minimum of* (5) *which is a solution of* (1)–(3).

If the coefficients and nonlinearities satisfy the inequalities

$$a_{ij}(x) \le \widehat{a}_{ij}, \quad b(x) \le \widehat{b}, \quad f(x,t)t \ge \widehat{f}(t)t, \quad \varphi(\xi,t)t \ge \widehat{\varphi}(t)t \tag{8}$$

then we have the following corollary.

Corollary 3. *Let* (8) *be satisfied, and assume that at least one of the first three inequalities in* (8) *is strict. If in addition*

$$\widehat{F}(t) - F(x,t) \le \tfrac{1}{2}\big(\widehat{f}(t)t - f(x,t)t\big) \quad \textit{and} \quad \widehat{\Phi}(t) - \Phi(\xi,t) \le \tfrac{1}{2}\big(\widehat{\varphi}(t)t - \varphi(\xi,t)t\big),$$

then $I < I^\infty$, i.e., every minimizing sequence of (4) *is relatively compact and there exists a nontrivial solution of* (1)–(3).

Proof. Let u_0 be a minimizer for I^∞, $J^\infty(u_0) = 0$, $\mathcal{E}^\infty(u_0) = I^\infty$. From the strictness of one of the inequalities in (8) we may assume that $J(u_0) = \alpha < 0$. Then

$$\begin{aligned}
\alpha &= \int_\Omega \Big(\sum a_{ij}(x)\partial_i u_0 \partial_j u_0 + b(x)u_0^2\Big)\,\mathrm{d}x - \int_\Omega \Big(\sum \widehat{a}_{ij}\partial_i u_0 \partial_j u_0 + \widehat{b}u_0^2\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega \big(\widehat{f}(u_0)u_0 - f(x,u_0)u_0\big)\,\mathrm{d}x + \int_\Gamma \big(\widehat{\varphi}(u_0)u_0 - \varphi(\xi,u_0)u_0\big)\,d\Gamma \\
&\ge \int_\Omega \Big(\sum a_{ij}(x)\partial_i u_0 \partial_j u_0 + b(x)u_0^2\Big)\,\mathrm{d}x - \int_\Omega \Big(\sum \widehat{a}_{ij}\partial_i u_0 \partial_j u_0 + \widehat{b}u_0^2\Big)\,\mathrm{d}x \\
&\quad + 2\int_\Omega \big(\widehat{F}(u_0) - F(x,u_0)\big)\,\mathrm{d}x + 2\int_\Gamma \big(\widehat{\Phi}(u_0) - \Phi(\xi,u_0)\big)\,\mathrm{d}\Gamma,
\end{aligned}$$

and we obtain

$$\begin{aligned}
I_\alpha &\le \mathcal{E}(u_0) - \frac{1}{2}J(u_0) = \mathcal{E}(u_0) - \frac{\alpha}{2} \\
&\le \frac{1}{2}\int_\Omega \Big(\sum \widehat{a}_{ij}\partial_i u_0 \partial_j u_0 + \widehat{b}u_0^2\Big)\,\mathrm{d}x - \int_\Omega \widehat{F}(u_0)\,\mathrm{d}x - \int_\Gamma \widehat{\Phi}(u_0)\,\mathrm{d}\Gamma = I^\infty.
\end{aligned}$$

Since $I < I_\alpha$, the proof is complete. □

References

[1] P. L. Lions, *The concentration-compactness method in the calculus of variations. The locally compact case. I, II*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 109–145, 223–283.

[2] Z. Nehari, *Characteristic values associated with a class of nonlinear second order differential equations*, Acta Math. **105** (1961), 141–175.

[3] A. A. Pankov, *Semilinear elliptic equations in $\mathbb{R}^n$ with nonstabilizing coefficients*, Ukrainian Math. J. **41** (1989), no. 9, 1075–1078; transl. from Ukraïn. Mat. Zh. **41** (1987), no. 9, 1247–1251.

[4] K. Pflüger, *Semilinear elliptic problems with nonlinear boundary conditions in unbounded domains*, Z. Anal. Anwendungen **14** (1995), 829–851.

[5] M. M. Vainberg, *Variational methods for the study of nonlinear operators*, Holden Day, San Francisco, 1964.