BOUNDARY VALUE PROBLEMS WITH COMPOSITION-TYPE NON-LINEARITIES IN GENERALIZED SOBOLEV SPACES

Thomas Runst

Introduction

The purpose of this paper is to describe some recent results of a joint work with J. Johnsen. We deal with the L_p-theory of certain semi-linear elliptic boundary value problems. For the sake of simplicity, we consider

$$\text{(P)} \qquad -\Delta u(x) + g[u(x)] = f(x) \text{ in } \Omega, \quad u_{\big|\partial\Omega} = 0.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain with boundary $\partial\Omega$ for some $n \geq 2$. The function $g(\cdot)$ is required to be well-behaved, i.e., we assume that g is a real-valued smooth function with $g^{(j)} \in L_\infty(\mathbb{R})$, $j \in \mathbb{N}$, and u and f belong to real-valued function spaces.

The aim is to study some effects caused by the non-linearity $g(u)$. We remark that the problem (P) is only a representative of a wider class of non-linear perturbations of elliptic boundary value problems in the Boutet de Monvel calculus. Hence operators of a pseudo-differential kind could also be considered in (P). Concerning the L_p-theory, we work here with generalized Sobolev spaces of Bessel-potential type H^s_p. We emphasize that one can extend the following results to more general function spaces of Besov type, $B^s_{p,q}$, and of Triebel-Lizorkin type, $F^s_{p,q}$, with $s \in \mathbb{R}$ and $0 < p, q \leq \infty$.

Main results and notation

Let $1 < p < \infty$ and $s > 1/p$, then

$$H^s_{p,0}(\Omega) = \{u \in H^s_p(\Omega) \colon u_{\big|\partial\Omega} = 0\}$$

denotes as usual the subspace of the Bessel-potential space $H^s_p(\Omega)$ which consists of all functions with vanishing trace on $\partial\Omega$. Then the following is known (see e.g. T. Runst, W. Sickel [RS]):

• $-\Delta$ yields an isomorphic map

$$\text{from } H^s_{p,0}(\Omega) \text{ onto } H^{s-2}_p(\Omega) \quad \text{if } s > 1/p.$$

(For $(s,p) = (2,2)$ this is just the usual H^2_0-realization (in L_2).)
• Mapping properties (boundedness, continuity) of composition (or Nemytskii) operators $g(\cdot)$ of the type

$$u(x) \to g[u(x)].$$

• Continuous embeddings

$$H^s_p(\Omega) \hookrightarrow H^t_r(\Omega) \quad \text{if } t < s \text{ and } s - \frac{n}{p} \geq t - \frac{n}{r},$$
$$H^s_p(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{if } s > \frac{n}{p}.$$

• Compact embeddings

$$H^{s+\delta}(\Omega) \hookrightarrow\hookrightarrow H^s_p(\Omega) \quad \text{for all } \delta > 0.$$

Once the class of admissible functions g is given, it is natural to ask for the parameters (s,p) such that $-\Delta u$ and $g(u)$ both make sense on $H^s_p(\Omega)$, and such that $g(\cdot)$ respects the mapping properties of $-\Delta$ on $H^s_p(\Omega)$. In this sense, our point of view is quite different from the one connected with the critical Sobolev exponents an Pokhozhaev identities, where non-linearities suitable for a fixed function space are sought-after.

Let $f \in H^{s-2}_p(\Omega)$. Then $u \in H^s_{p,0}(\Omega)$ is a solution of (P) if and only if $u \in H^s_{p,0}(\Omega)$ is a solution of the fixed point problem $u = -\Delta^{-1}[f - g(u)]$. We solve this equivalent problem by an application of the Leray-Schauder fixed point theorem. It means that the non-linear operator has to be completely continuous, i.e. continuous and compact. In this sense it is natural to introduce the set $\mathbb{D}$ given by

$$\begin{aligned}(1)\qquad \mathbb{D} = \{(s,p)\colon u_{\big|\partial\Omega} & \text{ makes sense for all } u \in H^s_p(\Omega) \text{ and} \\ & \exists \varepsilon > 0 \text{ such that } g(\cdot)\colon H^s_p(\Omega) \to H^{s-2+\varepsilon}_p(\Omega) \text{ is continuous}\},\end{aligned}$$

which provide a domain of parameters for the semi-linear operator in (P) in the sense that it could be considered on $H^s_{p,0}(\Omega)$ for each $(s,p) \in \mathbb{D}$ —even with a good control over $g(\cdot)$.

Then we can prove the following results:

(I) There exists a maximal non-convex region $\mathbb{D}$ of parameters (s,p) such that $u \mapsto g(u)$ is better behaved on $H^s_p(\Omega)$ than the Laplacian.

(II) There exists an inverse regularity theory.

(III) There exists a solvability theory for (P), also for non-invertible linear part.

We are looking for solutions u of (P) belonging to $H^s_p(\Omega)$. The earlier treatments mainly consider either the case $s > n/p$ (bounded functions) or weak solutions for $s = 1$. To our best knowledge the case with $1 < s < n/p$ has not been considered yet. Here this case is investigated among the others and in fact we set out to work with minimal assumptions on (s,p). That is, with the exception of the borderline case, we assume only that the trace operator makes sense on $H^s_p(\Omega)$ and that $g(\cdot)$ is better behaved than the Laplacian on these spaces.

Description of the domain $\mathbb{D}$

We consider the non-linear superposition operator $u \mapsto g(u)$ in real-valued spaces of type $H^s_p(\Omega)$. In T. Runst, W. Sickel [RS] the following mapping property was proved:

The non-linear operator $g(\cdot)\colon H^s_p(\Omega) \to H^\sigma_p(\Omega)$ is bounded when

$$\sigma(s,p) = \begin{cases} s & \text{for } 0 < s < 1 \text{ or } s > n/p, \\ s - \varepsilon & \text{for } s = n/p \text{ or } s = 1, \\ \dfrac{\frac{n}{p}}{\frac{n}{p} - s + 1} & \text{otherwise.} \end{cases} \tag{2}$$

Here $\varepsilon > 0$ can be arbitrarily small. Applying (2) we can measure the loss of smoothness $d(s)$ given by

$$d(s) = s - \sigma(s,p) = \frac{(s-1)(\frac{n}{p} - s)}{\frac{n}{p} - s + 1} \tag{3}$$

when $1 < s < n/p$, otherwise we have either $d(s) = 0$ or $d(s) = \varepsilon$. The function $d(\cdot)$ should be understood in the following sense:

$$\forall \varepsilon > 0 \ \exists u_\varepsilon \in H^s_p(\Omega) \ \text{ such that } \ \sin u_\varepsilon \notin H^{\sigma+\varepsilon}_p(\Omega).$$

It is not hard to check that $d(s)$ attains its maximum for fixed (n,p) at

$$s_0 = \frac{n}{p} - \sqrt{\frac{n}{p}} + 1 \quad \text{with} \quad d(s_0) = \left(\sqrt{\frac{n}{p}} - 1\right)^2.$$

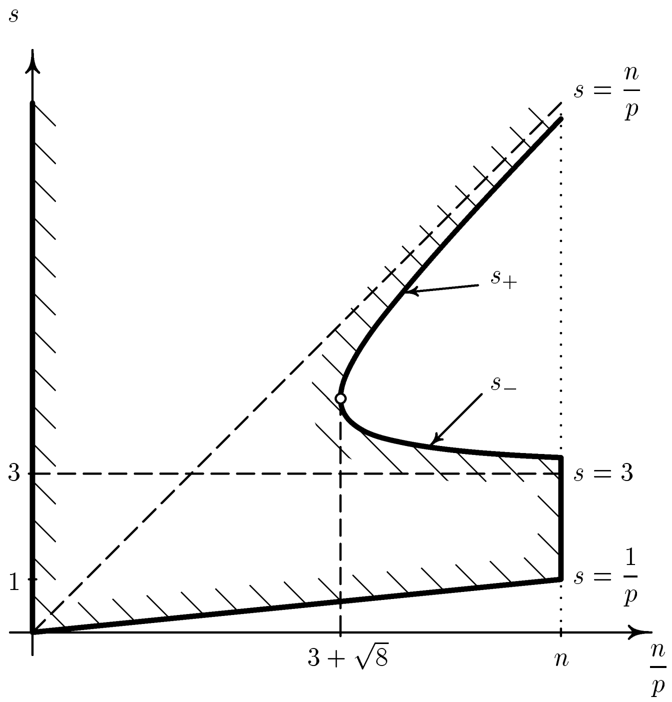
Note that $d(s_0) \to \infty$ as $n/p \to \infty$. This shows that the loss of smoothness may be greater than 2, i.e., $g(\cdot)$ is worse than $-\Delta$ on $H^s_p(\Omega)$ if $n \geq 6$ (we have $d(s_0) > 2$ if $n/p > 3 + \sqrt{8}$). As for the identity

$$\sigma(s,p) = s - 2 \tag{4}$$

(or for the level curve for the value $d(s) = 2$) one finds that

$$s_\pm = \frac{1}{2}\left(\frac{n}{p} + 3 \pm \sqrt{\left(\frac{n}{p} - 3\right)^2 - 8}\right) \quad \text{if} \quad \frac{n}{p} \geq 3 + \sqrt{8}. \tag{5}$$

In other words: the condition (4) or (5) determines a borderline of a region where the loss of smoothness equals or exceeds 2. Generally speaking this is correct, if $s_- < s < s_+$ and $n/p > 3 + \sqrt{8}$; then $u \mapsto \sin u$, for example, cannot map into $H^{s-2+\varepsilon}_p(\Omega)$ for any $\varepsilon > 0$.



The identity (4) describes a hyperbola in the $(n/p, s)$-halfplane, that lies entirely in the area with $s < n/p$, i.e., one gets the special shape of $\mathbb{D}$ given by (1) when one considers also unbounded solutions u of (P). Furthermore, one can show that $s = 3$ and $s = n/p$ are asymptotes of the hyperbola. Let $n \geq 6$ be fixed. Then the non-convex set $\mathbb{D}$ is depicted in the following figure. For every $n \geq 6$ there are certain $p > 1$ such that $n/p \geq 3 + \sqrt{8}$, i.e., the condition (5) is relevant, and thus for such dimensions restrictions occur also in the spaces W_p^s and H_p^s. Note that for $n \geq 12$ even the theory within the classical Sobolev spaces W_p^s is affected. The curved arc is truly a non-linear effect: without the non-linear function g the problem (P) makes sense if $s > 1/p$. As remarked before, the function $g(t) = \sin t$ shows that the definition of $\mathbb{D}$ is optimal. We believe that the non-convexity and the specific form of the domains $\mathbb{D}$ are new.

Inverse regularity

We shall prove inverse regularity results. The precise claim is the following: If $u \in H^s_{p,0}(\Omega)$, where $(s,p) \in \mathbb{D}$, is a given solution of (P) with $f \in H_r^{t-2}(\Omega)$ for some $(t,r) \in \mathbb{D}$, then $u \in H^t_{r,0}(\Omega)$ holds also, i.e., for $t > s$ one gets an improvement of smoothness. Briefly stated our regularity results are obtained from a bootstrap argument involving mapping properties of superposition and differential operators.

Rewriting (P) we get

$$u = -\Delta^{-1} f + \Delta^{-1}[g(u)] \tag{6}$$

for $u \in H^s_{p,0}(\Omega)$ and $f \in H_r^{t-2}(\Omega)$. Now (6) and the mapping properties show that

$$u \in H^t_{r,0}(\Omega) + H^{\sigma+2}_{p,0}(\Omega), \quad \sigma \text{ given by (2).} \tag{7}$$

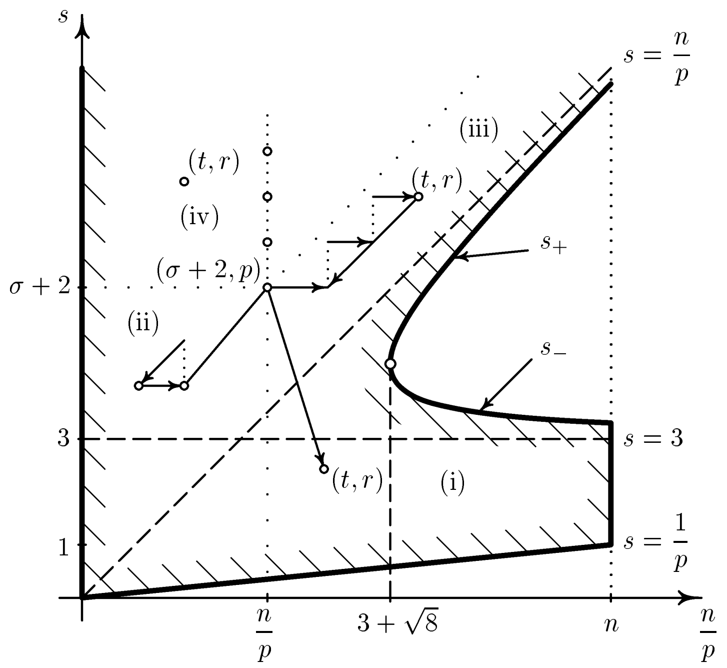
By the definition of $\mathbb{D}$ we have

$$\sigma(s,p) + 2 > s \text{ for all } (s,p) \in \mathbb{D} \text{ (even } \sigma = s \text{ unless } 1 \leq s \leq n/p). \tag{8}$$

This means that the non-linear operator $g(\cdot)$ also respects the inverse regularity of $-\Delta$ on every space $H^s_{p,0}(\Omega)$ for all $(s,p) \in \mathbb{D}$. To verify our claim we consider the cases

$$\text{(a)} \quad \frac{n}{p} < 3 + \sqrt{8} \qquad \text{and} \qquad \text{(b)} \quad \frac{n}{p} \geq 3 + \sqrt{8} \quad \text{for } n \geq 6.$$

Case (a): Now the procedure is far easier to sketch with a diagram than with words, so we refer to the next figure which displays the location of $H_p^{\sigma+2}(\Omega)$ and four examples of $H_r^t(\Omega)$ corresponding to the subdivision with respect to the embeddings ($t \lesseqgtr \sigma+2$ and $t - n/r \lesseqgtr \sigma + 2 - n/p$).



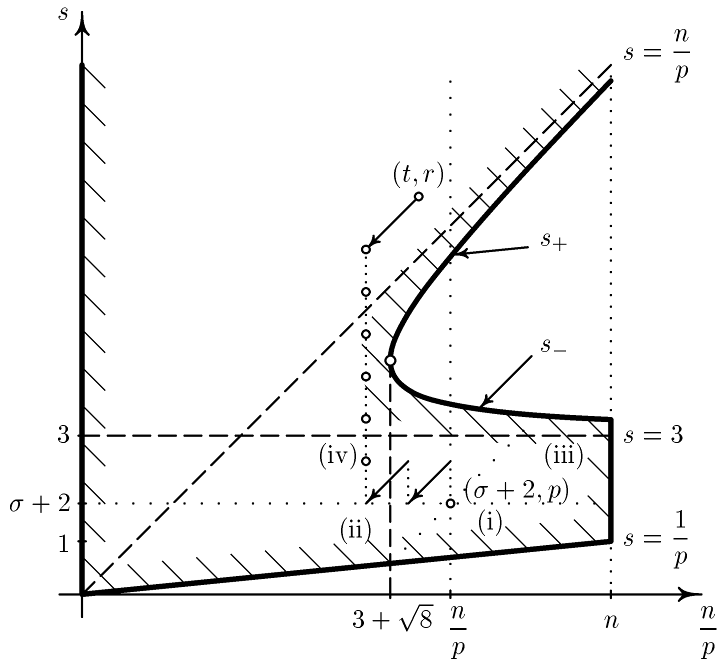
Case (a): $n/p < 3 + \sqrt{8}$

Here the arrows in full line indicate embeddings $H_p^{\sigma+2}(\Omega) \hookrightarrow H_{p_1}^{s_1}(\Omega)$, etc. The dotted lines indicate the improved knowledge of $\Delta^{-1}[g(u)]$ given by (8), etc. Note that one of the four cases is trivial, since for (i) we have

$$H_p^{\sigma+2}(\Omega) \hookrightarrow H_r^t(\Omega). \tag{9}$$

The other cases can be reduced in finitely many steps to (i) with either the "sawtooth" or the "staircase" manœuvres, see the above figure.

Case (b): Let $s > s_+$. The difference from (a) is only given when (i) holds (it is possible to cross $\partial\mathbb{D}$). But this case is trivial, as follows from the embedding (9). The situation is a little more complicated if $s < s_-$. Roughly speaking, it is clear that the cases (i), (ii) and (iii) from the case (a) can be carried out. Only the case (iv) is of interest. Because of the non-convexity of $\mathbb{D}$ the iterations may very well lie outside of $\mathbb{D}$, i.e., we will possibly have to go "around the corner", see the next figure. Altogether this leads to our result.



Case (b): $n/p \geq 3 + \sqrt{8}$

Solvability theory

The existence of solutions is derived from the Leray-Schauder fixed point theorem. When the associated linear problem is non-invertible, we establish Landesman-Lazer conditions. At first we consider the *invertible case*:

Theorem 1. *For all* $(s, p) \in \mathbb{D}$ *there exists a solution* $u \in H^s_{p,0}(\Omega)$ *of* (P) *if* $f \in H^{s-2}_p(\Omega)$.

Sketch. Applying mapping properties of g and $-\Delta$ and compactness arguments, one derives from Leray-Schauder fixed point theorem:

If $s = 1/p + \varepsilon$, $\varepsilon > 0$ is small enough, $(s,p) \in \mathbb{D}$, then there exists a solution $u \in H^s_{p,0}(\Omega)$ of (P) for each $f \in H^{s-2}_p(\Omega)$. It means that s is so small that $L_\infty(\Omega) \hookrightarrow H^{s-2}_p(\Omega)$. In this case it is easy to show a priori L_∞-estimates (g is bounded), and the Leray-Schauder theory is applicable. Then bootstrap arguments from (II) finish the proof. □

Now we consider the following semi-linear elliptic boundary value problem at resonance (*non-invertible case*), where $\lambda > 0$ is an eigenvalue of $-\Delta$ under Dirichlét condition:

$$-\Delta u - \lambda u + g(u) = f \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{Q}$$

The solvability of problems of type (Q) is more delicate. To prepare for this it is noted that the Fredholm properties of regular elliptic boundary value operators depend neither on the parameters $(s,p) \in \mathbb{D}$ nor on whether the spaces $H^s_p, B^s_{p,q}$ or $F^s_{p,q}$ are considered. Furthermore, the corresponding kernel is finite-dimensional and consists of smooth functions, we refer J. Franke, T. Runst [FR], and J. Johnsen [Jo]. Thus we are able to apply the method of bifurcation systems in the sense of Ljapunov-Schmidt, i.e., we split the original equation (P) by means of projections. At first one considers the case $L_\infty(\Omega) \hookrightarrow H^{s-2}_p(\Omega)$. We use the estimates (derived in the proof of Theorem 1) and the Landesman-Lazer condition for solvability in the sense of E. M. Landesman, A. C. Lazer [LL], and J. Nečas [Ne], in order to obtain a priori estimates of the solution set. Thus we can apply the Leray-Schauder theory. Finally, bootstrap arguments complete the proof of the next theorem.

Theorem 2. *Let g be such that*

$$g(\pm\infty) = \lim_{t\to\pm\infty} g(t) \tag{10}$$

both exist, and let $f \in H^{s-2}_p(\Omega)$, $(s,p) \in \mathbb{D}$. Moreover, let one of the following conditions be fulfilled:

(a) *the inequality*

$$\langle f, w\rangle < g(+\infty)\int_{\Omega_+} w + g(-\infty)\int_{\Omega_-} w \tag{11}$$

holds for all non-trivial $w \in \ker(-\Delta - \lambda)$, where $\Omega_+ = \{x \in \Omega : w(x) > 0\}$ and $\Omega_- = \Omega \setminus \Omega_+$.

(b) $g(\pm\infty) = 0$, *both* $G(\pm\infty) = \lim_{t\to\pm\infty} tg(t)$ *exist*, $\langle f, w\rangle = 0$ *and*

$$0 < G(+\infty)|\Omega_+| + G(-\infty)|\Omega_-| \tag{12}$$

holds for all non-trivial $w \in \ker(-\Delta - \lambda)$. *Then* (Q) *has at least one solution* $u \in H^s_p(\Omega)$.

Together with the above investigations we would finally get a result of the so-called Fredholm alternative for non-linear operators. These assertions extend the classical ones to the whole non-convex domain $\mathbb{D}$. Furthermore, we can also give a description of $\mathbb{D}$ with respect to the two scales of function spaces $B^s_{p,q}$ and $F^s_{p,q}$ (even $p, q \le 1$ is allowed).

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