# ON ANALYTIC CONTINUATION OF POWER SERIES WITH THE CONDITION ON THEIR GELFOND-LEONTEV DERIVATIVES

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## 1.

For $R \in [0, +\infty]$ we denote by $A(R)$ the class of power series $f(z) = \sum_{k=0}^{\infty} f_k z^k$ having radius of convergence $R$, and we say that $f \in A^+(R)$ if $f \in A(R)$ and $f_k > 0$ for all $k \in \mathbb{Z}_+$. For $f \in A(0)$ and $l \in A^+(0)$ we put

$$D_l^n f(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n}} f_{k+n} z^k.$$

The operator $D_l^n$ is called the Gelfond-Leontev operator. If $l(z) \equiv e^z$, then $D_l^n = \dfrac{d^n}{dz^n}$ is the usual differential operator. We can assume that $l_0 = 1$. We put

$$\varkappa_k = \frac{l_k^2}{l_{k+1} l_{k-1}}, \quad \eta_k = \frac{1}{k} \ln \frac{1}{l_k}, \quad \omega_k = \eta_{k+1} - \eta_k \quad (k \in \mathbb{N}).$$

It is easy to show [1] that if $\varkappa_k \searrow \varkappa$, then $\omega_k \searrow \omega \geq 0$ as $k \to \infty$.

Let $\mathfrak{N}$ be a class of increasing sequences $(n_p)$ of nonnegative integers, $n_0 = 0$, let $\Lambda$ be the class of all positive sequences, and let $\Lambda^* = \{\lambda \in \Lambda\colon \ln \lambda_k \leq a$ for every $k \in \mathbb{N}$ and some $a > 0\}$. We say that $f \in A_\lambda(R)$ if $f \in A(R)$ and $|f_k| \leq \lambda_k |f_1|$ for all $k \in \mathbb{N}$. In [1] the following theorem is proved.

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**Theorem 0.** *Suppose that $l \in A^+(\infty)$ and $\varkappa_k \searrow \varkappa$ as $k \to \infty$. In order that for every $(n_p) \in \mathfrak{N}$, $\lambda \in \Lambda^*$, and $f \in A(0)$ the inclusion $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$ implies the inclusion $f \in A(\infty)$, it is necessary and sufficient that $\omega > 0$.*

From the condition $l \in A^+(\infty)$ and $\varkappa_k \searrow \varkappa$ $(k \to \infty)$ it follows that $\ln(1/l_k) = O(k^2)$, $k \to \infty$, and if $\omega > 0$, then $\ln(1/l_k) \geq \omega k^2 + O(k)$, $k \to \infty$. Hence, if these three conditions hold, then $\ln(1/l_k) \asymp k^2$.

The question arises whether the sufficiency in Theorem 0 remains valid if $\ln(1/l_k)$ increases faster than $k^2$. It is clear that in this case the sequence $(\varkappa_k)$ is not nonincreasing. On the other hand, if the sequence $(\varkappa_k)$ is oscillating, then the statement of Theorem 0 can be obtained in general [1]. Therefore, we examine the case when the sequence $(\varkappa_k)$ is decreasing. It is possible to show that in this case it follows from the condition $l \in A^+(\infty)$ that $\varkappa_k > 1$ for all $k \geq k_0$. For simplicity we assume that $\varkappa_k > 1$ for all $k \geq 1$.

We prove the following assertion.

**Theorem 1.** *Suppose that $l \in A^+(\infty)$ and $1 < \varkappa_k \nearrow \varkappa \geq \infty$ as $k \to \infty$. Then for every $(n_p) \in \mathfrak{N}$, $\lambda \in \Lambda^*$, and $f \in A(0)$ the inclusion $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$ implies the inclusion $f \in A(\infty)$.*

## 2.

For the proof of Theorem 1 we need some lemmas.

**Lemma 1** ([1]). *Let $\lambda \in \Lambda$, $(n_p) \in \mathfrak{N}$, and $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$. Then for every $p \in \mathbb{N}$ and $k = 1, 2, \ldots, n_{p+1} - n_p + 1$, the following inequality holds*

$$|f_{n_p+k}| \leq |f_1| l_1^p l_{n_p+k} \frac{\lambda_k}{l_k} \prod_{j=1}^{p} \frac{\lambda_{n_j - n_{j-1}+1}}{l_{n_j - n_{j-1}+1}}.$$

**Lemma 2.** *If $1 < \varkappa_k \nearrow \varkappa \leq \infty$ as $k \to \infty$, then $\eta_k \nearrow \infty$ and $0 < \omega_k \nearrow \omega \leq \infty$ as $k \to \infty$.*

Indeed, $l_k / l_{k+1} = \frac{1}{l_1} \prod_{j=1}^{k} \varkappa_j$ and, therefore,

$$\ln \frac{1}{l_k} = \sum_{j=1}^{k-1} (k-j) \ln \varkappa_j + k \ln \frac{1}{l_1}, \quad k \geq 2.$$

Hence

$$\eta_k = \sum_{j=1}^{k-1} (1 - \frac{j}{k}) \ln \varkappa_j + \ln \frac{1}{l_1}, \quad k \geq 2,$$

and

$$\omega_k = \frac{1}{k(k+1)} \sum_{j=1}^{k} j \varkappa_j, \quad k \geq 1,$$

whence $\eta_k \nearrow \infty$ $(k \to \infty)$ provided that $l \in A^+(\infty)$. Now

$$\begin{aligned} \omega_{k+1} - \omega_k &= \frac{1}{k+2} \Big( \ln \varkappa_{k+1} - \frac{2}{k(k+1)} \sum_{j=1} kj \ln \varkappa_j \Big) \\ &\geq \frac{1}{k+2} \ln \varkappa_{k+1} \Big( 1 - \frac{2}{k(k+1)} \sum_{j=1}^{k} j \Big) = 0, \end{aligned}$$

whence $\omega_k \nearrow \omega \leq \infty$ $(k \to \infty)$. Recall that $\omega = \ln \varkappa$.

**Lemma 3.** *Let*

$$\gamma_k = \gamma_{k,n} = \frac{1}{n+k} \Big\{ \ln \frac{1}{l_{n+k}} - \ln \frac{1}{l_k} + k \ln \frac{1}{l_1} \Big\}, \quad n \in \mathbb{N}.$$

*If* $1 < \varkappa_k \nearrow \varkappa \leq \infty$ $(k \to \infty)$, *then* $\gamma_k \nearrow \gamma \leq \infty$ $(k \to \infty)$.

Indeed, according to Lemma 2 we have

$$\begin{aligned} \gamma_{k+1} - \gamma_k &= \omega_{n+k} + \frac{k}{n+k} \eta_k - \frac{k+1}{n+k+1} \eta_{k+1} + \frac{n}{(n+k)(n+k+1)} \ln \frac{1}{l_1} \\ &= \omega_{n+k} - \omega_k + \frac{n}{(n+k)(n+k+1)} \Big\{ (n+k)\omega_k - \sum_{j=1}^{k} \omega_j \Big\} > 0. \end{aligned}$$

## 3.

Now we turn to the proof of Theorem 1. Since $\lambda \in \Lambda^*$ we have $\ln \Big\{ \lambda_k \prod_{j=1}^{p} \lambda_{n_j - n_{j-1}+1} \Big\} \leq 2a(k+n_p)$ for every $p \in \mathbb{N}$ and $k = 1, 2, \ldots, n_p + 1$, and by Lemma 1

$$\frac{1}{n_p + k} \ln \frac{1}{|f_{n_p+k}|} \geq \gamma_{k,n_p} - \frac{1}{n_p + k} A_p + O(1), \quad p \to \infty,$$

where

$$A_p = \sum_{j=1}^{p} \ln \Big( \frac{1}{l_{n_j - n_{j-1}+1}} \Big).$$

Using Lemma 3 we get

$$\frac{1}{n_p+k}\ln\frac{1}{|f_{n_p+k}|}\ge \gamma_{1,n_p}-\frac{1}{n_p+1}A_p+\frac{(k-1)A_p}{(n_p+1)(n_p+k)}+O(1) \tag{1}$$

for all $k=1,2,\dots,n_{p+1}-n_p+1$ as $p\to\infty$. Since $\eta_k\nearrow\infty$ $(k\to\infty)$, we have $\ln(1/l_k)\ge 1$ $(k\ge k_0)$, and therefore

$$A_p=\Big(\sum_{\substack{j=1\\ n_j-n_{j-1}+1<k_0}}^{p}+\sum_{\substack{j=1\\ n_j-n_{j-1}+1\ge k_0}}^{p}\Big)\ln\frac{1}{l_{n_j-n_{j-1}+1}}=B_p+C_p,$$

where $B_p=O(p)$, $p\to\infty$, and $C_p\ge 0$. Hence it follows that the third summand on the right side of (1) can be omitted, and for $k=1,2,\dots,n_{p+1}-n_p+1$ we have

$$\frac{1}{n_p+k}\ln\frac{1}{|f_{n_p+k}|}\ge\frac{1}{n_p+1}\Big(\ln\frac{1}{l_{n_p}+1}-A_p\Big)+O(1),\quad p\to\infty.$$

Therefore, we have to prove that

$$\frac{1}{n_p+1}\Big(\ln\frac{1}{l_{n_p}+1}-\sum_{j=1}^{p}\frac{1}{l_{n_j-n_{j-1}+1}}\Big)\to+\infty,\quad p\to\infty.$$

But

$$\ln\frac{1}{l_k}=k\eta_k=k\Big(\sum_{j=1}^{k-1}\omega_j+\ln\frac{1}{l_1}\Big).$$

Hence we have to prove that

$$\frac{1}{n_p+1}\Big\{(n_p+1)\sum_{m=1}^{n_p}\omega_m-\sum_{j=1}^{p}(n_j-n_{j-1}+1)\sum_{m=1}^{n_j-n_{j-1}+1}\omega_m\Big\}\to+\infty, \tag{2}$$
$$\text{as } p\to\infty.$$

First let $\omega_m\nearrow\omega<\infty$ $(m\to\infty)$. Then it is easy to show that the condition (2) is equivalent to the condition

$$\frac{1}{n_p}\Big\{n_p\sum_{m=1}^{n_p}\omega_m-\sum_{j=1}^{p}(n_j-n_{j-1})\sum_{m=1}^{n_j-n_{j-1}}\omega_m\Big\}\to+\infty,\quad p\to\infty.$$

This condition holds provided that

$$\sum_{m=1}^{n_p}\omega_m-\max\Big\{\sum_{m=1}^{n_j-n_{j-1}}\omega_m\colon 1\le j\le p\Big\}\to+\infty,\quad p\to\infty,$$

that is,

$$\sum_{m=1}^{n_p}\omega_m-\sum_{m=1}^{n_j-n_{j-1}}\omega_m=\sum_{m=n_j-n_{j-1}}^{n_p}\omega_m\to\infty,\quad p\to\infty,\tag{3}$$

where $1\le j\le j_p\le p$. Since $\omega_1>0$, $\sum\limits_{m=k+1}^{n}\omega_m\ge(n-k)\omega_1$ and $n_p-n_j+n_{j-1}\to\infty$ for $1\le j\le j_p\le p$ as $p\to\infty$, we have (3), that is, Theorem 1 holds in the case $\omega<\infty$.

Let now $\omega_m\nearrow\infty$ $(m\to\infty)$. Then $\frac{1}{n_p+1}\sum_{j=1}^{n_p}\omega_m\to\infty$ $(p\to\infty)$, and since

$$\frac{1}{n_p+1}\Big\{(n_p+1)\sum_{m=1}^{n_p}\omega_m-\sum_{j=1}^{p}(n_j-n_{j-1}+1)\sum_{m=1}^{n_j-n_{j-1}+1}\omega_m\Big\}$$
$$=\frac{1}{n_p+1}\sum_{m=1}^{n_p}\omega_m+\frac{1}{n_p+1}\sum_{j=1}^{p}\Big\{(n_j-n_{j-1}+1)\sum_{m=n_j-n_{j-1}+2}^{n_p}\omega_m-\sum_{m=1}^{n_p}\omega_m\Big\},$$

it is sufficient to prove that

$$(n_j-n_{j-1}+1)\sum_{m=n_j-n_{j-1}}^{n_p}\omega_m-\sum_{m=1}^{n_p}\omega_m\ge 0\tag{4}$$

for all $p\ge j\ge j_0$. Let $j_0$ be such that $n_{j-1}\ge 4$ for $j\ge j_0$. Then $2<n_j-n_{j-1}+2\le n_p-2$, and the inequality (4) will be proved if we show that

$$k\sum_{m=k+1}^{n}\omega_m-\sum_{m=1}^{n}\omega_m\ge 0\quad(2\le k\le n-2).$$

The last inequality holds since $\omega_k$ is nondecreasing.

## References

[1] M. M. Sheremeta, *On power series with Gelfond-Leontev derivatives satisfying a special condition*, Matematicheskaya Fizika, Analiz i Geometriya (to appear). (Russian)