COMPOSITION OPERATORS ACTING ON SOBOLEV SPACES OF FRACTIONAL ORDER—A SURVEY ON SUFFICIENT AND NECESSARY CONDITIONS

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1. Introduction

What follows is a survey of recent results on sufficient and necessary conditions on composition operators to map one Sobolev space of fractional order into another. This report may be taken as a continuation of the contribution of G. Bourdaud given at the forerunner conference of this one, held in Friedrichroda 1992, cf. [Bo 5].

Composition operators are simple examples of nonlinear operators. Surprisingly enough, the behaviour of those operators even on Sobolev spaces itself is not completely understood up to now. As we hope to show, it turns out that

• the investigation of composition operators on generalized Sobolev spaces is an interesting field of research with several unexpected phenomena and

• the description of the image space of a Sobolev space has to make use of Sobolev spaces of fractional order in general.

So, it is meaningful to deal with fractional order spaces from the very beginning. From my point of view this part of the theory is just at the beginning and that will be reflected by several open problems, in a certain sense incomplete results, technicalities etc.

Let $G\colon \mathbb{R} \to \mathbb{R}$ be some continuous function. Then the associate composition operator T_G is given by $f \longrightarrow G(f)$. The properties of T_G like boundedness, continuity, differentiability etc., are strongly dependent on the domain. The situation in the case of Lebesgue spaces L_p, Hölder spaces C^s, and first order Sobolev spaces W^1_p is well understood and summarized in Appell and Zabrejko [AZ]. So we concentrate on higher order Sobolev spaces and its generalizations to the fractional case. More precisely, we shall deal here with Besov spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$

($s > 0$, $1 \le p, q \le \infty$), see Section 2 for definitions. If there is no need for a distinction between $B^s_{p,q}$ and $F^s_{p,q}$ then we shall use the notation $A^s_{p,q}$. So our interest could be stated as follows:

Problem 1. *Find necessary and sufficient conditions on G such that*

$$T_G(A^s_{p,q}) \subset A^s_{p,q} \tag{1}$$

holds.

As we shall immediately see, (1) will be sometimes too restrictive. Consider the mapping $f \to f^2$, that means $G(t) = t^2$. We test this mapping with very simple functions having a local singularity at the origin. Let

$$f_\alpha(x) = \psi(x)|x|^{-\alpha}, \quad \alpha > 0 \tag{2}$$

where ψ is a smooth cut-off function supported around the origin. Then it is well known (and not very complicated to prove) that in case $s > 0$

$$f_\alpha \in F^s_{p,q} \Longleftrightarrow 0 < \alpha < \frac{n}{p} - s \tag{3}$$

and

$$f_\alpha \in B^s_{p,q} \Longleftrightarrow \text{ either } \quad 0 < \alpha < \frac{n}{p} - s \quad \text{or} \quad \alpha = \frac{n}{p} - s \text{ and } q = \infty \tag{4}$$

hold, cf. [RS]. Because of f^2_α is a function of the same type as f_α itself, we find

$$f^2_\alpha \in F^s_{p,q} \Longleftrightarrow 0 < 2\alpha < \frac{n}{p} - s$$

and

$$f^2_\alpha \in B^s_{p,q} \Longleftrightarrow \text{ either } \quad 0 < 2\alpha < \frac{n}{p} - s \quad \text{or} \quad 2\alpha = \frac{n}{p} - s \quad \text{and} \quad q = \infty,$$

respectively. Let $s < n/p$. Choosing α sufficiently close to $\frac{n}{p} - s$ it follows that

$$T_{t^2}(F^s_{p,q}) \not\subset F^s_{p,q} \quad \text{and} \quad T_{t^2}(B^s_{p,q}) \not\subset B^s_{p,q}.$$

So, under certain circumstances we have to deal with

Problem 2. *Find necessary and sufficient conditions on G such that*

$$T_G(A^s_{p,q}) \subset A^r_{p,q}, \quad r \le s, \tag{5}$$

holds.

However, even the above example $G(t) = t^2$ could be taken as a motivation to consider

Problem 3. *Find necessary and sufficient conditions on G such that*

$$T_G(A^s_{p,q}) \subset A^s_{u,q}, \quad u \le p, \tag{6}$$

holds.

For this note that a mapping like $f \to f^2$ leads to a change in the integrability properties and not to a change of the regularity of f (in the sense of Hölder continuity).

Here we will mainly report on Problem 2. That is caused by the fact that the behaviour of the operators T_G becomes more interesting if the domain of definition contains unbounded functions, too, and then we are forced (at least if $s > 1$) to deal with Problem 2 instead of Problem 1. Problem 3 will only be touched here.

This note is organized as follows. In Section 2 we shall give the definitions of the function spaces under consideration here. Also some properties of these spaces are listed. In Section 3 we describe boundedness of composition operators in dependence on the parameters s, p and n in a very detailed form. It includes also a discussion of the corresponding inequalities.

Theorems, lemmas etc., are numbered with respect to subsections. So, Theorem 3.2/3 means Theorem 3 in Subsection 3.2.

The paper consists only of a survey of results, brief comments, and a few examples. In order to keep a reasonable length we omit all proofs, and the reader is referred to corresponding original papers and to the forthcoming book [RS].

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2. Function spaces

In general, all functions, distributions, etc., are defined on the Euclidean n-space $\mathbb{R}^n$. If there is no danger of confusion we will not indicate this. By $\mathbb{N}$ we denote the set of all natural numbers, by $\mathbb{N}_0$ the same set including 0.

To introduce Besov-Triebel-Lizorkin spaces we make use of the characterizations via differences and derivatives. Recall that if $f(x)$ is an arbitrary function and $h \in \mathbb{R}^n$, we put

$$\Delta_h^m f(x) = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell f\big(x + (m-\ell)h\big), \quad m = 1, 2, \dots .$$

Definition 1. Let $s > 0$, $1 \le q \le \infty$, and let $M \in \mathbb{N}_0$ be such that $M \le s < M + 1$.

(i) Suppose $1 \le p < \infty$. Then we put

$$F^s_{p,q} = \Big\{ f \in L_p \colon \|f \mid F^s_{p,q}\|$$
$$= \|f \mid L_p\| + \Big\| \Big(\int_0^1 t^{-sq} \Big(\frac{1}{t^n} \int_{|h|<t} |\Delta_h^{M+1} f(x)| \, \mathrm{d}h \Big)^q \frac{\mathrm{d}t}{t} \Big)^{1/q} \Big| L_p \Big\| < \infty \Big\}.$$

(ii) Suppose $1 \le p \le \infty$. Then we put

$$B^s_{p,q} = \Big\{ f \in L_p \colon$$
$$\|f \mid B^s_{p,q}\| = \|f \mid L_p\| + \Big(\int |h|^{-sq} \|\Delta_h^{M+1} f \mid L_p\|^q \frac{\mathrm{d}h}{|h|^n} \Big)^{1/q} < \infty \Big\}.$$

Also Bessel potential spaces will very often appear. As usual, $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

Definition 2. Let $s > 0$ and let $1 < p < \infty$. Then we put

$$H^s_p = \{ f \in L_p \colon \|f \mid H^s_p\| = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathcal{F} f(\xi)](\cdot) \mid L_p\| < \infty \}.$$

Remark 1. The spaces $F^s_{p,q}$, $B^s_{p,q}$ and H^s_p are Banach spaces. Nowadays there exists a rich literature on this subject. We refer to Frazier, Jawerth and Weiss [FJW], Besov, Il'yin, and Nikol'skii [BIN], Nikol'skii [Ni], Peetre [Pe 2] and Triebel [Tr 1], [Tr 2].

Remark 2. The advantage of the scales of Triebel-Lizorkin spaces $F^s_{p,q}$ and Besov spaces $B^s_{p,q}$ consists in the fact that they enable us to treat several "classical" function spaces from a common point of view. We have: $B^s_{\infty,\infty} = C^s$, $0 < s \neq integer$, (C^s are the Hölder spaces), $F^m_{p,2} = W^m_p$, $1 < p < \infty$, $m = 1, 2, \dots$, (W^m_p are the Sobolev spaces), $F^s_{p,p} = B^s_{p,p} = W^s_p$, $1 \le p < \infty$, $s > 0$, (W^s_p are the Slobodeckii spaces, known to include the trace classes of Sobolev spaces), and $F^s_{p,2} = H^s_p$, $1 < p < \infty$. All these spaces have in common that they are well-adapted to elliptic partial differential equations, cf. Triebel [Tr 1].

Convention 1. If there is no need for a distinction between $F^s_{p,q}$ and $B^s_{p,q}$, then we simply write $A^s_{p,q}$.

Convention 2. In theorems, propositions etc., only additional restrictions on the parameters s, p and q will be given. For example if there is no condition stated for s, then all values appearing in the definitions are admissible.

All the spaces defined above contain complex-valued functions. In general, $f \to G(f)$ makes no sense in such situation. Consequently, we shall deal here with real-valued parts of these spaces only.

Definition 3. $\mathbb{A}^s_{p,q}$ denotes the real-valued part of $A^s_{p,q}$.

Important for us will be several embedding relations. We have

$$H^{s+\varepsilon}_p \hookrightarrow (F^s_{p,q} \cap B^s_{p,q}) \hookrightarrow (F^s_{p,q} \cup B^s_{p,q}) \hookrightarrow H^{s-\varepsilon}_p$$

for any $\varepsilon > 0$. Hence, for fixed s and p all spaces are close to each other. The parameter q is playing the role of a fein-index. Further, we wish to mention that

$$F^s_{p,q} \hookrightarrow L_\infty \quad \text{if and only if either} \quad s > \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \text{ and } p = 1$$

and

$$B^s_{p,q} \hookrightarrow L_\infty \quad \text{if and only if either} \quad s > \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \text{ and } q = 1,$$

cf. [Tr 1, 2.3.2, 2.7.1]. More detailed information about embeddings is given in [ST].

3. Boundedness of composition operators

It follows from the results that the following subdivision of our survey is reasonable:

(a) $0 < s < 1$,
(b) $1 \leq s < 1 + \frac{1}{p}$,
(c) $1 < s < \frac{n}{p}$,
(d) $s > \frac{n}{p}$.

3.1. Boundedness in spaces of "low" smoothness.

Throughout this subsection we suppose $0 < s < 1$.

Theorem. (i) *Let* $\mathbb{A}^s_{p,q} \hookrightarrow L_\infty$. *Then the following assertions are equivalent:*

- $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^s_{p,q}$.
- *There exists a constant* c *such that*

$$\|G(f) \mid A^s_{p,q}\| \le c\|G' \mid L_\infty(f(\mathbb{R}^n))\| \, \|f \mid A^s_{p,q}\| \tag{1}$$

  holds for all $f \in \mathbb{A}^s_{p,q}$.
- $G(0) = 0$ *and* $G' \in \mathbb{L}^{\mathrm{loc}}_\infty(\mathbb{R})$.

(ii) *Let* $\mathbb{A}^s_{p,q} \not\subset L_\infty$. *Then the following assertions are equivalent:*

- $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^s_{p,q}$.
- *There exists a constant* c *such that*

$$\|G(f) \mid A^s_{p,q}\| \le c\|G' \mid L_\infty\| \, \|f \mid A^s_{p,q}\| \tag{2}$$

  holds for all $f \in \mathbb{A}^s_{p,q}$.
- $G(0) = 0$ *and* $G' \in \mathbb{L}_\infty(\mathbb{R})$.

Remark 1. It is clear that $f(\mathbb{R}^n) = \{y \colon \exists x \in \mathbb{R}^n \text{ with } f(x) = y\}$.

Remark 2. The sufficiency part is obvious. Because of $s < 1$ both scales $F^s_{p,q}$ and $B^s_{p,q}$ are defined by means of first order differences. It follows that

$$|G(f(x+h)) - G(f(x))| \le \|G' \mid L_\infty(\mathbb{R})\| \, |f(x+h) - f(x)|$$

and

$$|G(f(x))| = |G(f(x)) - G(0)| \le \|G' \mid L_\infty(\mathbb{R})\| \, |f(x)|,$$

which proves that the third assertion implies the first two. Conversely, testing T_G on $f \equiv 0$, we get $G(0) = 0$ as a necessary condition. The hard part of the equivalences stated above is to show the necessity of $G' \in \mathbb{L}^{\mathrm{loc}}_\infty(\mathbb{R})$ and $G' \in \mathbb{L}_\infty(\mathbb{R})$, respectively. That has been done by Igari [Ig] in a very special situation. The result as stated above has been proved by Bourdaud and his collaborators, cf. [Bo 3], [Bo 4], [BK 1], [BK 2].

Remark 3. The contents of the theorem could be stated as follows: whenever a composition operator yields a mapping into the space $\mathbb{A}^s_{p,q}$, $0 < s < 1$, then it is a bounded mapping, too.

3.2. Spaces with smoothness $s \in [1, 1 + 1/p)$.

We first deal with $s = 1$.

Theorem. (i) *Let $\mathbb{W}^1_p \hookrightarrow L_\infty$. Then the following assertions are equivalent:*

- $T_G(\mathbb{W}^1_p) \subset \mathbb{W}^1_p$.
- *There exists a constant c such that*

$$\|G(f) \mid W^1_p\| \le c\|G' \mid L_\infty(f(\mathbb{R}^n))\| \, \|f \mid W^1_p\| \tag{3}$$

  holds for all $f \in \mathbb{W}^1_p$.
- $G(0) = 0$ *and* $G' \in \mathbb{L}^{\mathrm{loc}}_\infty(\mathbb{R})$.

(ii) *Let $\mathbb{W}^1_p \not\subset L_\infty$. Then the following assertions are equivalent:*

- $T_G(\mathbb{W}^1_p) \subset \mathbb{W}^1_p$.
- *There exists a constant c such that* (3) *holds for all $f \in \mathbb{W}^1_p$.*
- $G(0) = 0$ *and* $G' \in \mathbb{L}_\infty(\mathbb{R})$.

(iii) *Consider the Hardy-Sobolev space $\mathbb{F}^1_{1,2}$. Then the following assertions are equivalent:*

- $T_G(\mathbb{F}^1_{1,2}) \subset \mathbb{F}^1_{1,2}$.
- $T_G(\mathbb{W}^1_1) \subset \mathbb{W}^1_1$.

Remark 1. Parts (i) and (ii) have to be traced to Marcus and Mizel [MM], whereas part (iii) has been proved by Janson [Ja]. Note that $F^1_{1,2}$ differs from W^1_1; in fact, it is a proper subspace.

The knowledge in all other cases with $s = 1$ is much less complete. Only partial answers are known. If we suppose not only differentiability properties of G but also some integrability properties, then we have the following extension.

Theorem 2. *Let $1 < p < \infty$ and let $1 \le s < 1 + 1/p$. Further, suppose*

$$G(0) = 0,\ G' \in \mathbb{L}_\infty(\mathbb{R}),\ A_G = \sup_{h>0} h^{-1/p} \Big\| \sup_{|\varrho|<h} |\Delta^1_\varrho G'(\cdot)| \ \Big|\ L_p \Big\| < \infty. \tag{4}$$

Then there exists some constant c such that

$$\|G(f) \mid B^s_{p,q}\| \le c\big(\|G' \mid L_\infty(\mathbb{R})\| + A_G\big) \|f \mid B^s_{p,q}\| \tag{5}$$

holds for all $f \in B^s_{p,q}$.

Remark 2. A proof of the Theorem has been given by Bourdaud and Kateb [BK 4].

To understand the conditions much better we add the following proposition, which is also taken over from [BK 4].

Proposition. *Let $1 < p < \infty$ and let $s > 1 + 1/p$.*

(i) *If $G \in \mathbb{B}^s_{p,\infty}(\mathbb{R})$ then $G' \in \mathbb{L}_\infty(\mathbb{R})$ and $A_G < \infty$.*

(ii) *If G is some function satisfying*

$$G(0) = 0, \quad G' \in \mathbb{L}_\infty(\mathbb{R}), \quad \text{and} \quad \sup_{t \neq 0} |t|\,|G''(t)| < \infty, \tag{6}$$

then G fulfils the conditions of Theorem 2.

(iii) *If G is some function satisfying*

$$G(0) = 0, \quad \text{and} \quad G'' \text{ is a bounded measure}, \tag{7}$$

then G fulfils the conditions of Theorem 2.

Remark 3. Now it becomes easy to select examples of functions satisfying the conditions of Theorem 2. So, any $G \in C_0^\infty(\mathbb{R})$ does.

Another very interesting example consists in $f \to |f|$. In this case one knows even more.

Theorem 3. *Suppose $s > 0$. The following assertions are equivalent:*

- $\{|f| \colon f \in \mathbb{B}^s_{p,q}\} \subset \mathbb{B}^s_{p,q}$.
- $\{|f| \colon f \in \mathbb{H}^s_p\} \subset \mathbb{H}^s_p$.
- $s < 1 + \frac{1}{p}$.

Remark 4. The proofs of the equivalence of the first and the third assertion have been given independently by Bourdaud and Meyer [BM] and Oswald [Os]. The equivalence with the second assertion has been established in Bourdaud and Kateb [BK 3]. For extensions to the general case of Triebel-Lizorkin spaces, see Bourdaud and Kateb [BK 3].

If we do not suppose integrability properties of G, then we are forced to consider spaces of bounded functions only (as long as we deal with Problem 1). For sufficient conditions we refer to the next subsection, especially to Theorem 3.3/1. Necessary conditions will be given in Subsections 3.3 and 3.4.

A final remark on $s \in [1, 1 + 1/p)$.

Up to now the mapping properties of composition operators on $F^s_{p,q}$ and $B^s_{p,q}$ are not well understood in this interval. The difference from the case $s < 1$ consists in the fact that we had to impose additional integrability conditions on G. But as in that case the corresponding estimates have on the right-hand side $c\|f \mid A^s_{p,q}\|$ with some constant independent of f. If

$s > 1 + 1/p$, such an estimate does not seem to be available. Then one has to add powers of $\|f \mid A^s_{p,q}\|$, cf. Subsections 3.3 and 3.4.

The case $s = 1$ itself, reduced to Sobolev spaces or to the Hardy-Sobolev space would be better placed in Subsection 3.1, but we believe that in general integrability properties of G have to be added to obtain boundedness.

3.3. Spaces of bounded functions.

In this subsection we investigate the behaviour of composition operators either on $\mathbb{A}^s_{p,q}$ with $s > n/p$ or on $\mathbb{A}^s_{p,q} \cap \mathbb{L}_\infty$.

Theorem 1. *Suppose* $1 \le s < \mu$. *Let* $G \in C^\mu(\mathbb{R})$ *and let* $G(0) = 0$.

(i) *Then there exists a constant* c *such that*

$$\|G(f) \mid F^s_{p,q}\| \le c\|f \mid F^s_{p,q}\|(1 + \|f \mid L_\infty\|^{\mu-1}) \tag{8}$$

holds for all $f \in \mathbb{F}^s_{p,q}$.

(ii) *Then there exists a constant* c *such that*

$$\|G(f) \mid B^s_{p,q}\| \le c\|f \mid B^s_{p,q}\|(1 + \|f \mid L_\infty\|^{\mu-1}) \tag{9}$$

holds for all $f \in \mathbb{B}^s_{p,q}$.

Remark 1. For the proof we refer to [RS], see also [Ru 1]. But Theorem 1 itself has a lot of forerunners, even in the fractional case. The first reference in the case of the Bessel potential spaces $\mathbb{H}^s_2$ seems to be Mizohata [Mi]. Later on also Peetre [Pe 1] ($\mathbb{B}^s_{p,q} \cap \mathbb{L}_\infty$), Adams [Ad] ($\mathbb{H}^s_p$), Meyer [Me] ($\mathbb{H}^s_p$), Yamazaki [Ya] (general case) and Marschall [Ma] (also general case) have contributed to this subject. An elegant approach to the regularity of compositions via pseudodifferential operators, originally introduced by Meyer [Me], later on applied extensively by several authors, may be found also in the monograph by Taylor [Tay].

Remark 2. The proof given in [RS] is based on a maximal function approach, which is a standard tool in the framework of Triebel-Lizorkin spaces. More precisely, the Fefferman-Stein-Peetre maximal function (cf. [Tr 1, p. 30], [RS, 2.1.6]) is used. One starts with the Taylor expansion of the composition $G(f(x))$. Then the Taylor polynomial can be estimated by means of pointwise multiplier assertions and the remainder can be handled with the help of the Fefferman-Stein-Peetre maximal function and certain integral means $I^\mu_k f$. Here

$$I^\mu_k f(x) = \int_{|z| \le 2^{-k}} |f(x+z) - f(x)|^\mu \, \mathrm{d}z, \quad x \in \mathbb{R}^n,\ k \in \mathbb{Z}^n,\ \mu > 0.$$

The means I_k^μ have been introduced by Runst [Ru 1]. A detailed investigation of $\|\{2^{ks} I_k^\mu f\}_k \mid L_p(\ell_q)\|$ has been done in [RS].

There are several improvements of Theorem 1 in special situations. Before we list some of them we collect some necessary conditions.

Theorem 2. (i) *Let $A^s_{p,q}$ be a subspace of L_∞. Then*

$$T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^s_{p,q} \tag{10}$$

implies $G' \in \mathbb{L}^{\mathrm{loc}}_\infty(\mathbb{R})$.

(ii) *Suppose that $A^s_{p,q}$ is not a subspace of L_∞. Then* (10) *implies $G' \in \mathbb{L}_\infty(\mathbb{R})$.*

(iii) *If we assume*

$$T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^r_{p,v} \tag{11}$$

then $G \in \mathbb{A}^{r,\mathrm{loc}}_{p,v}(\mathbb{R})$ follows.

Remark 3. Parts (i) and (ii) have been proved by Bourdaud, and Bourdaud and Kateb [Bo 3], [Bo 4], [BK 1], [BK 2]. Part (iii) may be found in [Si 4], cf. also [RS].

Comparing Theorem 1 with Theorem 2 one observes that the conditions on the regularity of G are too restrictive. Let us start with the limit case of Theorem 1. We investigate whether $s = \mu$ is admissible.

Proposition. *Suppose $s > 1$, $G(0) = 0$, and $G \in C^s(\mathbb{R})$. Then there exists a constant c such that*

$$\|G(f) \mid F^s_{p,\infty}\| \le c \|f \mid F^s_{p,\infty}\| \big(1 + \|f \mid L_\infty\|^{s-1}\big) \tag{12}$$

holds for all $f \in \mathbb{F}^s_{p,\infty}$.

Remark 4. A proof is given in [RS]. Note that (12) can not be true with $\mathbb{F}^s_{p,\infty}$ replaced by $\mathbb{F}^s_{p,q}$ with $q < \infty$. That follows from Theorem 2 (iii) and $C^{s,\mathrm{loc}}(\mathbb{R}) \not\subset F^{s,\mathrm{loc}}_{p,q}(\mathbb{R})$, $q < \infty$.

Remark 5. A somewhat weaker result is known in the case of potential spaces. Then the inequality (8) is improved, but the price one has to pay is a higher regularity of G, cf. Subsection 3.5.

A final answer is known in case of Sobolev spaces, cf. Bourdaud [Bo 2].

Theorem 3. *Suppose $W_p^m \hookrightarrow L_\infty$. Then the following assertions are equivalent:*

- $T_G(\mathbb{W}_p^m) \subset \mathbb{W}_p^m$.
- *There exists a constant c such that*

$$\|G(f) \mid W_p^m\| \le c\|G \mid W_p^m(f(\mathbb{R}^n))\| \big(\|f \mid W_p^m\| + \|f \mid W_p^m\|^m\big) \tag{13}$$

  holds for all $f \in \mathbb{W}_p^m$.
- $G(0) = 0$ *and* $G \in W_p^{m,\mathrm{loc}}(\mathbb{R})$.

In view of the last theorem and our knowledge of necessary conditions we have a conjecture about the general case:

Conjecture. Let $s > \max(n/p, 1 + 1/p)$. Then (10) holds if and only if $G(0) = 0$ and $G \in \mathbb{A}_{p,q}^{s,\mathrm{loc}}(\mathbb{R})$.

Remark 6. The restriction $s > 1+1/p$ comes from $G' \in \mathbb{L}_\infty(\mathbb{R})$. But observe that to take $G' \in \mathbb{A}_{p,q}^{s-1,\mathrm{loc}}(\mathbb{R}) \cap \mathbb{L}_\infty^{\mathrm{loc}}(\mathbb{R})$ would not be sufficient. That follows from $G(t) = |t| \in B_{p,\infty}^{1+1/p,\mathrm{loc}}(\mathbb{R})$, cf. Theorem 3.2/3.

3.4. The critical case $1 < s < n/p$.

We call this case critical because of the fact that there exist functions $G \in C_0^\infty(\mathbb{R})$ such that the corresponding composition operator T_G does not map the function space into itself, provided that s is taken from this critical interval (for references see below). Obviously, this restriction implies that we have to deal with $n > p \ge 1$, so it looks purely multi-dimensional. However, if one is interested also in spaces with $p < 1$ (which makes sense even in case of nonlinear mappings) then the following phenomena will appear in the one-dimensional case, too. To describe the situation we have to introduce the following nonlinear quantity:

$$\varrho(s,p,n,\delta) = \frac{\frac{n}{p} + \frac{1}{p}(1-\delta)(\frac{n}{p} - s)}{\frac{n}{p} - s + 1}, \quad 0 \le \delta \le 1. \tag{14}$$

Theorem 1. *Suppose $1 < s < n/p$. Further, we assume $G(0) = 0$ and $G \in C^{\varrho(s,p,n,1)}(\mathbb{R})$. Then there exists a constant c such that*

$$\|G(f) \mid F_{p,r}^{\varrho(s,p,n,1)}\| \le c\big(\|f \mid F_{p,q}^s\| + \|f \mid F_{p,q}^s\|^{\varrho(s,p,n,1)}\big) \tag{15}$$

holds for all $f \in \mathbb{F}_{p,q}^s$.

Remark 1. A proof has been given in [Si 2], cf. also [RS].

Observe that $1 < \varrho(s,p,n,1) < s$ as long as $1 < s < n/p$. Hence, (15) indicates some loss of regularity under the action of T_G. The following example seems to be helpful to make this more transparent. Consider $G(t) = \sin t$ and let f_α be the family of unbounded functions defined in (2). Then

$$\frac{\partial^m}{\partial x_1^m} G(f_\alpha(x)) \sim \sin^{(m)}(|x|^{-\alpha})(|x|^{-\alpha-2}x_1)^m + \text{lower order terms}$$

on a neighbourhood of the origin. Compare this with the m-th order derivative of the original function. It turns out that

$$\frac{\partial^m}{\partial x_1^m} f_\alpha(x) \sim |x|^{-\alpha-2m} x_1^m + \text{lower order terms} \quad (|x| \to 0).$$

Since we have $(\alpha+1)m > \alpha+m$ if $m \geq 2$ this shows that the local singularity of the composition becomes stronger than the singularity of the original function f_α. This reflects the fact that some loss of regularity under the action of the composition operator $T_{\sin t}$ does occur necessarily. Applying more sophisticated test functions one obtains the following.

Theorem 2. *Suppose* $1 + 1/p < s < n/p$. *Then* $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^s_{p,q}$ *implies* $G(t) = ct$ *for some constant* c.

Remark 2. The phenomenon, described in Theorem 2 is well known since Dahlberg [Da] had published his short note in 1978. While he had dealt with Sobolev spaces only, the extensions to Besov and Triebel-Lizorkin spaces have been given by Bourdaud [Bo 1], [Bo 5]. Extensions to values $p, q < 1$ are given in Runst [Ru 1] and [Si 3].

Looking at this degeneracy result a natural question consists in asking for the best possible image space.

Theorem 3. *Suppose* $1 + 1/p < s < n/p$. *Let* G *be non-polynomial. Then for every* $r > \varrho(s,p,n,0)$ *there exists a compactly supported function* $f_r \in \mathbb{A}^s_{p,q}$ *such that the composition* $G(f_r)$ *does not belong to* $\mathbb{B}^r_{p,\infty}$.

Remark 3. For the proof we refer to [Si 2], [Si 3], [RS]. The proof there is delivered for smooth G, but the same argument applies in the case of less smooth functions, at least if the underlying spaces are Banach spaces.

Remark 4. Observe that if $1+1/p < s < n/p$, then $1+1/p < \varrho(s,p,n,0) < s$. Further, $\varrho(s,p,n,1) < \varrho(s,p,n,0)$. Whether there exist nonlinear functions G such that $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{B}^{\varrho(s,p,n,0)}_{p,\infty}$ is still an open question.

Remark 5. It turns out that the following observation is of some interest. We study the difference

$$d(s) = s - \varrho(s,p,n,0) = s - \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s)}{\frac{n}{p} - s + 1}, \quad 1 + \frac{1}{p} \le s \le \frac{n}{p},$$

for fixed n and p $(n > p)$. Obviously, $d(1+1/p) = d(n/p) = 0$ and $d(s) > 0$ if $1+1/p < s < n/p$. Moreover, the function $d(s)$ is concave on this interval, hence, it attains a maximal value $d(s_0)$ there. We have

$$s_0 = \frac{n}{p} + 1 - \sqrt{\frac{n-1}{p}} \quad \text{and} \quad d(s_0) = \Big(\sqrt{\frac{n-1}{p}} - 1\Big)^2 + \frac{p-1}{\sqrt{p(n-1)}}.$$

This shows that $d(s)$ has a bound depending on p and n, but it does not have an a priori bound for fixed p and independent of n.

More qualitative necessary conditions on G.

If one adds additional conditions on G, then one can improve Theorem 2. So, in [Si 3] we introduced classes of functions A_δ^m and B_δ^m $(0 \le \delta \le 1)$ by means of explicit conditions on the "thickness" and distribution of the set $\{t\colon |G^{(m)}(t)| \ge A > 0\}$. Then it turns out that a counterpart of Theorem 2 is true with $\varrho(s,p,n,0)$ replaced by $\varrho(s,p,n,\delta)$. We omit details and refer to [Si 3]. Here we shall follow a slightly different approach to improve the knowledge about necessary conditions. From Theorem 3.3/2 we know that $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{A}^r_{p,v}$ implies $G \in \mathbb{A}^{r,\mathrm{loc}}_{p,v}(\mathbb{R})$. In view of this a natural ansatz for a necessary condition would consist in a growth condition of the norm in the image space taken of a certain restriction of G. In the fractional case it will not be clear from the very beginning what is meant by this, while in case of Sobolev spaces it has an immediate interpretation: if $\mathbb{A}^r_{p,v} = \mathbb{W}^m_p$ is our image space, then we ask for necessary conditions on the growth of

$$\int_{-w}^{w} |G^{(m)}(t)|^p \, \mathrm{d}t$$

implied by this.

Theorem 4. *Let us suppose that*

$$T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{W}^m_p, \tag{16}$$

where we require $0 \le m \le s$ *and* $0 < s < n/p$. *Then*

$$\sup_{w \ge 1} w^{-\gamma} \left(\int_{-w}^{w} |G^{(m)}(y)|^p \, \mathrm{d}y \right)^{1/p} < \infty \tag{17}$$

for all

$$\gamma > \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - m(\frac{n}{p} - s + 1)}{\frac{n}{p} - s}. \tag{18}$$

Remark 6. The proof is given in [Si 4].

We shall illustrate the sharpness of the above theorem by a natural example; we study the mapping $f \to |f|^\mu$, $\mu > 1$, cf. [Si 2], [RS].

Proposition. *Let* $\mu > 1$. *Suppose that* $0 < s < \min(\frac{n}{p}, \mu)$. *We put* $s_\mu = s - (\mu - 1)(\frac{n}{p} - s)$. *If* $s_\mu > 0$, *then there exists a constant* c *such that*

$$\| |f|^\mu \mid F^{s_\mu}_{p,r} \| \le c \|f \mid F^s_{p,q}\|^\mu \tag{19}$$

holds for all $f \in F^s_{p,q}$.

Remark 7. The restriction $s < \mu$ comes from the finite regularity of $G(t) = |t|^\mu$. The second restriction, namely $s_\mu > 0$, becomes transparent by plugging in f_α, defined in (2), cf. (3). Then $s_\mu > 0$ means $|f_\alpha|^\mu \in L_p$.

Remark 8. It is not hard to check that

$$\left(\int_{-w}^{w} |G_\mu^{(m)}(y)|^p \, \mathrm{d}y \right)^{1/p} \sim w^{\mu - m + 1/p}$$

for $w \to \infty$ and $G_\mu(t) = |t|^\mu$. Therefore, as a consequence of Theorem 4, we need to have

$$\frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - m(\frac{n}{p} - s + 1)}{\frac{n}{p} - s} \ge \mu - m + \frac{1}{p},$$

which is equivalent to

$$m \le s - \big(\mu - 1\big)\big(\frac{n}{p} - s\big) = s_\mu.$$

Hence, Theorem 4 shows that the estimate given in the Proposition is the best possible (recall $F^m_{p,2} = W^m_p$ if $1 < p < \infty$).

Next we consider the fractional case. For technical reasons we restrict ourselves to $\mathbb{B}^r_{p,p}$ as the image space. Unfortunately we have to add a growth condition on G in that case (probably also for technical reasons only). For simplicity we take $G \in \mathbb{L}_\infty(\mathbb{R})$, for more general conditions see [Si 4].

As we explained above, there are several versions by what to replace $\int_{-w}^{w} |G^{(m)}(y)|^p \, \mathrm{d}y$ in the fractional case. For the first version we introduce the following abbreviation:

$$\|G \mid \dot{a}_p^r([-w,w])\| = \left(\int_0^w |h|^{-rp-1} \int_{-w}^{w} |\Delta_h^{m+1} G(y)|^p \, \mathrm{d}y \, \mathrm{d}h \right)^{1/p} \tag{20}$$

if $m \in \mathbb{N}_0$ and $m+1 > r$. The notation $\|G \mid \dot{a}_p^r([-w,w])\|$ reminds of a norm in a homogeneous Besov space. In some sense it is an incomplete one. Whether one can replace $\|G \mid \dot{a}_p^r([-w,w])\|$ by an appropriate homogeneous norm will be discussed later on.

Theorem 5. *Suppose $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{B}^r_{p,p}$ and $0 < r \le s < n/p$. Let $G \in \mathbb{L}_\infty(\mathbb{R})$. Then it follows that*

$$\sup_{w \ge 1} w^{-\gamma} \|G \mid \dot{a}_p^r([-w,w])\| < \infty \tag{21}$$

for all

$$\gamma > \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - r(\frac{n}{p} - s + 1)}{\frac{n}{p} - s}. \tag{22}$$

Remark 9. For the proof we refer to [Si 4].

Remark 10. Let us comment on the condition (22). Fix n and p and consider $s \to n/p$. Then the lower bound for γ tends to infinity, which means that the necessary condition (21) becomes less restrictive. That is connected with the fact that local singularities in spaces with $s \uparrow n/p$ become weaker and weaker, cf. (2) and (3). If we fix also s and consider $r \downarrow 0$, then the necessary condition (21) becomes weaker again (since the lower bound of γ increases). Clearly, that corresponds to the fact that the spaces $\mathbb{B}^r_{p,p}$ become larger. A similar observation gives a converse result for $r \uparrow s$.

Remark 11. First essential conclusion of Theorem 5 is obtained by observing that $\|G \mid \dot{a}_p^r([-w,w])\|$ is a non-decreasing function in w. So, whenever G is not a polynomial of a low degree, $\|G \mid \dot{a}_p^r([-w,w])\|$ is bounded from below by a positive constant. Then Theorem 5 says (in the case that G is bounded) that

$$\frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - r(\frac{n}{p} - s + 1)}{\frac{n}{p} - s} \ge 0 \Longleftrightarrow r \le \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s)}{\frac{n}{p} - s + 1} = \varrho(s,p,n,0),$$

which also follows from Theorem 3. Hence, Theorem 5 represents an extension of Theorem 3, at least a partial one.

Before we discuss Theorem 5 by means of some further examples we shall consider with two further variants of replacing $\int_{-w}^{w} |G^{(m)}(y)|^p \, dy$. One variant is based on weighted function spaces. The only weights we are interested in are polynomial weights: for any $\gamma \in \mathbb{R}$ we define

$$\varrho_\gamma(y) = (1 + |y|)^{-\gamma}, \quad y \in \mathbb{R}. \tag{23}$$

Definition. (i) We put

$$L_p(\mathbb{R}, \varrho_\gamma) = \Big\{ f : f \text{ Lebesgue measurable}, \quad \|f \mid L_p(\mathbb{R}, \varrho_\gamma)\| = \Big(\int_{-\infty}^{\infty} |f(y) \varrho_\gamma(y)|^p \, \mathrm{d}y \Big)^{1/p} \Big\}. \tag{24}$$

(ii) Let M be a natural number. Suppose $\sigma_p < s < M \leq s + 1$. Then

$$B^s_{p,p}(\mathbb{R}, \varrho_\gamma) = \Big\{ f \in \mathcal{S}' : \quad \|f \mid B^s_{p,p}(\mathbb{R}, \varrho_\gamma)\| = \ \|f \mid L_p(\mathbb{R}, \varrho_\gamma)\| + \|f \mid \dot{B}^s_{p,p}(\mathbb{R}, \varrho_\gamma)\| < \infty \Big\}, \tag{25}$$

with

$$\|f \mid \dot{B}^s_{p,p}(\mathbb{R}, \varrho_\gamma)\| = \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h|^{-sp-1} |\varrho_\gamma(y) \Delta_h^M f(y)|^p \, \mathrm{d}y \ \mathrm{d}h \Big)^{1/p}. \tag{26}$$

Remark 12. Of course, in the case $\gamma = 0$ we are back in the nonweighted case. For the properties of these classes we refer to Schmeisser and Triebel [ScTr, Chapter 5].

With the help of these weighted Besov spaces it becomes easy to formulate necessary conditions in a more compact and convenient form.

Theorem 6. *Suppose $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{B}^r_{p,p}$ and $0 < r \leq s < n/p$. Let $G \in \mathbb{L}_\infty(\mathbb{R})$ be non-trivial. Then it follows that*

- $r \leq \dfrac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s)}{\frac{n}{p} - s + 1}$,
- $G(0) = 0$,
- $G \in \mathbb{B}^{r,\mathrm{loc}}_{p,p}(\mathbb{R})$,
- $\|G \mid \dot{\mathbb{B}}^r_{p,p}(\mathbb{R}, \varrho_\gamma)\| < \infty$ *for all*

$$\gamma > \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - r(\frac{n}{p} - s + 1)}{\frac{n}{p} - s}. \tag{27}$$

Remark 13. The proof is given in [Si 4].

Under certain circumstances one can also use homogeneous Besov spaces without weights. We need some preliminary work. Let

$$\omega(z) = \begin{cases} c \exp\left(-\frac{1}{1-|z|^2}\right) & \text{if } |z| \le 1, \\ 0 & \text{otherwise} \end{cases}$$

with c determined by $\int \omega(z)\, \mathrm{d}z = 1$. Further, let $\chi_w(z)$ denote the characteristic function of the interval $[-w, w]$. Then we put

$$\psi_w(x) = \int_{-1}^{1} \chi_w(x+z)\omega(z)\, \mathrm{d}z. \tag{28}$$

Hence, $\psi_w \in C_0^\infty(\mathbb{R})$ and

$$\sup_{w \ge 1} \sup_x |\psi_w^{(\ell)}(x)| \le c_\ell < \infty \quad \text{for all } \ell \in \mathbb{N}_0.$$

Theorem 7. *Let $G \in \mathbb{L}_\infty(\mathbb{R})$. Suppose $T_G(\mathbb{A}^s_{p,q}) \subset \mathbb{B}^r_{p,p}$, where*

$$1 < p < \infty, \quad \frac{1}{p} \le r \le \frac{\frac{n}{p}}{\frac{n}{p} - s + 1}, \quad \text{and} \quad r \le s < \frac{n}{p}. \tag{29}$$

Then it follows that

- $G(0) = 0$,
- $G \in \mathbb{B}^{r,\mathrm{loc}}_{p,p}(\mathbb{R})$,
- $\sup_{w \ge 1} w^{-\gamma} \|\psi_w G \mid \dot{\mathbb{B}}^r_{p,p}(\mathbb{R})\| < \infty$ *for all*

$$\gamma > \frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - r(\frac{n}{p} - s + 1)}{\frac{n}{p} - s}. \tag{30}$$

Remark 14. Again the proof may be found in [Si 4].

Example 1. Consider $G(t) = \sin t$. Then $\Delta_h^{m+1} G$ is a periodic function, too. Hence, $\Delta_h^{m+1} G \in L_p(\mathbb{R}, \varrho_\mu)$ if and only if $\mu > 1/p$. For $r > 0$ some explicit calculations give

$$\sin t \in \dot{\mathbb{B}}^r_{p,p}(\mathbb{R}, \varrho_\mu) \iff \varrho_\mu \in L_p(\mathbb{R}) \iff \mu p > 1.$$

Then Theorem 3.4/6 says that

$$\frac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s) - r(\frac{n}{p} - s + 1)}{\frac{n}{p} - s} \geq \frac{1}{p},$$

which is equivalent to

$$r \leq \frac{\frac{n}{p}}{\frac{n}{p} - s + 1}.$$

Putting this together with Theorem 1 we have proved the following quite satisfactory corollary.

Corollary. *The following assertions are equivalent:*

- $T_{\sin t}(\mathbb{F}^s_{p,q}) \subset \mathbb{F}^r_{p,\infty}$.
- $T_{\sin t}(\mathbb{F}^s_{p,\infty}) \subset \mathbb{F}^r_{p,v}$.
- $r \leq \frac{\frac{n}{p}}{\frac{n}{p} - s + 1}$.

Remark 15. The Corollary remains true for all periodic, non-constant, bounded and sufficiently smooth ($\in \mathbb{B}^{r,\mathrm{loc}}_{p,p}(\mathbb{R})$) functions G satisfying $G(0) = 0$. This reflects the optimality of Theorem 1, Theorem 5 and Theorem 6. In addition, it shows that the required integrability properties in Theorem 3.2/2 are not superfluous.

Example 2. Consider $G(t) = (1 + t^2)^{-\alpha/2} \sin t$, $\alpha > 0$. Observe that G is infinitely differentiable, bounded with all its derivatives. In case $r > 0$ some explicit calculations give

$$G \in \dot{\mathbb{B}}^r_{p,p}(\mathbb{R}, \varrho_\mu) \Longleftrightarrow (1 + t^2)^{-\alpha/2} \varrho_\mu \in L_p(\mathbb{R}) \Longleftrightarrow \mu > \frac{1}{p} - \alpha.$$

As a conclusion of Theorem 6 we find that

$$r \leq \begin{cases} \dfrac{\frac{n}{p} + \alpha(\frac{n}{p} - s)}{\frac{n}{p} - s + 1} & \text{if } 0 < \alpha < \frac{1}{p}, \\[2ex] \dfrac{\frac{n}{p} + \frac{1}{p}(\frac{n}{p} - s)}{\frac{n}{p} - s + 1} & \text{if } \alpha \geq \frac{1}{p}. \end{cases} \tag{31}$$

The case $\alpha \geq 1/p$ is also covered by Theorem 3.

Remark 16. Let us mention once again that more general classes of functions G than those discussed in Examples 1,2 are treated in [Si 3], [Si 4].

A final remark to $1 < s < n/p$.

We have determined an upper bound for the best possible regularity of $G(f)$, denoted by $\varrho(s,p,n,0)$. Moreover, we have shown that periodic functions G lead to composition operators with "bad" mapping properties, reflected by $\varrho(s,p,n,1)$. Restricted to those operators, our results are optimal, cf. the Corollary and Remark 15. This makes it clear that the image space of a Sobolev space, e.g. under the mapping $T_{\sin t}$, is contained in a fractional order Sobolev space in general. In addition we remark that all counterexamples have compact support. This means that turning from spaces defined on $\mathbb{R}^n$ to spaces defined on compact domains does not change the necessary conditions which we have derived in Theorems 2–7. One open problem consists in an improvement of Theorem 1 by adding integrability properties of the function G. A second one is given by the question if there are further necessary conditions in the case $1 \le s \le 1 + 1/p$.

3.5. The composition operator on subspaces.

We have seen that for sufficiently regular G the corresponding operator T_G acts on $\mathbb{A}^s_{p,q} \cap L_\infty$ also in the case $s < n/p$, cf. Theorem 3.3/1. Furthermore, as we know from Theorem 3.4/3, T_G (induced by a nonlinear G) can not act on the whole space $\mathbb{A}^s_{p,q}$, at least if $1 + 1/p < s < n/p$. The aim of this subsection consists in a calculation of larger subspaces of $\mathbb{A}^s_{p,q}$ than $\mathbb{A}^s_{p,q} \cap L_\infty$ such that T_G acts on this subspace for all reasonable G. To begin, we employ Theorem 3.4/5 and Corollary 3.4 to derive some information about the admissible subspaces. For $0 \le r < s$ assume

$$T_G(\mathbb{A}^s_{p,q} \cap \mathbb{A}^r_{t,q}) \subset \mathbb{B}^s_{p,\infty}. \tag{32}$$

for all reasonable G, say $G' \in C^\infty(\mathbb{R})$ satisfying $G(0) = 0$. Of course, $\mathbb{A}^s_{p,q}$ should not be a subspace of $\mathbb{A}^r_{t,q}$, not even locally. That can be expressed by the condition $t > n/(r - s + n/p)$. Because of

$$\mathbb{A}^{r+n(\frac{1}{p}-\frac{1}{t})}_{p,q} \hookrightarrow \mathbb{A}^s_{p,q} \cap \mathbb{A}^r_{t,q}$$

Corollary 3.4 implies that

$$\begin{aligned} s \le \varrho\big(r + n(\tfrac{1}{p} - \tfrac{1}{t}), p, n, 1\big) &= \frac{\frac{n}{p}}{\frac{n}{p} - r - n(\frac{1}{p} - \frac{1}{t}) + 1} \\ \Longrightarrow t &\ge \frac{n}{\frac{n}{ps} + r - 1}. \end{aligned} \tag{33}$$

Putting $r = 1$ we get $t \ge ps$. This motivates us to consider intersections like $\mathbb{A}^s_{p,q} \cap \mathbb{A}^1_{ps,q}$.

Theorem 1. *Let $G' \in C^\infty(\mathbb{R})$ and let $G(0) = 0$. Suppose $s > 1$ and $s \notin \mathbb{N}$. Then for sufficiently small v the operator T_G maps $\mathbb{F}^s_{p,q} \cap \mathbb{F}^1_{ps,v}$ into $\mathbb{F}^s_{p,q}$. Moreover, there exists a constant c such that*

$$\|G(f) \mid F^s_{p,q}\| \le c(\|f \mid F^s_{p,q}\| + \|f \mid F^1_{ps,v}\|^s) \tag{34}$$

holds for all $f \in \mathbb{F}^s_{p,q} \cap \mathbb{F}^1_{ps,v}$.

Remark 1. The proof of this theorem is given in [RS]. It is based on pointwise multiplier assertions for $F^s_{p,q}$-spaces and Theorem 3.1.

The proof also yields a first estimate for which the values of v are sufficient. In the case $q = 2$ one may choose even $v = 2$. Further, if s is a natural number and $q = 2$, then we are in case of ordinary Sobolev spaces and in this situation one can overcome the difficulties appearing in the general case.

Corollary 1. *Let $G' \in C^\infty(\mathbb{R})$ and let $G(0) = 0$. Suppose $s > 1$. Then T_G acts on $\mathbb{H}^s_p \cap \mathbb{H}^1_{ps}$. Moreover, there exists a constant c such that*

$$\|G(f) \mid H^s_p\| \le c(\|f \mid H^s_p\| + \|f \mid H^1_{ps}\|^s) \tag{35}$$

holds for all $f \in \mathbb{H}^s_p \cap \mathbb{H}^1_{ps}$.

Remark 2. The idea to consider T_G on $\mathbb{H}^s_p \cap \mathbb{H}^1_{ps}$ goes back to Adams and Frazier [AF 1], [AF 2]. Corollary 1 has been proved there.

A second corollary is obtained by observing that

$$\|f \mid F^1_{ps,v}\| \le c\|f \mid F^s_{p,q}\|^\theta \|f \mid B^0_{\infty,\infty}\|^{1-\theta}, \quad \theta = \frac{1}{s},$$

cf. [Ru 1], [RS].

Corollary 2. *Let $G' \in C^\infty(\mathbb{R})$ and let $G(0) = 0$. Suppose $s > 1$ and $s \notin \mathbb{N}$. Then T_G acts on $\mathbb{F}^s_{p,q} \cap \mathbb{L}_\infty$. Moreover, there exists a constant c such that*

$$\|G(f) \mid F^s_{p,q}\| \le c(\|f \mid F^s_{p,q}\| + \|f \mid F^s_{p,q}\| \, \|f \mid L_\infty\|^{s-1}) \tag{36}$$

holds for all $f \in \mathbb{F}^s_{p,q} \cap \mathbb{L}_\infty$.

Remark 3. This improves also the estimate stated in Theorem 3.3/1(i).

3.6. Some remarks to Problem 3.

Sometimes composition operators do not lead to loss of smoothness but to a change of integrability properties. A good example of such a mapping consists in $f \to f^m$, $m \in \mathbb{N}$, $m \ge 2$, cf. [Si 1] or more generally in $f \to |f|^\mu$, $\mu > 1$, cf. [Si 1]. Here our aim will be an investigation of smooth and bounded functions G. Then simple examples show that we have to concentrate on the compact case.

Theorem. *Let $G' \in C^\infty(\mathbb{R})$ and let $G(0) = 0$. Suppose that*

$$1 < s < \frac{n}{p} \quad \text{and} \quad s(\frac{n}{p} - s) < n. \tag{37}$$

Let

$$t < \frac{n}{s + s(\frac{n}{p} - s)} \tag{38}$$

and put

$$\eta = \frac{1}{t} - \frac{1}{p}. \tag{39}$$

Then for every $\mu > s$ there exists a constant such that

$$\|G(f) \mid F^s_{t,q}\| \le c(1 + |\operatorname{supp} f|)^\eta (\|f \mid F^s_{p,q}\| + \|f \mid F^s_{p,q}\|^\mu) \tag{40}$$

holds for all compactly supported $f \in \mathbb{F}^s_{p,q}$.

Remark 1. Some technicalities: Of course, the two conditions in (37) are interacting. We give some reformulations. If $n \le 4p^2$, then $s(n/p - s) < n$ is automatically fulfilled. If $n > 4p^2$ then either we must have $s < s_1 = n/(2p) - \sqrt{n^2/(4p^2) - n}$ or $s > s_2 = n/(2p) + \sqrt{n^2/(4p^2) - n}$. Because of $s_2 < n/p$ in any case we end up with the following equivalence

$$(37) \Longleftrightarrow \begin{cases} 1 < s < \frac{n}{p} & \text{if } p \ge \frac{\sqrt{n}}{2}, \\ s_2 < s < \frac{n}{p} & \text{if } p < \frac{\sqrt{n}}{2}, \\ 1 < s < s_1 & \text{if } p < \frac{\sqrt{n}}{2} \text{ and } p > \frac{n}{n+1}. \end{cases} \tag{41}$$

For the interesting dimensions $n = 2, 3$, the condition (37) reduces to $1 \le p < 2$, $1 < s < 2/p$ and $1 \le p < 3$, $1 < s < 3/p$, respectively.

There is an interesting limit case.

Proposition 1. *Let $G' \in C^\infty(\mathbb{R})$ and let $G(0) = 0$. Suppose that (37) holds. We put*

$$t = \frac{n}{s + s(\frac{n}{p} - s)} \quad \text{and} \quad \eta = \frac{(s-1)(\frac{n}{p} - s)}{n}. \tag{42}$$

Then there exists a constant c such that

$$\|G(f) \mid F^s_{t,\infty}\| \le c(1 + |\operatorname{supp} f|)^\eta(\|f \mid F^s_{p,\infty}\| + \|f \mid F^s_{p,\infty}\|^s) \tag{43}$$

holds for all compactly supported $f \in \mathbb{F}^s_{p,\infty}$.

Remark 2. The proof of Theorem and also of Proposition 1 may be found in [RS]. Similar results are obtained in Colin [Co 1], [Co 2].

Remark 3. Observe that $F^s_{t,\infty} \not\subset F^{\varrho(s,p,n,1)}_{p,\infty}$. That means that neither Proposition 1 nor Theorem improve Theorem 3.4/1; they only supplement it.

Because of the preceding remark the next question consists of determination of the best possible value of t.

Proposition 2. *Suppose that* (37) *holds. Suppose further that*

$$t > \frac{n}{s + s(\frac{n}{p} - s)}.$$

Let G be a smooth, periodic and non-constant function. Then there exists a compactly supported function $f_t \in \mathbb{A}^s_{p,q}$ such that $G(f_t)$ does not belong to $\mathbb{B}^s_{t,\infty}$.

Remark 4. The propositions show a good correspondence. Again the periodic functions G lead to composition operators with "bad" mapping properties. Also extensions of Theorem and Proposition 1 to Besov spaces are known, but then with several additional restrictions.

Final remark.

Also the last two subsections indicate that the theory of mapping properties of composition operators in Sobolev spaces of fractional order is just at the beginning. Up to now we have said nothing about extensions to the more general operators

$$(f_1, \dots, f_m) \to G(\cdot, f_1, \dots, f_m),$$

usually called Nemytskii operators. Also continuity and differentiability properties are of interest. Some results in these directions may be found in [RS]. There, in Johnsen and Runst [JR] and in Runst [Ru 2] also applications to nonlinear partial differential equations are discussed.

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