ON THE ADJOINT OF THE MAXIMAL FUNCTION

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1. INTRODUCTION

Let A_p, $p > 1$, denote the class of functions satisfying Muckenhoupt's condition, i.e.,

$$\int_Q w \left(\int_Q w^{1-q} \right)^{p-1} \leq C|Q|^p, \quad p + q = pq.$$

It is clear that $w \in A_p$ if and only if $w^{1-q} \in A_q$. This means that if Mf is the usual maximal operator $Mf(x) = \sup\limits_{x \in Q} \frac{1}{|Q|} \int_Q |f|$, then [Mu]

$$\int (Mf)^p w \leq C \int |f|^p w \quad \forall f \in L_p(w), \tag{1}$$

if and only if

$$\int (Mf)^q w^{1-q} \leq C \int |f|^q w^{1-q} \quad \forall f \in L_q(w^{1-q}), \tag{2}$$

but this is quite surprising because the equivalence of (1) and (2) is natural only for self-adjoint operators.

If instead of A_p we consider the classes A_p^+, i.e., the functions w for which there exists C so that for any $a < b < c$ the following holds:

$$\int_a^b w \left(\int_b^c w^{1-q} \right)^{p-1} \leq C(c-a)^p,$$

then we see that now $w \in A_p^+$ if and only if $w^{1-q} \in A_q^-$ (the definition of A_p^- is the same as A_p^+ except that we reverse the orientation). Since we know that these classes characterize the weights for which the one-sided maximal operators $M^+f(x) = \sup\limits_{h>0} \frac{1}{h} \int_x^{x+h} f$ and $M^-f(x) = \sup\limits_{h>0} \frac{1}{h} \int_{x-h}^{x} f$

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are bounded [Sa], now we have that

$$\int (M^+f)^p w \le C \int |f|^p w \quad \forall f \in L_p(w), \tag{3}$$

if and only if

$$\int (M^-f)^q w^{1-q} \le C \int |f|^q w^{1-q} \quad \forall f \in L_q(w^{1-q}). \tag{4}$$

This means that M^- behaves as the "adjoint" of M^+. Our aim is to prove that, although certainly the operator M is not "self-adjoint", it controls the adjoints of the linearizations of M. We will also prove that for the one-sided operator M^+, what happens is that the adjoints of the linearizations of M^+ are controlled by M^-. This will explain why the maximal operator behaves as a self-adjoint operator, while M^+ behaves as the adjoint of M^-. Throughout the paper C will denote an absolute constant that may change from line to line. By A_∞ we will denote the union of the A_p weights, $p > 1$.

2. Linearizations of the maximal operator

Let ϕ be a smooth non-negative function, radial, decreasing, with support on $\{x \in \mathbb{R}^n : |x| \le 1\}$ and normalized so that $\int \phi = 1$. It is well known [SW] that then the maximal operator

$$M_\phi f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int \Big| \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y) \Big| \, \mathrm{d}y = \sup_{\varepsilon>0} |f| * \phi_\varepsilon(x),$$

is pointwise equivalent to Mf. It is also clear that we can replace the sup in this definition by $\sup\limits_{\varepsilon_i}$, where ε_i runs over a countable family.

Now to each partition $P = (A_i)$, of $\mathbb{R}^n$ we can associate a linear operator $T_P f(x) = \sum_i \chi_{A_i}(x)(f * \phi_{\varepsilon_i})(x)$. It is clear that

$$|T_P f(x)| \le \sum_i \chi_{A_i}(x) |f * \phi_{\varepsilon_i}(x)| \le \sum_i \chi_{A_i} M_\phi f(x) = M_\phi f(x).$$

Conversely, for any $f \ge 0$ there exists a partition $P = (A_i)$ of $\mathbb{R}^N$ such that $M_\phi f(x) \le 2(f * \phi_{\varepsilon_i})(x)$ for any $x \in A_i$, and then

$$M_\phi f = \sum_i \chi_{A_i} M_\phi f \le 2 \sum_i \chi_{A_i} (f * \phi_{\varepsilon_i}) = 2T_P f.$$

From these observations we obtain that (1) is equivalent to

$$\int (T_P|f|)^p w \le C \int |f|^p w \quad \forall f \in L^p(w), \tag{5}$$

where C is a constant independent of the particular linearization.

Let T_P^* be the adjoint of T_P, i.e.,

$$T_P^* g(x) = \sum_i (g\chi_{A_i}) * \phi_{\varepsilon_i}(x).$$

If we can control T_P^* by M in the sense that there exists C, independent of the partition P, such that

$$\int |T_P^* g|^r w \le C \int (Mg)^r w, \quad r > 0,\ w \in A_\infty, \tag{6}$$

then we will have that (1) implies

$$\int |T_P^* g|^p w \le C \int |g|^p w, \tag{7}$$

By duality this is equivalent to

$$\int |T_P f|^q w^{1-q} \le C \int |f|^q w^{1-q},$$

where C does not depend on P. Since this is equivalent to (2), we will have shown that the equivalence between (1) and (2) is due to the fact that, although the maximal operator is not self-adjoint, it controls the adjoints of all its linearizations. To prove (6) we will make use of the sharp maximal function, defined as follows:

$$f^\sharp(x) = \sup_{x\in B} \frac{1}{|B|} \int_B |f - m_B f|,$$

where $m_B f = \frac{1}{|B|}\int_B f$. It is easy to see that in this definition one can replace $m_B f$ by any number C_B depending only on the ball B. The main fact about $f^\sharp$, due to C. Fefferman and E. Stein, is that for weights in A_∞ the following inequality holds,

$$\int (Mf)^r w \le C \int {f^\sharp}^r w, \quad r > 0, \tag{8}$$

and this clearly implies

$$\int |f|^r w \le C \int {f^\sharp}^r w. \tag{9}$$

We will control the adjoint of the linearizations of M_ϕ in the following precise sense.

Theorem 1. *Let T_P^* be the adjoint of some linearization of M_ϕ, then there exists C independent of the particular linearization such that*

$$(T_P^* |g|)^\sharp(x) \le CMg(x)$$

for any x and g.

This result, together with (9), implies (6) as we wanted.

Proof. Let $g \geq 0$. It is enough to prove that for any fixed x and any ball that contains x, $B = B(x_B, r)$, there exists C_B so that

$$\frac{1}{|B|}\int_B |T_P^* g(z) - C_B| \,\mathrm{d}z \leq CMg(x).$$

Let us write $g = g_1 + g_2$ with $g_1 = g\chi_{3B}$, where $3B$ denotes a ball with center x_B and radius $3r$. Let us choose $C_B = T_P^* g_2(x_B)$, then

$$\begin{aligned}\frac{1}{|B|}\int_B &|T_P^* g(z) - C_B| \,\mathrm{d}z \\ &\leq \frac{1}{|B|}\int_B |T_P^* g_1(z)| + \frac{1}{|B|}\int_B |T_P^* g_2(z) - T_P^* g_2(x_B)| \\ &= I_1 + I_2.\end{aligned}$$

For I_1 we simply use the fact that the operator T_P^* is obviously bounded in L^1 and obtain,

$$I_1 \leq \frac{1}{|B|}\int g_1 = \frac{1}{|B|}\int_{3B} g \leq CMg(x).$$

For I_2 we will prove that there exists C such that for any $z \in B$

$$|T_P^* g_2(z) - T_P^* g_2(x_B)| \leq CMg(x).$$

Indeed,

$$\begin{aligned}|T_P^* g_2(z) - T_P^* g_2(x_B)| &= \Big|\sum_i (g_2\chi_{A_i}) * \phi_{\varepsilon_i}(z) - (g_2\chi_{A_i}) * \phi_{\varepsilon_i}(x_B)\Big| \\ &\leq \sum_i \int_{y\in\mathbb{R}^n\setminus 3B} g_2(y)\chi_{A_i}(y)\big|\phi_{\varepsilon_i}(z-y) - \phi_{\varepsilon_i}(x_B - y)\big| \,\mathrm{d}y.\end{aligned} \tag{10}$$

Now our assumptions on the support of ϕ imply, first of all, that only those ε_i that are greater then r give a non zero contribution to this last sum. Furthermore, for each $y \in \mathbb{R}^n \setminus 3B$ one has that $\phi_{\varepsilon_i}(z-y) = \phi_{\varepsilon_i}(x_B - y) = 0$ unless $\varepsilon_i \geq \frac{1}{4}|y-x|$, since otherwise for any $z \in B$

$$\varepsilon_i < \frac{1}{4}|y-x| \leq \frac{1}{4}(|y-z| + |z-x|) \leq \frac{1}{4}(|y-z| + 2r) \leq \frac{1}{4}|y-z| + \frac{\varepsilon_i}{2},$$

and this implies $|y-z|/\varepsilon_i > 2$. If we now use that ϕ is smooth we can dominate our last term in (10) by

$$C\sum_i \int \chi_{A_i}(y) g_2(y) \frac{|z-x_B|}{|y-x|^{n+1}} \,\mathrm{d}y \leq rC\int_{\mathbb{R}^n\setminus 3B} g(y)\frac{1}{|y-x|^{n+1}} \,\mathrm{d}y.$$

This can be dominated by $g * h(x)$ where $h(y) = rC\chi_{\mathbb{R}^n \setminus B(0,2r)}(y)|y|^{-(n+1)}$, and this is dominated by $CMg(x)$ because h is radial, decreasing and integrable. □

Remark. Observe that, in particular, we have proved that the adjoints of the linearizations of the maximal operator map L_∞ into BMO, i.e., near L^1 they behave as a nice convolution operator, but near L^∞ they behave as singular integrals.

3. One-sided operators

The ingredients of the proof in the previous case are the sharp maximal inequality for A_∞ weights in (8), and Theorem 1. For the one-sided case one needs to consider A_∞^+ weights. This class, and the corresponding A_∞^-, were introduced and studied in [MPT]. The lateral version of the sharp maximal function was introduced in [MT], where a substitute of (8) was proved, i.e., for any $w \in A_\infty^+$,

$$\int (M^+ f)^r w \leq C \int \left(f^{\sharp +}\right)^r w,$$

where the lateral sharp function is defined as

$$f^{\sharp +}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f - a_I|,$$

where a_I is a constant that depends only on the interval $I = (x, x+h)$. (Actually in [MT], a smaller sharp function is used, but for our purpose this is enough.) Since the operator M^- is equivalent to

$$M_\phi^- f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)\ \mathrm{d}y = \sup_{\varepsilon>0} f * \phi_\varepsilon(x)$$

with ϕ supported in $[0,1)$ integrable and decreasing, it is enough to prove a lateral version of Theorem 1 but with respect to the adjoint of the linearizations of this operator. Observe that any linearization of M_ϕ^- is of the form $T_P^- f(x) = \sum_i \chi_{A_i}(x)(f * \phi_{\varepsilon_i})(x)$, but due to the lack of symmetry in ϕ, now the adjoint is of the form

$$T_P^* g(x) = \sum_i (g\chi_{A_i}) * \phi_{\varepsilon_i}(x),$$

where ϕ is increasing and supported on $(-1, 0]$.

Theorem 2. *Let T_P^* be the adjoint of some linearization of M_ϕ^-, then there exists C independent of the particular linearization such that*

$$(T_P^*|g|)^{\sharp+}(x) \leq CM^+g(x)$$

for any x and g.

Proof. The proof runs along the same lines as the proof of Theorem 1. For fixed x and $h > 0$ we consider the interval $I = [x, x+h]$, and write $g = g_1 + g_2 + g_3$, where $g_1 = g\chi_{(-\infty,x)}$, $g_2 = g\chi_{[x,x+3h)}$, and choose $a_I = T_P^* g_3(x)$. The fact that ϕ is supported on $(-1, 0]$ implies that $T_P^* g_1(z) = 0$, for $z \in I$, and then

$$\frac{1}{h}\int_I |T_P^* g(z) - a_I| \, \mathrm{d}z \leq \frac{1}{h}\int_I |T_P^* g_2(z)| + \frac{1}{h}\int_I |T_P^* g_3(z) - T_P^* g_3(x)| = I_1 + I_2.$$

For I_1 we use again that our operator is bounded in L^1 and obtain $I_1 \leq \frac{1}{h}\int_x^{x+3h} g \leq CM^+g(x)$. For I_2 the same reasoning as before gives

$$\begin{aligned}
|T_P^* g_3(z) - T_P^* g_3(x)| &= \Big|\sum_i (g_3\chi_{A_i}) * \phi_{\varepsilon_i}(z) - (g_3\chi_{A_i}) * \phi_{\varepsilon_i}(x_B)\Big| \\
&\leq \sum_i \int_{y > x+3h} g(y)\chi_{A_i}(y)\big|\phi_{\varepsilon_i}(z-y) - \phi_{\varepsilon_i}(x_-y)\big| \, \mathrm{d}y, \\
&\leq C\sum_i \int \chi_{A_i}(y) g_3(y) \frac{(z-x)}{(y-x)^2} \, \mathrm{d}y \\
&\leq hC \int_{y > x+3h} g(y) \frac{1}{(y-x)^2} \, \mathrm{d}y \leq CM^+g(x).
\end{aligned}$$

□

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