Extension theory for Sobolev spaces on open sets with Lipschitz boundaries

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Introduction

The main aim of this paper[1] is to give a complete exposition of the extension theory for Sobolev spaces for the case of open sets with Lipschitz boundaries.

Some results are known for more general open sets, and for some particular values of the parameters necessary and sufficient conditions are known ensuring validity of the extension theorem.[2] However, in that case the theory is not complete. On the other hand the case of open sets with Lipschitz boundaries is, and will be, the most important from the point of view of applications. For all these reasons we have restricted ourselves to this case.

The extension theory consists of three interconnected parts. The first, main, part is the construction of an extension operator

$$T : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n), \quad (Tf)(x) = f(x), \qquad x \in \Omega, \tag{1}$$

[1] This work was supported by the grants of European Association for Promotion of Fundamental Research (INTAS - 881 - 94) and Russian Foundation of Basic Research (RFBR - 96 - 01 - 01380).

[2] At the recent conference "Functions spaces, partial differential equations and applications" in Rostock I have promised to pay DM $1,000$ to a mathematician who would give a complete solution of the extension problem for all admissible values of the parameters.

which is linear and bounded, where $W_p^l(\Omega)$ is the Sobolev space and Ω is an open set with a Lipschitz boundary. The second part is the construction of analogous operators for the semi-normed Sobolev spaces and other variants of Sobolev spaces. The third part is dedicated to sharp estimates of the minimal norm of extension operators, mentioned above.

We present here only those proofs, which are related directly to the extension procedure, and omit the proofs of auxiliary statements (giving references, where to find them).

The exposition is based mostly on the papers of the author, his pupils and co-authors [6], [10], [21], [11], [12], [32], [16], [28], [14], [25], [26], [27]. A brief information about other extension methods will also be given. As for further results, including the case of irregular open sets, for which an extension with preservation of smoothness is impossible, we shall give a very brief survey and references.

1 Notation

We shall use the following standard notation for sets:

$\mathbb{N}$ – the set of all natural numbers,
$\mathbb{N}_0$ – the set of all non-negative integers,
$\mathbb{R}$ – the set of all real numbers,
$\mathbb{N}_0^n = \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{n-\text{times}}$ – the set of multi-indices (n is the natural number which will be used exclusively to denote the dimension),
$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}$,
$B(x,r)$ – the open ball of radius $r > 0$ centred at the point $x \in \mathbb{R}^n$.

For $\alpha \in \mathbb{N}_0^n$, $\alpha \neq 0$, we shall write:

$D^\alpha f \equiv \dfrac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ – the (ordinary) derivative of the function f of order α and
$D_w^\alpha f \equiv \left(\dfrac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right)_w$ – the weak derivative of the function f of order α.

For an open nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$C(\Omega)$ – the space of functions continuous on Ω,

$\overline{C}(\Omega)$ – the Banach space of functions f uniformly continuous and bounded on Ω with the norm

$$\|f\|_{C(\Omega)} = \sup_{x\in\Omega} |f(x)|,$$

$C^l(\Omega)$ $(l \in \mathbb{N})$ – the space of functions f defined on Ω such that $\forall\alpha \in \mathbb{N}_0^n$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n = l$, and $\forall x \in \Omega$ the derivatives $(D^\alpha f)(x)$ exist and $D^\alpha f \in C(\Omega)$,

$\overline{C}^l(\Omega)$ $(l \in \mathbb{N})$ – the Banach space of functions $f \in \overline{C}(\Omega)$ such that $\forall\alpha \in \mathbb{N}_0^n$, where $|\alpha| = l$, and $\forall x \in \Omega$ the derivatives $(D^\alpha f)(x)$ exist and $D^\alpha f \in \overline{C}(\Omega)$ with the norm

$$\|f\|_{C^l(\Omega)} = \|f\|_{C(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha f\|_{C(\Omega)},$$

$C^\infty(\Omega) = \bigcap\limits_{l=0}^{\infty} C^l(\Omega)$ – the space of functions infinitely continuously differentiable on Ω,

$C_0^\infty(\Omega)$ – the space of functions in $C^\infty(\Omega)$ compactly supported in Ω.

Further notation will be introduced in the text.

2 Sobolev spaces

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $l \in \mathbb{N}$, $1 \le p \le \infty$. The function f belongs to the *Sobolev space* $W_p^l(\Omega)$ if $f \in L_p(\Omega)$, if it has weak derivatives $D_w^\alpha f$ on Ω for all $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| = l$ and

$$\|f\|_{W_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)} < \infty.$$

Here

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \, dx\right)^{1/p}$$

for $1 \le p < \infty$ and

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Remark 2.1. In the one-dimensional case this definition is equivalent to the following: the function f is equivalent to a function h on Ω, for which the (ordinary) derivative $h^{(l-1)}$ is locally absolutely continuous on Ω and

$$\|f\|_{W_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \|f_w^{(l)}\|_{L_p(\Omega)} = \|h\|_{L_p(\Omega)} + \|h^{(l)}\|_{L_p(\Omega)} < \infty.$$

Moreover, if $\Omega = (a,b)$ is a finite interval, the limits $\lim\limits_{x\to a+} h(x)$ and $\lim\limits_{x\to b-} h(x)$ exist and one may define h on $[a,b]$ by setting $h(a)$ and $h(b)$ to be equal to those limits. Then $h^{(s)}$, $s = 1,\dots,l-1$, exist and $h^{(l-1)}$ is absolutely continuous on $[a,b]$. This follows from the Taylor expansion

$$h^{(s)}(x) = \sum_{k=0}^{l-s-1} \frac{h^{(s+k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{(l-s-1)!}\int_{x_0}^{x} (x-u)^{l-s-1} h^{(l)}(u)\, du,$$

where $x, x_0 \in (a,b)$ and $s = 1,\dots,l-1$. Since $h^{(l)} \in L_p(a,b)$, hence $h^{(l)} \in L_1(a,b)$, the limits $\lim\limits_{x\to a+} h(x)$ and $\lim\limits_{x\to b-} h(x)$ exist. Consequently, the right derivatives $h^{(s)}(a)$ and the left derivatives $h^{(s)}(b)$ exist and $h^{(s)}(a) = \lim\limits_{x\to a+} h^{(s)}(x)$, $h^{(s)}(b) = \lim\limits_{x\to b-} h^{(s)}(x)$. Finally, since $h^{(l-1)}(x) = h^{(l-1)}(x_0) + \int_{x_0}^{x} h^{(l)}(u)\, du$ for all $x, x_0 \in [a,b]$ and $h^{(l)} \in L_1(a,b)$, it follows that $h^{(l-1)}$ is absolutely continuous on $[a,b]$.

Other variants of Sobolev spaces $V_p^l(\Omega)$, $W_p^{l,\dots,l}(\Omega)$ and $V_p^{l,\dots,l}(\Omega)$ are also of interest. They are characterized by the finiteness of the following norms:

$$\|f\|_{V_p^l(\Omega)} = \sum_{|\alpha|\le l} \|D_w^\alpha f\|_{L_p(\Omega)},$$

$$\|f\|_{W_p^{l,\dots,l}(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{n} \left\| \left(\frac{\partial^l f}{\partial x_j^l}\right)_w \right\|_{L_p(\Omega)},$$

$$\|f\|_{V_p^{l,\dots,l}(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{n}\sum_{m=1}^{l} \left\| \left(\frac{\partial^m f}{\partial x_j^m}\right)_w \right\|_{L_p(\Omega)}$$

respectively.

For a wide class of open sets with a quasi-resolved boundary (see definition in Section 5) $W_p^l(\Omega) = V_p^l(\Omega)$ and the norms $\|\cdot\|_{W_p^l(\Omega)}$ and $\|\cdot\|_{V_p^l(\Omega)}$

are equivalent. Moreover, if Ω is a bounded domain with a quasi-resolved boundary, these norms are equivalent to

$$\|f\|_{L_p^l(\Omega)} = \|f\|_{L_1(B)} + \sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)},$$

where $B \subset \Omega$ is an arbitrary ball. (See, for example, [19], Chapter 4.)[3]

Definition 2.2. Let Ω be an open set, $l \in \mathbb{N}$, $1 \le p \le \infty$. The function f belongs to the *semi-normed Sobolev space* $w_p^l(\Omega)$ if $f \in L_1^{\mathrm{loc}}(\Omega)$, if it has weak derivatives $D_w^\alpha f$ on Ω for all $\alpha \in \mathbb{N}_0^n$ satisfying $|\,\alpha\,| = l$ and

$$\|f\|_{w_p^l(\Omega)} = \sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)} < \infty.$$

Another variant of semi-normed Sobolev spaces $w_p^{l,\dots,l}(\Omega)$ is also of interest. It is defined by the finiteness of the semi-norm

$$\|f\|_{w_p^{l,\dots,l}(\Omega)} = \sum_{j=1}^n \left\| \left(\frac{\partial^l f}{\partial x_j^l} \right)_w \right\|_{L_p(\Omega)}.$$

Remark 2.2. The initial idea of S.L. Sobolev was to study the spaces $L_p^l(\Omega)$, defined by the finiteness of the norm $\|\cdot\|_{L_p^l(\Omega)}$, hence to study the properties of functions, for which only the L_p-norms of the weak derivatives of order l are finite, without additional assumptions such as finiteness of the L_p-norm of f in case of the space $W_p^l(\Omega)$ or finiteness of the L_p-norms of all weak derivatives of order less than l in case of the spaces $V_p^l(\Omega)$. One can verify that if $\sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)} < \infty$, then $\|f\|_{L_1(B)} < \infty$ for each ball $B \subset \Omega$. Thus $\|f\|_{L_1(\Omega)}$ is added only to make the space $L_p^l(\Omega)$ a normed space, which is so if Ω is a domain. Moreover in this case for different balls $B \subset \Omega$ the L_p^l-norms are equivalent. (If Ω is a disconnected open set, then $\|\cdot\|_{L_p^l(\Omega)}$ is only a semi-norm.) Thus, the study of the spaces $L_p^l(\Omega)$ is "purely" related to the behaviour of the weak derivatives of order l.

However, as was noted above, the spaces $L_p^l(\Omega)$ differ from the spaces $W_p^l(\Omega)$ or $V_p^l(\Omega)$ only for "very irregular open sets". For this reason usually the spaces $W_p^l(\Omega)$ or $V_p^l(\Omega)$ are considered, just because their definition is simpler in the sense that an additional ball is not involved.

[3] The proof is direct and is based essentially on the one-dimensional inequalities for the norms of intermediate derivatives. Application of extension theorems in this case is not of interest because the statement is valid for such Ω, for which the extension theorem could be invalid.

Still more "pure" variant is the study of the spaces $w_p^l(\Omega)$. However, the fact that they are not normed spaces is sometimes inconvenient. On the other hand in some other cases it is useful to work with semi-norms $\|\cdot\|_{w_p^l(\Omega)}$ themselves.

As for the spaces $W_p^{l,\dots,l}(\Omega)$, $V_p^{l,\dots,l}(\Omega)$ and $w_p^{l,\dots,l}(\Omega)$, they are to a certain extent anisotropic and are particular cases of purely anisotropic spaces $W_p^{l_1,\dots,l_n}(\Omega)$, $V_p^{l_1,\dots,l_n}(\Omega)$ and $w_p^{l_1,\dots,l_n}(\Omega)$, whose definitions involve weak derivatives of different orders l_j with respect to different variables x_j. (They will not be considered in this article.) If Q is a parallelepiped with the faces parallel to the coordinate planes, finite or infinite, then $W_p^{l,\dots,l}(Q) = V_p^{l,\dots,l}(Q) \quad (= V_p^l(Q))$ for $1 \le p \le \infty$ and $W_p^{l,\dots,l}(Q) = W_p^l(Q)$ for $1 < p < \infty$, and the appropriate norms are equivalent. The first statement easily follows from the one-dimensional inequalities for intermediate derivatives (see, for example, [19]), while the second one, which is much more complicated, follows from the Marcinkiewicz multiplier theorem (see, for example, [56]).

If Ω is an arbitrary open set, then from the first statement it follows that for $1 \le p \le \infty$, $\beta \in \mathbb{N}_0$ satisfying $|\beta| < l$ and[4] for each $\delta > 0$,

$$\|D_w^\beta f\|_{L_p(\Omega_\delta)} \le c_\delta \|f\|_{W_p^{l,\dots,l}(\Omega)}, \tag{2}$$

where $c_\delta > 0$ is independent of f.

In a continuation of this paper, for open sets Ω satisfying the cone condition (see definition in Section 5), it will be proved that $W_p^{l,\dots,l}(Q) = V_p^{l,\dots,l}(Q) \quad (= V_p^l(Q))$ for $1 \le p \le \infty$ and $W_p^{l,\dots,l}(Q) = W_p^l(Q)$ for $1 < p < \infty$, and the appropriate norms are equivalent. The proof will be based on the fact, mentioned above, that for $\Omega = \mathbb{R}^n$ these equalities are valid, and on the extension theorem for the spaces $W_p^{l,\dots,l}(\Omega)$ for open sets with a Lipschitz boundary (see definition in Section 5).

We also note that clearly $W_p^l(\Omega) \subset w_p^l(\Omega)$ and $W_p^{l,\dots,l}(\Omega) \subset w_p^{l,\dots,l}(\Omega)$. Moreover, locally these spaces coincide, i.e., for each open set G with compact closure in Ω, $W_p^l(\Omega)\Big|_G = w_p^l(\Omega)\Big|_G$ and $W_p^{l,\dots,l}(\Omega)\Big|_G = w_p^{l,\dots,l}(\Omega)\Big|_G$. In the continuation of this paper it will be proved that for $1 \le p \le \infty$ for bounded open sets satisfying the cone condition, $W_p^l(\Omega) = w_p^l(\Omega)$ and $W_p^{l,\dots,l}(\Omega) = w_p^{l,\dots,l}(\Omega)$. (These equalities are equalities of the sets of functions; the equivalence of appropriate norms and semi-norms is impossible.) Again the proofs will be based on the extension theorems, now for the spaces $w_p^l(\Omega), w_p^{l,\dots,l}(\Omega)$ respectively, for open sets with a Lipschitz boundary.

[4] For $\delta > 0$, we denote $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \delta\}$ and $\Omega^\delta = \bigcup_{x \in \Omega} B(x, \delta)$.

Finally, we recall that, due to the closedness of the weak differentiation, all considered spaces are complete: $W_p^l(\Omega), V_p^l(\Omega), W_p^{l,\dots,l}(\Omega)$, $V_p^{l,\dots,l}(\Omega)$ are Banach spaces[5] and $w_p^l(\Omega)$, $w_p^{l,\dots,l}(\Omega)$ are semi-Banach spaces.

Remark 2.3. The first exposition of the theory of Sobolev spaces was given by S. L. Sobolev himself in his book [60], later an extended exposition was given in his other book [61].

There are several books dedicated directly to different aspects of the theory of Sobolev spaces: R. A. Adams [1], V. G. Maz'ya [48], A. Kufner [45], S. V. Uspenskii, G. V. Demidenko and V. G. Perepelkin [66], V. G. Maz'ya and S. V. Poborchii [53], V. I. Burenkov [19]. In some other books the theory of Sobolev spaces is included into a more general framework of the theory of function spaces: S. M. Nikol'skii [56], O. V. Besov, V. P. Il'in and S. M. Nikol'skii [4], A. Kufner, O. John and S. Fučík [46], E. M. Stein [63], H. Triebel [64], [65]. Moreover, in many other books, especially on the theory of partial differential equations and functional analysis, there are chapters containing exposition of different topics of the theory of Sobolev spaces, adjusted to the aims of those books (we do not name them here). Also throughout the years a number of survey papers were published, containing exposition of the results on the theory of Sobolev spaces. We name some of them: S. L. Sobolev and S. M. Nikol'skii [62], S. M. Nikol'skii [55], V. I. Burenkov [7], O. V. Besov, V. P. Il'in, L. D. Kudryavtsev, P. I. Lizorkin and S. M. Nikol'skii [3], S. K. Vodop'yanov, V. M. Gol'dshtein and Yu. G. Reshetnyak [68], L. D. Kudryavtsev and S. M. Nikol'skii [44], V. G. Maz'ya [48].

3 General applications of extension theorems

The existence of a bounded extension operator (1) ensures that a number of properties of the space $W_p^l(\mathbb{R}^n)$ or $W_p^l(G)$, where G is a ball or a cube, are inherited by the space $W_p^l(\Omega)$ (in the last case $\Omega \subset G$). In this section we shall prove and discuss a series of simple statements showing possible applications of different variants of extension theorems. We start with the simplest possible case.

Lemma 3.1. *Let $l \in \mathbb{N}$, $\beta \in \mathbb{N}_0^n$, $|\beta| \le l$, $1 \le p, q \le \infty$. Suppose that $\forall f \in W_p^l(\mathbb{R}^n)$ the inequality*

$$\|D_w^\beta f\|_{L_q(\mathbb{R}^n)} \le c_1 \|f\|_{W_p^l(\mathbb{R}^n)}, \tag{3}$$

[5] As usual, saying a "Banach space", we ignore here the fact that the condition $\|f\| = 0$ is equivalent to the condition $f \sim 0$ on Ω and not to the condition $f = 0$ on Ω.

where $c_1 > 0$ is independent of f, is valid. Let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a bounded extension operator (1)*.*

Then there exists $c_2 > 0$ such that $\forall f \in W_p^l(\Omega)$,

$$\|D_w^\beta f\|_{L_q(\Omega)} \leq c_2 \|f\|_{W_p^l(\Omega)}. \tag{4}$$

Proof. Since T is an extension operator, we have

$$\begin{aligned}\|D_w^\beta f\|_{L_q(\Omega)} &\leq \|D_w^\beta (Tf)\|_{L_q(\mathbb{R}^n)} \leq c_1 \|Tf\|_{W_p^l(\mathbb{R}^n)} \\ &\leq c_1 \|T\|_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)} \|f\|_{W_p^l(\Omega)} = c_2 \|f\|_{W_p^l(\Omega)}.\end{aligned}$$

□

The above simple argument allows obtaining a more profound estimate for the norm of an extension operator from below.

For $l, n \in \mathbb{N}, 1 \leq p \leq \infty$, let[6]

$$M_{l,n,p} = \{(\beta, q) : \beta \in \mathbb{N}_0^n,\ |\beta| < l,\ 1 \leq q \leq \infty \text{ and (3) is valid}\}$$

and let $C^*(\mathbb{R}^n, n, l, \beta, p, q)$ and $C^*(\Omega, n, l, \beta, p, q)$ be the sharp (minimal possible) values of c_1, c_2 respectively.

Lemma 3.2. *Let $l \in \mathbb{N}$, $\beta \in \mathbb{N}_0^n$, $|\beta| \leq l$, $1 \leq p, q \leq \infty$. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and that inequality* (4) *holds.*

If inequality (3) *does not hold, then a bounded extension operator* (1) *does not exist.*

If inequality (3) *holds and there exists an extension operator* (1)*, then*

$$\|T\|_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)} \geq \sup_{(\beta,q) \in M_{l,n,p}} \frac{C^*(\Omega, n, l, \beta, p, q)}{C^*(\mathbb{R}^n, n, l, \beta, p, q)}.$$

Proof. Since inequality (3) is valid with $c_1 = C^*(\mathbb{R}^n, n, l, \beta, p, q)$, from the proof of Lemma 3.1 it follows that

$$\|D_w^\beta f\|_{L_q(\Omega)} \leq C^*(\mathbb{R}^n, n, l, \beta, p, q) \|T\|_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)} \|f\|_{W_p^l(\Omega)}.$$

Hence,

$$C^*(\Omega, n, l, \beta, p, q) \leq C^*(\mathbb{R}^n, n, l, \beta, p, q) \|T\|_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)},$$

and the desired inequality follows. □

[6] It is well known that $(\beta, q) \in M_{l,n,p}$ if, and only if, $1 \leq q \leq \infty$ if $1 \leq p \leq \infty$ and $|\beta| < l - n/p$ or $p = 1$ and $|\beta| \leq l - n$, $1 \leq q < \infty$ if $1 < p < \infty$ and $|\beta| = l - n/p$ and $1 \leq q \leq pn/(n - pl)$ if $1 \leq p < \infty$ and $|\beta| > l - n/p$.

Lemma 3.2 will be used in Section 4 for establishing sharp estimates for the minimal norm of an extension operator.

Next we note that, by the properties of the spaces $L_q(\Omega)$ and $W_p^l(\Omega)$, the class of open sets in Lemma 3.1 can be widened and, starting with open sets for which a bounded extension operator (1) exists, inequality (4) can be proved for some open sets for which an extension operator (1) does not exist.

Lemma 3.3. *Let $l \in \mathbb{N}$, $\beta \in \mathbb{N}_0^n$, $|\beta| \le l$, $1 \le p, q \le \infty$. Suppose that inequality* (3) *is valid and let $\Omega = \bigcup_{k=1}^{s} \Omega_k$, where $s \in \mathbb{N}$ or $s = \infty$ if $p \le q$ and $s \in \mathbb{N}$ if $q < p$, and Ω_k are open sets such that in the case $s = \infty$ the multiplicity of the covering $\{\Omega_k\}_{k=1}^{\infty}$ is finite.*

Suppose that for each $k = \overline{1, s}$ there exists a bounded extension operator $T_k : W_p^l(\Omega_k) \to W_p^l(\mathbb{R}^n)$. If $s = \infty$, let also $\sup_{k \in \mathbb{N}} \|T_k\| < \infty$.

Then inequality (4) *is also valid.*

Proof. By the properties of the weak derivatives $D_w^\beta f$ exists on Ω. Furthermore, if $q < \infty$, then by (3) and Jensen's or Hölder's inequality for sums,

$$\begin{aligned}
\|D_w^\beta f\|_{L_q(\Omega)} &\le \left(\sum_{k=1}^{s} \int_{\Omega_k} |D_w^\beta f|^q \, dx \right)^{1/q} \\
&\le \left(\sum_{k=1}^{s} \int_{\mathbb{R}^n} |D_w^\beta (T_k f)|^q \, dx \right)^{1/q} \le c_1 \left(\sum_{k=1}^{s} \|T_k f\|_{W_p^l(\mathbb{R}^n)}^q \right)^{1/q} \\
&\le c_1 M_1 \left(\sum_{k=1}^{s} \|T_k f\|_{W_p^l(\mathbb{R}^n)}^p \right)^{1/p} \le c_1 M_1 \left(\sum_{k=1}^{s} \|T_k\|^p \|f\|_{W_p^l(\Omega_k)}^p \right)^{1/p} \\
&\le c_1 M_1 \sup_{k \in \mathbb{N}} \|T_k\| \left(\sum_{k=1}^{s} \|f\|_{W_p^l(\Omega_k)}^p \right)^{1/p},
\end{aligned}$$

where $M_1 = 1$ if $p \le q$ and $M_1 = s^{1/q - 1/p}$ if $q < p$.

Let $\varkappa$ be the multiplicity of the covering $\{\Omega_k\}_{k=1}^{s}$. Then

$$\begin{aligned}
\left(\sum_{k=1}^{s} \|f\|_{W_p^l(\Omega_k)}^p \right)^{1/p} &\le l^n \left(\sum_{k=1}^{s} \int_{\Omega_k} |f|^p \, dx + \sum_{|\alpha|=l} \sum_{k=1}^{s} \int_{\Omega_k} |D_w^\alpha f|^p \, dx \right)^{1/p} \\
&\le l^n \varkappa^{1/p} \left(\int_{\Omega} |f|^p \, dx + \sum_{|\alpha|=l} \int_{\Omega} |D_w^\alpha f|^p \, dx \right)^{1/p} \\
&\le l^n \varkappa^{1/p} \|f\|_{W_p^l(\Omega)},
\end{aligned}$$

and the statement of the lemma follows. $\square$

Lemma 3.4. *Let $l \in \mathbb{N}$, $m \in \mathbb{N}_0$, $m < l$, $1 \le p, q \le \infty$ and let $\Omega = \bigcup_{k=1}^{s} \Omega_k$, where $s \in \mathbb{N}$ and Ω_k are bounded open sets for which there exist bounded extension operators $T_k : W_p^l(\Omega_k) \to W_p^l(\mathbb{R}^n)$.*

Suppose that the embedding $W_p^l(Q) \subset L_q(Q)$ is compact for each cube Q, whose faces are parallel to the coordinate planes. Then the embedding $W_p^l(\Omega) \subset L_q(\Omega)$ is also compact.

Proof. As in Lemma 3.3, it is enough to prove that the embedding $W_p^l(\Omega_k) \subset W_q^m(\Omega_k)$ is compact for each $k = \overline{1, s}$. Let the cube Q_k, whose faces are parallel to coordinate planes, be such that $\Omega_k \subset Q_k$, and I_{Q_k} and I_{Ω_k} be the embedding operators, corresponding to the embeddings $W_p^l(Q_k) \subset W_q^m(Q_k)$, $W_p^l(\Omega_k) \subset W_q^m(\Omega_k)$ respectively, i.e., say, $I_{Q_k} : W_p^l(Q_k) \to L_q(Q_k)$ and $I_{Q_k} f = f$ for $f \in W_p^l(Q_k)$. In order to prove that the operator I_{Q_k} is compact we note that

$$W_p^l(\Omega) \xrightarrow{T_k} W_p^l(\mathbb{R}^n) \xrightarrow{R_{1k}} W_p^l(Q_k) \xrightarrow{I_{Q_k}} L_q(Q_k) \xrightarrow{R_{2k}} L_q(\Omega_k),$$

where $R_{1k} : W_p^l(\mathbb{R}^n) \to W_p^l(Q_k)$ and $R_{2k} : L_q(Q_k) \to L_q(\Omega_k)$ are the restriction operators. Thus,

$$I_{\Omega_k} = R_{2k} I_{Q_k} R_{1k} T_k.$$

Since the operators T_k, R_{1k} and R_{2k} are bounded and the operator I_{Q_k} is compact, it follows that the operator I_{Ω_k} is compact. □

Next let us define a variant of (p, l)-capacity. For an open set $\Omega \subset \mathbb{R}^n$ we put

$$c_{p,l}(\Omega) = \inf\{\|f\|^p_{W_p^l(\mathbb{R}^n)} : f \in W_p^l(\mathbb{R}^n),\ f = 1 \text{ on } \Omega\}.$$

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure and let $l \in \mathbb{N}$, $1 \le p < \infty$.*

Then for each extension operator (1),

$$\|T\|_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)} \ge \left(\frac{c_{p,l}(\Omega)}{\text{meas}\, \Omega} \right)^{1/p}. \tag{5}$$

Proof. Since

$$c_{p,l}(\Omega) \le \|T(1)\|^p_{W_p^l(\mathbb{R}^n)} \le \|T\|^p_{W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)} \|1\|^p_{W_p^l(\Omega)},$$

(5) follows. □

Inequality (5) may be applied both for establishing estimates from above for (p,l)-capacity and for estimates from below for the norm of an extension operator. In [51], [52], [42] it was used for obtaining sharp estimates from below of the minimal norm of an extension operator.

Lemma 3.6. *Let $l \in \mathbb{N}$ and let $\Omega = \bigcup_{k=1}^{s} \Omega_k$, where $s \in \mathbb{N}$ or $s = \infty$ and Ω_k are open sets such that in the case $s = \infty$ the multiplicity of the covering $\{\Omega_k\}_{k=1}^{\infty}$ is finite and for each $k = \overline{1,s}$ there exists a bounded extension operator $T_k : W_p^{l,\dots,l}(\Omega_k) \to W_p^{l,\dots,l}(\mathbb{R}^n)$. If $s = \infty$, let also $\sup\limits_{k\in\mathbb{N}} \|T_k\| < \infty$.*

Then for $1 \le p \le \infty$ $W_p^{l,\dots,l}(\Omega) = V_p^{l,\dots,l}(\Omega)$ and for $1 < p < \infty$ $W_p^{l,\dots,l}(\Omega) = W_p^l(\Omega)$.

Proof. Let $1 < p < \infty$, $f \in W_p^{l,\dots,l}(\Omega)$ and let $\beta \in \mathbb{N}_0^n$ satisfy $|\beta| = l$. By the properties of weak derivatives $D_w^\beta f$ exists on Ω. Furthermore, since (see Section 2) $W_p^{l,\dots,l}(\mathbb{R}^n) = W_p^l(\mathbb{R}^n)$ and the norms are equivalent, we have

$$\|D_w^\beta f\|_{L_p(\Omega)}^p \le \sum_{k=1}^{s} \int_{\Omega_k} |D_w^\beta f|^p\,dx \le \sum_{k=1}^{s} \int_{\mathbb{R}^n} |D_w^\beta (T_k f)|^p\,dx$$

$$\le M_1 \sum_{k=1}^{s} \|T_k f\|_{W_p^{l,\dots,l}(\mathbb{R}^n)}^p \le M_1 \sum_{k=1}^{s} \|T_k\|^p\, \|f\|_{W_p^{l,\dots,l}(\Omega_k)}^p$$

$$\le M_1 \sup_{k=\overline{1,s}} \|T_k\|^p \sum_{k=1}^{s} \|f\|_{W_p^{l,\dots,l}(\Omega_k)}^p$$

$$\le M_1 (n+1)^{p-1} \sup_{k=\overline{1,s}} \|T_k\|^p \left(\sum_{k=1}^{s} \int_{\Omega_k} |f_k|^p\,dx + \sum_{j=1}^{n} \sum_{k=1}^{s} \int_{\Omega_k} \left| \left(\frac{\partial^l f}{\partial x_j^l} \right)_w \right|^p dx \right)$$

$$\le M_2\, \varkappa \sup_{k=\overline{1,s}} \|T_k\|^p \left(\int_{\Omega} |f|^p\,dx + \sum_{j=1}^{n} \int_{\Omega} \left| \left(\frac{\partial^l f}{\partial x_j^l} \right)_w \right|^p dx \right)$$

$$\le M_2 \varkappa \sup_{k=\overline{1,s}} \|T_k\|^p \|f\|_{W_p^{l,\dots,l}(\Omega)}^p,$$

where $M_1 > 0$ and $M_2 = M_1(n+1)^{p-1}$ depend only on n, l and p.

Hence the second statement of the lemma follows. The first statement is proved in a similar way. (If $p = \infty$, then the proof is simpler and there is no necessity to suppose in the case $s = \infty$ that the multiplicity of the covering $\{\Omega_k\}_{k=1}^{\infty}$ is finite.) $\square$

Lemma 3.7. 1. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$ and let $\Omega = \bigcup_{k=1}^{s} \Omega_k$, where $s \in \mathbb{N}$ and Ω_k are bounded open sets such that for each $k = \overline{1,s}$ and for some*

$\delta_k > 0$ there exists an extension operator $T_k : w_p^l(\Omega_k) \to w_p^l((\Omega_k)^{\delta_k})$. Then $W_p^l(\Omega) = w_p^l(\Omega)$.
2. The statement 1 is also valid if we replace $w_p^l(\cdot)$, $W_p^l(\cdot)$ by $w_p^{l,\dots,l}(\cdot)$, $W_p^{l,\dots,l}(\cdot)$ respectively.

Proof. Let $f \in w_p^l(\Omega)$. Since $T_k f \in w_p^l((\Omega_k)^{\delta_k})$ and $\Omega_k \subset ((\Omega_k)^{\delta_k})_{\delta_k}$, by (2) we have $f = T_k f \in L_p(\Omega_k)$. Hence $f \in L_p(\Omega)$ and the statement of the lemma follows. □

Next we give several examples showing the advantage of constructing of an extension operator which is bounded both as

$$T : L_p(\Omega) \to L_p(\mathbb{R}^n) \quad \text{and} \quad T : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n). \tag{6}$$

First we note that from inequality (3), where $1 \le p = q \le \infty$ and $\beta \in \mathbb{N}_0^n$ satisfies $|\beta| \le l-1$, it follows that for all $\varepsilon > 0$ and $\forall f \in W_p^l(\mathbb{R}^n)$

$$\|D_w^\beta f\|_{L_p(\mathbb{R}^n)} \le c_1 \varepsilon^{-|\beta|/(l-|\beta|)} \|f\|_{L_p(\mathbb{R}^n)} + \varepsilon \|f\|_{w_p^l(\mathbb{R}^n)}$$

and

$$\|D_w^\beta f\|_{L_p(\mathbb{R}^n)} \le c_3 \|f\|_{L_p(\mathbb{R}^n)}^{1-|\beta|/l} \|f\|_{w_p^l(\mathbb{R}^n)}^{|\beta|/l},$$

where $c_3 = c_1^{1-|\beta|/l} \left(\frac{|\beta|}{l}\right)^{-|\beta|/l} \left(1 - \frac{|\beta|}{l}\right)^{-(1-|\beta|/l)}$.

Lemma 3.8. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $\beta \in \mathbb{N}_0^n$ satisfy $|\beta| \le l-1$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a bounded extension operator (6). Then*
1) there exists $c_4 > 0$ such that for all $\varepsilon > 0$, $\forall f \in W_p^l(\Omega)$,

$$\|D_w^\beta f\|_{L_p(\Omega)} \le c_4 \varepsilon^{-|\beta|/(l-|\beta|)} \|f\|_{L_p(\Omega)} + \varepsilon \|f\|_{W_p^l(\Omega)},$$

2) for each $\varepsilon_0 > 0$ there exists $c_5 = c_5(\varepsilon_0) > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$, $\forall f \in W_p^l(\Omega)$,

$$\|D_w^\beta f\|_{L_p(\Omega)} \le c_5 \varepsilon^{-|\beta|/(l-|\beta|)} \|f\|_{L_p(\Omega)} + \varepsilon \|f\|_{w_p^l(\Omega)},$$

3) there exists $c_6 > 0$ such that $\forall f \in W_p^l(\Omega)$

$$\|D_w^\beta f\|_{L_p(\Omega)} \le c_6 \|f\|_{L_p(\Omega)}^{1-|\beta|/l} \|f\|_{W_p^l(\Omega)}^{|\beta|/l}.$$

Proof. Let $\|T\|_0 = \|T\|_{L_p(\Omega)\to L_p(\mathbb{R}^n)}$ and $\|T\|_l = \|T\|_{W_p^l(\Omega)\to W_p^l(\mathbb{R}^n)}$.
1) By (6) for all $\delta > 0$,

$$\begin{aligned}\|D_w^\beta f\|_{L_p(\Omega)} &\le \|D_w^\beta (Tf)\|_{L_p(\mathbb{R}^n)} \\ &\le c_1 \delta^{-\frac{|\beta|}{l-|\beta|}} \|Tf\|_{L_p(\mathbb{R}^n)} + \delta \|Tf\|_{W_p^l(\mathbb{R}^n)} \\ &\le c_1 \delta^{-\frac{|\beta|}{l-|\beta|}} \|T\|_0 \|f\|_{L_p(\Omega)} + \delta \|T\|_l \|f\|_{W_p^l(\Omega)}.\end{aligned}$$

Setting $\delta\|T\|_l = \varepsilon$, we obtain that for all $\varepsilon > 0$,

$$\|D_w^\beta f\|_{L_p(\Omega)} \le \left(c_1 \|T\|_0 \|T\|_l^{|\beta|/(l-|\beta|)}\right) \varepsilon^{-|\beta|/(l-|\beta|)} \|f\|_{L_p(\Omega)} + \varepsilon \|f\|_{W_p^l(\Omega)}.$$

2) Also for $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned}\|D_w^\beta f\|_{L_p(\Omega)} &\le \left(c_1 \|T\|_0 \|T\|_l^{|\beta|/(l-|\beta|)} \varepsilon^{-|\beta|/(l-|\beta|)} + \varepsilon\right) \|f\|_{L_p(\Omega)} + \varepsilon \|f\|_{w_p^l(\Omega)} \\ &\le \left(c_1 \|T\|_0 \|T\|_l^{|\beta|/(l-|\beta|)} + \varepsilon_0^{l/(l-|\beta|)}\right) \varepsilon^{-|\beta|/(l-|\beta|)} \|f\|_{L_p(\Omega)} + \varepsilon \|f\|_{w_p^l(\Omega)}.\end{aligned}$$

3) Finally,

$$\begin{aligned}\|D_w^\beta f\|_{L_p(\Omega)} &\le \|D_w^\beta (Tf)\|_{L_p(\mathbb{R}^n)} \le c_3 \|Tf\|_{L_p(\mathbb{R}^n)}^{1-|\beta|/l} \|Tf\|_{w_p^l(\mathbb{R}^n)}^{|\beta|/l} \\ &\le c_3 \|T\|_0^{1-|\beta|/l} \|T\|_l^{|\beta|/l} \|f\|_{L_p(\Omega)}^{1-|\beta|/l} \|f\|_{W_p^l(\Omega)}^{|\beta|/l}.\end{aligned}$$

□

Let us consider the K-functional

$$K(t, f, \Omega) = \inf_{g \in W_p^l(\Omega)} \left(\|g\|_{L_p(\Omega)} + t \|f - g\|_{W_p^l(\Omega)} \right),$$

where $t > 0$, $l \in \mathbb{N}$, $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ is an arbitrary open set. Let $\big(L_p(\Omega), W_p^l(\Omega)\big)_{\theta,q}$, where $0 < q < \infty$, $1 \le \theta \le \infty$, be an interpolation space with the norm

$$\|f\|_{(L_p(\Omega), W_p^l(\Omega))_{\theta,q}} = \left(\int_0^\infty (t^{-q} K(t, f, \Omega))^\theta \, \frac{dt}{t} \right)^{1/\theta}$$

if $1 \le \theta < \infty$ and

$$\|f\|_{(L_p(\Omega), W_p^l(\Omega))_{\infty,q}} = \sup_{t>0} (t^{-q} K(t, f, \Omega)),$$

if $\theta = \infty$. $((L_p(\Omega), W_p^l(\Omega))_{\theta,q}$ is the closure of $W_p^l(\Omega)$ with respect to this norm.)

Lemma 3.9. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a linear bounded extension operator* (6).

Then for all $0 < m < l$, $1 \le \theta \le \infty$

$$T : (L_p(\Omega), W_p^l(\Omega))_{\theta,m} \to (L_p(\mathbb{R}^n), W_p^l(\mathbb{R}^n))_{\theta,m}$$

and

$$\|T\|_{(L_p(\Omega), W_p^l(\Omega))_{\theta,m} \to (L_p(\mathbb{R}^n), W_p^l(\mathbb{R}^n))_{\theta,m}} \le \max(\|T\|_0, \|T\|_l).$$

Proof. Since the operator T is linear,

$$\begin{aligned} K(t, Tf, \mathbb{R}^n) &\le \|Tg\|_{L_p(\mathbb{R}^n)} + t\,\|Tf - Tg\|_{W_p^l(\mathbb{R}^n)} \\ &= \|Tg\|_{L_p(\mathbb{R}^n)} + t\,\|T(f-g)\|_{W_p^l(\mathbb{R}^n)} \\ &\le \|T\|_0\,\|g\|_{L_p(\Omega)} + t\,\|T\|_l\,\|f-g\|_{W_p^l(\Omega)} \\ &\le \max(\|T\|_0, \|T\|_l)\,(\|g\|_{L_p(\Omega)} + t\,\|f-g\|_{W_p^l(\Omega)}). \end{aligned}$$

Hence,

$$K(t, Tf, \mathbb{R}^n) \le \max(\,\|T\|_0,\, \|T\|_l\,) K(t, f, \Omega).$$

Consequently,

$$\|Tf\|_{(L_p(\mathbb{R}^n), W_p^l(\mathbb{R}^n))_{\theta,m}} \le \max\,(\,\|T\|_0,\, \|T\|_l\,)\,\|f\|_{(L_p(\Omega), W_p^l(\Omega))_{\theta,m}}.$$

$\square$

It is well known (see, for example, [64]) that $(L_p(\mathbb{R}^n), W_p^l(\mathbb{R}^n))_{\theta,m}$, where $1 \le p, \theta \le \infty, 0 < m < l$, coincides with the Nikol'skii-Besov space $B_{p,\theta}^m(\mathbb{R}^n)$ defined by the finiteness of the norm

$$\|f\|_{B_{p,\theta}^m(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \left(\frac{\|\Delta_h^\sigma f\|_{L_p(\mathbb{R}^n)}}{|h|^m}\right)^\theta \frac{dh}{|h|^n}\right)^{1/\theta},$$

where $\sigma \in \mathbb{N}, \sigma > m$ and $\Delta_h^\sigma f$ is the difference of order σ of the function f with step $h \in \mathbb{R}^n$. If $\theta = \infty$, then $(\int_{\mathbb{R}^n} (\cdot)^\theta |h|^{-n})\,dh^{1/\theta}$ should be replaced by $\sup_{h \in \mathbb{R}^n, h \ne 0}(\cdot)$.

The definition is easily extended to open sets $\Omega \subset \mathbb{R}^n$: one should replace $\|\cdot\|_{L_p(\mathbb{R}^n)}$ by $\|\cdot\|_{L_p((\Omega)_{\sigma|h|})}$.

Lemma 3.10. *Let $m > 0$, $1 \le p, \theta \le \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a linear extension operator T which is bounded as $T : B^m_{p,\theta}(\Omega) \to B^m_{p,\theta}(\mathbb{R}^n)$ and, for some $l \in \mathbb{N}$, $l > m$, is bounded in the sense* (6).

Then $(L_p(\Omega), W^l_p(\Omega))_{\theta,m} = B^m_{p,\theta}(\Omega)$ and the norms are equivalent.

Proof. 1. Let $f \in W^l_p(\Omega)$ and $t > 0$. Then for each $g \in W^l_p(\mathbb{R}^n)$,

$$K(t, f, \Omega) \le \|g\|_{L_p(\Omega)} + t\,\|f - g\|_{W^l_p(\Omega)} \le \|g\|_{L_p(\mathbb{R}^n)} + t\,\|Tf - g\|_{W^l_p(\mathbb{R}^n)}.$$

Hence,

$$K(t, f, \Omega) \le K(t, Tf, \mathbb{R}^n).$$

Consequently, say for $1 \le \theta < \infty$,

$$\begin{aligned}
\|f\|_{(L_p(\Omega), W^l_p(\Omega))_{\theta,m}} &= \left(\int_0^\infty \left(t^{-m} K(t, f, \Omega)\right)^\theta \frac{dt}{t}\right)^{1/\theta} \\
\le \left(\int_0^\infty \left(t^{-m} K(t, Tf, \mathbb{R}^n)\right)^\theta \frac{dt}{t}\right)^{1/\theta} &= \|T\|_{(L_p(\mathbb{R}^n), W^l_p(\mathbb{R}^n))_{\theta,m}} \\
\le\; M_1\,\|Tf\|_{B^m_{p,\theta}(\mathbb{R}^n)} \;\le\; M_1\,\|T\|_{B^m_{p,\theta}(\Omega) \to B^m_{p,\theta}(\mathbb{R}^n)} \|f\|_{B^m_{p,\theta}(\Omega)},
\end{aligned}$$

where $M_1 > 0$ is independent of f.

2. On the other hand $\forall g \in W^l_p(\Omega)$,

$$\begin{aligned}
\|f\|_{B^m_{p,\theta}(\Omega)} &\le \|Tf\|_{B^m_{p,\theta}(\mathbb{R}^n)} \le M_2\,\|Tf\|_{(L_p(\mathbb{R}^n), W^l_p(\mathbb{R}^n))_{\theta,m}} \\
&= M_2 \left(\int_0^\infty \left(t^{-m} K(t, f, \mathbb{R}^n)\right)^\theta \frac{dt}{t}\right)^{1/\theta} \\
&\le M_2 \max(\|T\|_0,\, \|T\|_l) \left(\int_0^\infty \left(t^{-m} K(t, f, \Omega)\right)^\theta \frac{dt}{t}\right)^{1/\theta} \\
&= M_2 \max(\|T\|_0,\, \|T\|_l)\, \|f\|_{(L_p(\Omega), W^l_p(\Omega))_{\theta,m}},
\end{aligned}$$

where $M_2 > 0$ is independent of f. □

Sometimes the following corollary of Lemma 3.10 is useful.

Lemma 3.11. *Let $m > 0$, $l \in \mathbb{N}$, $l > m$, $1 \le p, \theta \le \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a linear extension operator T bounded in sense* (6).

If there exists a bounded linear extension operator

$$T_1 : B^m_{p,\theta}(\Omega) \to B^m_{p,\theta}(\mathbb{R}^n),$$

then also

$$T : B^m_{p,\theta}(\Omega) \to B^m_{p,\theta}(\mathbb{R}^n)$$

and is bounded.

Proof. Let $f \in W^l_p(\Omega) \subset B^m_{p,\theta}(\Omega)$. Then by Lemmas 3.9, 3.10,

$$\begin{aligned}\|Tf\|_{B^m_{p,\theta}(\mathbb{R}^n)} &\le M_1 \,\|Tf\|_{(L_p(\mathbb{R}^n),W^l_p(\mathbb{R}^n))_{\theta,m}} \\ &\le M_1 \max(\|T\|_0, \|T\|_l)\,\|f\|_{(L_p(\Omega),W^l_p(\Omega))_{\theta,m}} \le M_2\,\|f\|_{B^l_{p,\theta}(\Omega)},\end{aligned}$$

where M_1, M_2 are independent of f. □

Finally, we give some comments about applications of extension operators which are bounded as

$$T : L_p(\Omega) \to L_p(\mathbb{R}^n) \quad \text{and} \quad T : w^l_p(\Omega) \to w^l_p(\mathbb{R}^n). \tag{7}$$

First we note that if Ω is bounded or unbounded and such that for some $\alpha_0 \in \mathbb{N}^n_0$ satisfying $|\alpha_0| = l-1$, $\|x^{\alpha_0}\|_{L_p(\Omega)} < \infty$, then operators T satisfying (7) do not exist. Indeed, if it were so, then by the same argument as in the proof of Lemma 3.8 one could prove that

$$\|D^\beta_w f\|_{L_p(\Omega)} \le \, c_7\,\|f\|^{1-|\beta|/l}_{L_p(\Omega)}\;\|f\|^{|\beta|/l}_{w^l_p(\Omega)}, \tag{8}$$

where $c_7 > 0$ is independent of f. Taking $f(x) = x^{\alpha_0}$ (by the above assumptions $x^{\alpha_0} \in W^l_p(\Omega)$), we arrive to a contradiction.

The analogue of Lemma 3.7 has the form:

Lemma 3.12. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $\beta \in \mathbb{N}^n_0$ satisfy $|\beta| \le l-1$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that there exists a bounded extension operator* (7)*. Then*
1) *there exists $c_8 > 0$ such that for all $\varepsilon > 0$, $\forall f \in W^l_p(\Omega)$,*

$$\|D^\beta_w f\|_{L_p(\Omega)} \le \, c_8\,\varepsilon^{-|\beta|/(l-|\beta|)}\,\|f\|_{L_p(\Omega)} + \varepsilon\,\|f\|_{w^l_p(\Omega)},$$

2) *there exists $c_7 > 0$ such that $\forall f \in W^l_p(\Omega)$ inequality* (8) *is valid.*

The proof is essentially the same as that of Lemma 3.8.

4 The one-dimensional case

We start with the simplest case of Sobolev spaces $W_p^l(a,b)$, in which it is possible to give sharp two-sided estimates of the minimal norm of an extension operator $T: W_p^l(a,b) \longrightarrow W_p^l(-\infty,\infty)$.

Lemma 4.1. *Let $-\infty < a < b < c < \infty$. If f is defined on $[a,c]$ and is absolutely continuous on $[a,b]$ and $[b,c]$, then f is absolutely continuous on $[a,c]$.*

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite system of disjoint intervals $(\alpha_i^{(1)},\beta_i^{(1)}) \subset [a,b]$ and $(\alpha_i^{(2)},\beta_i^{(2)}) \subset [b,c]$ satisfying the inequalities $\sum\limits_i(\beta_i^{(j)} - \alpha_i^{(j)}) < \delta,\ j = 1,2$, the inequalities $\sum\limits_i |f(\alpha_i^{(j)}) - f(\beta_i^{(j)})| < \varepsilon/2,\ j = 1,2$, hold. Now let $(\alpha_i,\beta_i) \subset [a,c]$ be a finite system of disjoint intervals satisfying $\sum\limits_i(\beta_i - \alpha_i) < \delta$. If one of them contains b, denote it by (α^*,β^*) . Then

$$\begin{aligned}\sum_i |f(\alpha_i) - f(\beta_i)| \le & \sum_{i:\,(\alpha_i,\beta_i)\subset[a,b]} |f(\alpha_i) - f(\beta_i)| + |f(\alpha^*) - f(b)| \\ & + |f(b) - f(\beta^*)| + \sum_{i:\,(\alpha_i,\beta_i)\subset[b,c]} |f(\alpha_i) - f(\beta_i)| \\ & < \varepsilon.\end{aligned}$$

(If there is no such interval (α^*,β^*), then the summands $|f(\alpha^*) - f(\beta^*)|$ and $|f(b) - f(\beta^*)|$ must be omitted.) $\square$

Lemma 4.2. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty < a < b < c < \infty$, $f \in W_p^l(a,b)$ and $g \in W_p^l(b,c)$. Then the pasted function*

$$h = \begin{cases} f & \text{on } (a,b), \\ g & \text{on } (b,c), \end{cases}$$

belongs to $W_p^l(a,c)$ if, and only if,

$$f_w^{(s)}(b-) = g_w^{(s)}(b+), \qquad s = 0,1,\dots,l-1, \tag{9}$$

where $f_w^{(s)}(b-)$ and $g_w^{(s)}(b+)$ are boundary values of $f_w^{(s)}$ and $g_w^{(s)}$ (see Remark 2.1).

If (9) is satisfied, then

$$\|h\|_{W_p^l(a,c)} \le \|f\|_{W_p^l(a,b)} + \|g\|_{W_p^l(b,c)}. \tag{10}$$

Proof. Let f_1 and g_1 be the functions, equivalent to f and g, whose derivatives $f_1^{(l-1)}$, $g_1^{(l-1)}$ exist and are absolutely continuous on $[a,b]$, $[b,c]$ respectively. Then $f_1^{(s)}(b) = f_w^{(s)}(b-)$ and $g_1^{(s)}(b) = g_w^{(s)}(b+), s = 0,1,\dots,l-1$. If (9) is satisfied, then the function

$$h_1 = \begin{cases} f_1 & \text{on } [a,b], \\ g_1 & \text{on } [b,c], \end{cases}$$

is such that $h_1^{(l-1)}$ exists and is absolutely continuous on $[a,b]$. Consequently, the weak derivative $h_w^{(l)}$ exists on (a,b) and

$$h_w^{(l)} = \begin{cases} f_w^{(l)} & \text{on } (a,b), \\ g_w^{(l)} & \text{on } (b,c). \end{cases}$$

Hence, inequality (10) follows.

If (9) is not satisfied, then for any function h_2 defined on $[a,b]$, coinciding with f_1 on $[a,b)$ and with g_1 on $(b,c]$, the ordinary derivative $h_2^{(l-1)}(b)$ does not exist. Hence, the weak derivative $h_w^{(l-1)}$ does not exist on (a,c) and h is not in $W_p^{(l)}(a,c)$. □

Lemma 4.3. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$. Then there exists a linear extension operator $T : W_p^l(-\infty,0) \longrightarrow W_p^l(-\infty,\infty)$, such that*

$$\|T\|_{W_p^l(-\infty,0)\to W_p^l(-\infty,\infty)} \le 8^l. \tag{11}$$

Idea of the proof. If $l = 1$, it is enough to consider the reflection operator, i.e., to set

$$(T_1 f)(x) = f(-x), \qquad x > 0. \tag{12}$$

If $l \ge 2$, define $(T_2 f)(x)$ for $x > 0$ as a linear combination of reflection and dilations:

$$(T_2 f)(x) = \sum_{k=1}^{l} \alpha_k (T_1 f)(\beta_k x) = \sum_{k=1}^{l} \alpha_k f(-\beta_k x), \tag{13}$$

where $\beta_k > 0$ and α_k are chosen in such a way that

$$(T_2 f)_w^{(s)}(0+) = f_w^{(s)}(0-), \qquad s = 0,1,\dots,l-1. \tag{14}$$

Verify that $\|T_2\|_{W_p^l(-\infty,0)\longrightarrow W_p^l(-\infty,\infty)} < \infty$ and choose $\beta_k = k/l$, $k = 1,\dots,l$, in order to prove (11). □

Proof. Equalities (14) are equivalent to

$$\sum_{k=1}^{l} \alpha_k (-\beta_k)^s = 1, \qquad s = 0, 1, \ldots, l-1.$$

Consequently, by Cramer's rule and the formula for Van-der-Monde's determinant,

$$\begin{aligned} \alpha_k &= \frac{\prod\limits_{1 \le i < j \le l} (\beta_i - \beta_j) \mid_{\beta_k = -1}}{\prod\limits_{1 \le i < j \le l} (\beta_i - \beta_j)} \\ &= \frac{\prod\limits_{1 \le i < k} (\beta_i + 1) \prod\limits_{k < j \le l} (-1 - \beta_j)}{\prod\limits_{1 \le i < k} (\beta_i - \beta_j) \prod\limits_{k < j \le l} (\beta_k - \beta_j)} \\ &= \prod_{1 \le j \le l, j \ne k} \frac{1 + \beta_j}{\beta_j - \beta_k}, \qquad k = 1, \ldots, l. \end{aligned} \tag{15}$$

If $\beta_k = k/l$, $k = 1, \ldots, l$, then

$$\alpha_k = \frac{(-1)^{k-1} k}{l+k} \binom{2l}{l} \binom{l}{k}$$

and

$$|\alpha_k| \le 4^l \, \frac{k}{l} \binom{l}{k}.$$

Therefore, setting $y = -\beta_k x$, we have

$$\begin{aligned} &\|T_2 f\|_{W_p^l(0,\infty)} = \|T_2 f\|_{L_p(0,\infty)} + \|(T_2 f)_w^{(l)}\|_{L_p(0,\infty)} \\ &\le \left(\sum_{k=1}^{l} |\alpha_k| \beta_k^{-1/p} \right) \|f\|_{L_p(-\infty,0)} + \left(\sum_{k=1}^{l} |\alpha_k| \beta_k^{l-1/p} \right) \|f_w^{(l)}\|_{L_p(-\infty,0)} \\ &\le \left(\sum_{k=1}^{l} |\alpha_k| \beta_k^{-1/p} \right) \|f\|_{W_p^l(-\infty,0)} \le 4^l \left(\sum_{k=1}^{l} \left(\frac{k}{l}\right)^{1-1/p} \binom{l}{k} \right) \|f\|_{W_p^l(-\infty,0)} \\ &\le (8^l - 1) \|f\|_{W_p^l(-\infty,0)}. \end{aligned}$$

Hence, inequality (11) follows if we take into account Lemma 4.2. $\square$

Remark 4.1. It follows from the above proof that the inequalities

$$\|T_2\|_{w_p^m(-\infty,0)\to w_p^m(-\infty,\infty)} \le 8^l, \qquad m \in \mathbb{N}_0\,,\ m \le l,$$

also hold.

Corollary 4.1. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty < a < b < \infty$. Then there exists a linear extension operator $T : W_p^l(a,b) \longrightarrow W_p^l(2a-b, 2b-a)$, such that*

$$\|T\|_{W_p^l(a,b)\to W_p^l(2a-b,2b-a)} \le 2\cdot 8^l. \tag{16}$$

Idea of the proof. Define

$$(T_3 f)(x) = \begin{cases} \sum\limits_{k=1}^{l} \alpha_k f(a+\beta_k(a-x)) & \text{for } x \in (2a-b, a),\\ f(x) & \text{for } x \in (a,b),\\ \sum\limits_{k=1}^{l} \alpha_k f(b+\beta_k(b-x)) & \text{for } x \in (b, 2b-a), \end{cases} \tag{17}$$

where α_k and β_k are the same as in (13), observe that $T_3 f$ is defined on $(2a-b, 2b-a)$ since $0 < \beta_k \le 1$, and apply the proof of Lemma 4.3. $\square$

Corollary 4.2. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty < a < b < \infty$. Then there exists a linear extension operator $T : W_p^l(a,b) \longrightarrow W_p^l(a-1, b+1)$ such that*

$$\|T\|_{W_p^l(a,b)\to W_p^l(a-1,b+1)} \le 2\cdot 8^l\big(1+(b-a)^{-l+1/p'}\big). \tag{18}$$

Proof. Let $\delta = \min\{1, b-a\}$ and define

$$(T_4 f)(x) = \begin{cases} \sum\limits_{k=1}^{l} \alpha_{k,\delta} f(a+\delta\beta_k(a-x)) & \text{for } x \in (a-1, a),\\ f(x) & \text{for } x \in (a,b),\\ \sum\limits_{k=1}^{l} \alpha_{k,\delta} f(b+\delta\beta_k(b-x)) & \text{for } x \in (b, b+1), \end{cases} \tag{19}$$

where β_k are the same as in (17) and $\alpha_{k,\delta}$ are such that $\sum\limits_{k=1}^{l} \alpha_{k,\delta}(-\delta\beta_k)^s = 1$, $s = 0, \ldots, l-1$. Since by (14) $|\alpha_{k,\delta}| \le (b-a)^{-l+1}|\alpha_k|$, applying the proof of Lemma 4.3 one arrives at (18). $\square$

In order to estimate the norm of an extension operator $T : W_p^l(-\infty, 0) \to W_p^l(-\infty, \infty)$ from below we prove the following statement, which reduces this problem to a certain type of extremal boundary-value problems.

For given $a_0, \dots, a_{l-1} \in \mathbb{R}$ let

$$G_{p,l}^+(a_0, \dots, a_{l-1}) = \inf_{\substack{f \in W_p^l(0,\infty) \\ f_w^{(k)}(0+) = a_k,\, k=0,\dots,l-1}} \|f\|_{W_p^l(0,\infty)}.$$

The quantity $G_{p,l}^-(a_0, \dots, a_{l-1})$ is defined in a similar way with $(-\infty, 0)$ replacing $(0, \infty)$. Let

$$Q_{p,l} = \sup_{|a_0|+\dots+|a_{l-1}|>0} \frac{G_{p,l}^+(a_0, a_1, \dots, a_{l-1})}{G_{p,l}^-(a_0, a_1, \dots, a_{l-1})}$$

$$= \sup_{|a_0|+\dots+|a_{l-1}|>0} \frac{G_{p,l}^+(a_0, a_1, \dots, a_{l-1})}{G_{p,l}^+(a_0, -a_1, \dots, (-1)^{l-1} a_{l-1})}. \tag{20}$$

The latter equality follows if the argument x is replaced by $-x$ in the definition of $G_{p,l}^-$. Moreover, it follows from (20) that for $1 \le p \le \infty$,

$$Q_{p,l} \ge 1,\ l \in \mathbb{N}, \quad Q_{p,1} = 1. \tag{21}$$

Lemma 4.4. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$. Then*

$$\left(1 + Q_{p,l}^p\right)^{1/p} \le \inf_T \|T\|_{W_p^l(-\infty,0) \to W_p^l(-\infty,\infty)} \le 1 + Q_{p,l}. \tag{22}$$

(If $p = \infty$, then $(1 + Q_{p,l}^p)^{1/p}$ must be replaced by $Q_{\infty,l}$.)

Idea of the proof. Apply the inequality

$$\begin{aligned}\left(\|f\|_{W_p^l(-\infty,0)}^p + \|Tf\|_{W_p^l(0,\infty)}^p\right)^{1/p} \\ \le \|Tf\|_{W_p^l(-\infty,\infty)} \le \|f\|_{W_p^l(-\infty,0)} + \|Tf\|_{W_p^l(0,\infty)}.\end{aligned} \tag{23}$$

In order to prove the first inequality (22) apply also the inequality

$$\|Tf\|_{W_p^l(0,\infty)} \ge G_{p,l}^+(a_0, \dots, a_{l-1}), \tag{24}$$

which, by the definition of $G_{p,l}^+$, holds for all $a_0, \dots, a_{l-1}$ and for each extension operator T. In order to prove the second inequality (22) define,

$\forall \varepsilon > 0$, the extension operator T_ε setting $T_\varepsilon f = g_\varepsilon$ for $x \in (0,\infty)$, where $g_\varepsilon \in W_p^l(0,\infty)$ is any function, which is such that $g_{\varepsilon,w}^{(k)}(0+) = f_w^{(k)}(0-)$, $k = 0,\dots,l-1$, and

$$\|g_\varepsilon\|_{W_p^l(0,\infty)} \le G_{p,l}^+(f(0-),\dots,f_w^{(l-1)}(0-)) + \varepsilon\,\|f\|_{W_p^l(-\infty,0)}. \tag{25}$$

$\square$

Proof. 1. The second inequality in (23) is trivial since

$$\|h\|_{L_p(-\infty,\infty)} \le \|h\|_{L_p(-\infty,0)} + \|h\|_{L_p(0,\infty)}.$$

The first inequality in (23) follows from Minkowski's inequality for finite sums because

$$\begin{aligned}
\|h\|_{W_p^l(-\infty,\infty)} &= \Big(\|h\|^p_{L_p(-\infty,0)} + \|h\|^p_{L_p(0,\infty)}\Big)^{1/p} \\
&\quad + \Big(\|h_w^{(l)}\|^p_{L_p(-\infty,0)} + \|h_w^{(l)}\|^p_{L_p(0,\infty)}\Big)^{1/p} \\
&\ge \Big\{\Big(\|h\|_{L_p(-\infty,0)} + \|h_w^{(l)}\|_{L_p(-\infty,0)}\Big)^p \\
&\quad + \Big(\|h\|_{L_p(0,\infty)} + \|h_w^{(l)}\|_{L_p(0,\infty)}\Big)^p\Big\}^{1/p} \\
&= \Big(\|h\|^p_{W_p^l(-\infty,0)} + \|h\|^p_{W_p^l(0,\infty)}\Big)^{1/p}.
\end{aligned}$$

2. It follows from (23) and (24) that for each $a_0,\dots,a_{l-1} \in \mathbb{R}$ such that $|a_0| + \dots + |a_{l-1}| > 0$,

$$\begin{aligned}
&\|T\|_{W_p^l(-\infty,0)\to W_p^l(-\infty,\infty)} = \sup_{f\in W_p^l(0,\infty),\, f\nsim 0} \frac{\|Tf\|_{W_p^l(-\infty,\infty)}}{\|f\|_{W_p^l(-\infty,0)}} \\
&\ge \Bigg(1 + \sup_{\substack{f\in W_p^l(-\infty,0)\\ f_w^{(k)}(0-)=a_k,\, k=0,\dots,l-1}} \left(\frac{\|Tf\|_{W_p^l(0,\infty)}}{\|f\|_{W_p^l(-\infty,0)}}\right)^p\Bigg)^{1/p} \\
&\ge \Bigg(1 + \Big(G_{p,l}^+(a_0,\dots,a_{l-1})\Big)^p \sup_{\substack{f\in W_p^l(-\infty,0)\\ f_w^{(k)}(0-)=a_k,\, k=0,\dots,l-1}} \frac{1}{\|f\|^p_{W_p^l(-\infty,0)}}\Bigg)^{1/p} \\
&= \Bigg(1 + \left(\frac{G_{p,l}^+(a_0,\dots,a_{l-1})}{G_{p,l}^-(a_0,\dots,a_{l-1}}\right)^p\Bigg)^{1/p},
\end{aligned}$$

and we arrive at the first inequality in (22).

3. Given $\varepsilon > 0$, by (23) and (26) we have

$$\begin{aligned}\|T_\varepsilon\| &\le 1 + \sup_{f\in W_p^l(-\infty,0),\, f\not\sim 0} \frac{\|g_\varepsilon\|_{W_p^l(0,\infty)}}{\|f\|_{W_p^l(-\infty,0)}} \\ &\le 1 + \varepsilon + \sup_{\substack{a_0,\dots,a_{l-1}\in\mathbb{R}:\\ |a_0|+\cdots+|a_{l-1}|>0}} \sup_{\substack{f\in W_p^l(-\infty,0)\\ f_w^{(k)}(0-)=a_k,\, k=0,\dots,l-1}} \frac{G_{p,l}^+(a_0,\dots,a_{l-1})}{\|f\|_{W_p^l(-\infty,0)}} \\ &= 1 + Q_{p,l} + \varepsilon\end{aligned}$$

and the second inequality in (22) follows. □

Corollary 4.3. *Let* $1 \le p \le \infty$. *Then*

$$\inf_T \|T\|_{W_p^1(-\infty,0)\to W_p^1(-\infty,\infty)} = 2^{1/p}.$$

Proof. By (22) and (21), $\|T\|_{W_p^l(-\infty,0)\to W_p^l(-\infty,\infty)} \ge 2^{1/p}$ for each extension operator T. On the other hand it is clear that for the extension operator T_1 defined by (12), $\|T_1\|_{W_p^l(-\infty,0)\to W_p^l(-\infty,\infty)} = 2^{1/p}$. □

Lemma 4.5. *Let* $l \in \mathbb{N}$, $1 \le p \le \infty$ *and* $f \in W_p^l(0,\infty)$. *Then*

$$\|f\|_{W_p^l(0,\infty)} \ge \left\| \sum_{k=0}^{l-1} \frac{f^{(k)}(0+)}{k!} x^k \right\|_{L_p(0,(l!)^{1/l})} \tag{26}$$

Proof. Let $f \in W_p^l(0,\infty)$. Then for almost every $x \in (0,\infty)$

$$f(x) = \sum_{k=0}^{l-1} \frac{f_w^{(k)}(0+)\, x^k}{k!} + \frac{1}{(l-1)!}\int_0^x (x-u)^{l-1} f_w^{(l)}(u)\, du,$$

where the $f_w^{(k)}(0+), k = 0, 1, \dots, l-1$, are the boundary values of the weak derivatives $f_w^{(k)}$. (See Remark 2.1.) Hence, by the triangle inequality for each $a > 0$,

$$\begin{aligned}\left\| \sum_{k=0}^{l-1} \frac{f_w^{(k)}(0+)\, x^k}{k!} \right\|_{L_p(0,a)} &\le \|f\|_{L_p(0,a)} \\ &\quad + \frac{1}{(l-1)!} \left\| \int_0^x (x-u)^{l-1} f_w^{(l)}(u)\, du \right\|_{L_p(0,a)}.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\left\| \int_0^x (x-u)^{l-1} f_w^{(l)}(u)\, du \right\|_{L_p(0,a)} \\
&\quad \le \left\| \left(\frac{x^{(l-1)p'+1}}{(l-1)p'+1} \right)^{1/p'} \|f_w^{(l)}\|_{L_p(0,x)} \right\|_{L_p(0,a)} \\
&\quad \le ((l-1)p'+1)^{-1/p'} \|x^{l-1/p}\|_{L_p(0,a)} \|f_w^{(l)}\|_{L_p(0,a)} \\
&\quad = a^l (lp)^{-1/p} ((l-1)p'+1)^{-1/p'} \|f_w^{(l)}\|_{L_p(0,a)} \le \frac{a^l}{l} \|f_w^{(l)}\|_{L_p(0,a)} \,.
\end{aligned}$$

Consequently,

$$\left\| \sum_{k=0}^{l-1} \frac{f_w^{(k)}(0+)x^k}{k!} \right\|_{L_p(0,a)} \le \|f\|_{L_p(0,a)} + \frac{a^l}{l!} \|f_w^{(l)}\|_{L_p(0,a)}.$$

Setting $a = (l!)^{1/l}$, we get (26). □

Corollary 4.4. *For all $l \in \mathbb{N}$, $1 \le p \le \infty$, $a_0, \dots, a_{l-1} \in \mathbb{R}$,*

$$G^+_{p,l}(a_0, \dots, a_{l-1}) \ge \left\| \sum_{k=0}^{l-1} \frac{a_k}{k!} x^k \right\|_{L_p(0,(l!)^{1/l})}. \tag{27}$$

Lemma 4.6. *For all $l \in \mathbb{N}$, $1 \le p \le \infty$ and every extension operator $T : W_p^l(-\infty, 0) \to W_p^l(-\infty, \infty)$,*

$$\|T\|_{W_p^l(-\infty,0) \to W_p^l(-\infty,\infty)} \ge 0.3 \cdot 2^l l^{-1/(2p)}. \tag{28}$$

Proof. In view of Corollary 4.3 it is enough to consider the case $l \ge 2$. We set

$$f_l(x) = \begin{cases} 0 & \text{for } -\infty < x \le -\nu a, \\ (x+\nu a)^l & \text{for } -\nu a \le x \le 0, \end{cases}$$

where $a = (l!)^{1/l}$ and $\nu = (pl+1)^{1/(pl)}$. With this choice of a and ν,

$$\|f_l\|_{W_p^l(-\infty,0)} = \frac{(\nu a)^{l+1/p}}{(pl+1)^{1/p}} + l!(\nu a)^{1/p} = 2\frac{(\nu a)^{l+1/p}}{(pl+1)^{1/p}}.$$

Since

$$\sum_{k=0}^{l-1}\frac{f_l^{(k)}(0)}{k!}x^k=(x+\nu a)^l-x^l\geq(1-(1+\nu)^{-l})(x+\nu a)^l$$
$$\geq(1-2^{-l})(x+\nu a)^l,$$

by (20) and (27) we have

$$\begin{aligned}Q_{p,l}&\geq\frac{G^+_{p,l}(f_l(0),\dots,f_l^{(l-1)}(0))}{\|f_l\|_{W_p^l(-\infty,0)}}\geq\frac{\left\|\sum\limits_{k=0}^{l-1}\frac{f_l^{(k)}(0)x^k}{k!}\right\|_{L_p(0,a)}}{\|f_l\|_{W_p^l(-\infty,0)}}\\&\geq(1-2^{-l})\frac{\|(x+\nu a)^l\|_{L_p(0,a)}}{\|f_l\|_{W_p^l(-\infty,0)}}=\frac{1-2^{-l}}{2}\left[\left(1+\frac{1}{\nu}\right)^{pl+1}-1\right]^{1/p}\\&=\frac{1-2^{-l}}{2}\left[(1+(pl+1)^{-\frac{1}{pl}})^{pl+1}-1\right]^{1/p}\\&=\frac{2^l-1}{2}l^{-1/(2p)}p^{-1/(2p)}\left\{2^{-pl}\sqrt{pl}\left[(1+(pl+1)^{-\frac{1}{pl}})^{pl+1}-1\right]\right\}^{1/p}\\&\geq\frac{2^{-l}-1}{2}l^{-1/(2p)}p^{-1/(2p)}\xi^{1/p},\end{aligned}$$

where $\xi=\min\limits_{x\geq2}\varphi(x)$ and for $x>0$,

$$\varphi(x)=2^{-x}\sqrt{x}\left[\left(1+(1+x)^{-1/x}\right)^{1+x}-1\right].$$

One can prove that $\xi=\varphi(2)>1.03396$ and $\varphi(x)\to2$ as $x\to+\infty$. Hence

$$\begin{aligned}Q_{p,l}&\geq\frac{2^{-l}-1}{2}l^{-1/(2p)}p^{-1/(2p)}\geq\frac{2^{-l}-1}{2}l^{-1/2p}e^{-1/(2e)}\\&\geq0.4\cdot(2^l-1)l^{-1/(2p)}\geq0.3\cdot2^l l^{-1/(2p)}\end{aligned}$$

and by (21) inequality (27) follows. □

Remark 4.2. Estimate (27) is slightly better than in [19], p. 255, and [27].

Lemma 4.7. *Let $l\in\mathbb{N}$, $-\infty<a<b<\infty$, $\varepsilon>0$. Then there exists a "cap-shaped" function $\eta\in C_0^\infty(\mathbb{R})$ such that $0\leq\eta\leq1$, $\eta=1$ on (a,b), $\operatorname{supp}\eta\subset(a-\varepsilon,b+\varepsilon)$ and*

$$|\eta^{(k)}(x)|\leq(4l)^k\varepsilon^{-k},\qquad x\in\mathbb{R},\quad k=0,\dots,l. \tag{29}$$

Idea of the proof. Set

$$\eta = \widetilde{\omega}_{\frac{\gamma\varepsilon}{4(l+\gamma)}} * \underbrace{\omega_{\frac{\varepsilon}{2(l+\gamma)}} * \cdots * \omega_{\frac{\varepsilon}{2(l+\gamma)}}}_{l\text{-times}} * \chi_{(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2})}, \tag{30}$$

where $\chi_{(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2})}$ is the characteristic function of the interval $(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2})$, $\omega(x) = 1-|x|$ if $|x| \le 1$, $\omega(x) = 0$ if $|x| > 1$, $\widetilde{\omega}$ is any non-negative infinitely differentiable kernel of mollification[7] and γ is a sufficiently small positive number. Apply Young's inequality and the equality

$$\big\|\big(\omega_{\frac{\varepsilon}{2(l+\gamma)}} * \chi_{(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2})}\big)'\big\|_{L_\infty(\mathbb{R})} = \|\omega_{\frac{\varepsilon}{2(l+\gamma)}}\|_{L_\infty(\mathbb{R})}.$$

(One can find a detailed proof in [26] or [19] (Chapter 6).) □

Corollary 4.5. *In the one-dimensional case $\forall l \in \mathbb{N}$ there exists a non-negative infinitely differentiable kernel of mollification μ such that*

$$|\mu^{(k)}(x)| \le (4l)^k, \qquad x \in \mathbb{R}, \quad k = 0, \dots, l.$$

Proof. Define η by (30), where $a = b = 0$ and $\varepsilon = 1$, and apply the equality $\|f * g\|_{L_1(\mathbb{R})} = \|f\|_{L_1(\mathbb{R})} \cdot \|g\|_{L_1(\mathbb{R})}$ for non-negative $f, g \in L_1(\mathbb{R})$. □

Lemma 4.8. *There exists $c_2 > 0$ such that for all $l, m \in \mathbb{N}$, $m < l$, $1 \le p, q \le \infty$, $-\infty < a < b < \infty$ and $\forall f \in W_p^l(a,b)$,*

$$\begin{aligned} \|f_w^{(m)}\|_{L_q(a,b)} \le c_2^l\, (b-a)^{1/q-1/p} \Big(\Big(\frac{l}{b-a}\Big)^m \|f\|_{L_p(a,b)} \\ + \Big(\frac{b-a}{l}\Big)^{l-m} \|f_w^{(l)}\|_{L_p(a,b)} \Big). \end{aligned} \tag{31}$$

The proof can be found in [19] (Chapters 3, 6).

Corollary 4.6. *If, in addition to the assumptions of Lemma 4.8, $b-a \le 1$, then*

$$\|f_w^{(m)}\|_{L_q(a,b)} \le {c_2}^l\, l^m\, (b-a)^{-m+1/q-1/p}\, \|f\|_{W_p^l(a,b)}.$$

If, in addition to the assumptions of Lemma 4.8, $b-a \ge 1$ and $q \ge p$, then

$$\|f_w^{(m)}\|_{L_q(a,b)} \le 2^{1/q}\, {c_2}^l\, l^m\, \|f\|_{W_p^l(a,b)}. \tag{32}$$

[7] I.e., $\widetilde{\omega} \in C_0^\infty(\Omega)$, $\operatorname{supp} \widetilde{\omega} \subset \overline{B(0,1)}$ and $\int_{\mathbb{R}^n} \widetilde{\omega}\, dx = 1$.

Lemma 4.9. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty < a < b < \infty$, $b - a \le 1$. There exists a linear operator $T : W_p^l(a,b) \to W_p^l(-\infty,\infty)$, such that*

$$\|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \le \frac{{c_3}^l\, l^l}{(b-a)^{l-1/p'}}, \tag{33}$$

where c_3 is a constant greater than 1.

Idea of the proof. Consider the operator

$$\big(T_5 f\big)(x) = \big(T_4 f\big)(x)\,\eta(x), \qquad x \in \mathbb{R},$$

where η is the function constructed in Lemma 4.7 for $\varepsilon = 1$ and T_4 is defined by (19), assuming that $(T_5 f)(x) = 0$ for $x \notin (a-1, b+1)$, and apply Corollary 4.6. □

Proof. It follows by the Leibniz formula, (29), (32) and (18) that

$$\begin{aligned}
&\|T_5 f\|_{W_p^l(-\infty,\infty)} = \|\eta T_4 f\|_{L_p(a-1,b+1)} + \|(\eta T_4 f)_w^{(l)}\|_{L_p(a-1,b+1)} \\
&\le \|T_4 f\|_{L_p(a-1,b+1)} + \sum_{m=0}^{l} \binom{l}{m} \|\eta^{(l-m)}\|_{L_\infty(-\infty,\infty)} \|(T_4 f)_w^{(l)}\|_{L_p(a-1,b+1)} \\
&\le \|T_4 f\|_{L_p(a-1,b+1)} + \left(\sum_{m=0}^{l} \binom{l}{m} (4l)^{l-m} (2c_2)^l l^m\right) \|T_4 f\|_{W_p^l(a-1,b+1)} \\
&\le \big(1 + (16\, c_2\, l)^l\big) \|T_4 f\|_{W_p^l(a-1,b+1)} \\
&\le 4\,\big(1 + (16\, c_2\, l)^l\big)\, 8^l\, (b-a)^{-l+1/p'}\, \|f\|_{W_p^l(a,b)} \\
&\le c_3^l\, l^l\, (b-a)^{-l+1/p'}\, \|f\|_{W_p^l(a,b)}\,,
\end{aligned}$$

where $c_3 = 32\,(1 + 16 c_2)$. Hence we obtain (33). □

Lemma 4.10. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty < a < b < \infty$, $b - a \ge 1$. There exists a linear extension operator $T : W_p^l(a,b) \to W_p^l(-\infty,\infty)$ such that*

$$\|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \le c_4^l \left(1 + \frac{l^l}{(b-a)^{l-1/p'}}\right), \tag{34}$$

where c_4 is a constant greater than 1.

Idea of the proof. Consider the operator

$$(T_6 f)(x) = (T_3 f)(x)\,\eta(x),$$

where η is the function constructed in Lemma 4.7 for $\varepsilon = b - a$ and T_3 is defined by (17), and apply Lemma 4.8. □

Proof. It follows by the Leibniz formula, (29), (31) and (16) that

$$
\begin{aligned}
&\|T_6 f\|_{W_p^l(-\infty,\infty)} = \|\eta T_3 f\|_{L_p(2a-b,2b-a)} + \|(\eta T_3 f)_w^{(l)}\|_{L_p(2a-b,2b-a)} \\
&\quad \le \|T_3 f\|_{L_p(2a-b,2b-a)} \\
&\qquad + \sum_{m=0}^{l} \binom{l}{m} \|\eta^{(l-m)}\|_{L_\infty(-\infty,\infty)} \|(T_3 f)_w^{(m)}\|_{L_p(2a-b,2b-a)} \\
&\quad \le \|T_3 f\|_{L_p(2a-b,2b-a)} \\
&\qquad + \sum_{m=0}^{l} \binom{l}{m} (4l)^{l-m} (b-a)^{m-l}\, {c_2}^m \left(\left(\frac{l}{b-a} \right)^m \|T_3 f\|_{L_p(2a-b,2b-a)} \right. \\
&\qquad\quad \left. + \left(\frac{b-a}{l} \right)^{l-m} \|(T_3 f)_w^{(l)}\|_{L_p(2a-b,2b-a)} \right) \\
&\quad \le \|T_3 f\|_{L_p(2a-b,2b-a)} \\
&\qquad + (4l)^l \left(\sum_{m=0}^{l} \binom{l}{m} {c_2}^m \right) (b-a)^{-l}\, \|T_3 f\|_{L_p(2a-b,2b-a)} \\
&\qquad + 4^l \sum_{m=0}^{l} \binom{l}{m} c_2^m\, \|(T_3 f)_w^{(l)}\|_{L_p(2a-b,2b-a)} \\
&\quad \le \big(1 + (4\,(1+c_2))\big)^l \big(1 + l^l (b-a)^{-l}\big) \|T_3 f\|_{W_p^l(2a-b,2b-a)} \\
&\quad \le 2\, \big(1 + (4\,(1+c_2))\big)^l\, 8^l\, \big(1 + l^l (b-a)^{-l}\big) \|f\|_{W_p^l(a,b)} \\
&\quad \le c_4^l\, \big(1 + l^l (b-a)^{-l}\big) \|f\|_{W_p^l(a,b)} \le\ c_4^l\, \left(1 + l^l (b-a)^{-l+1/p'}\right) \|f\|_{W_p^l(a,b)}\,,
\end{aligned}
$$

where $c_4 = 16\,(1 + 4\,(1 + c_2))$. Hence we obtain (34). □

Remark 4.3. It follows from the proofs of Lemmas 4.9 and 4.10 that for all $-\infty < a < b < \infty$ there exists an extension operator T such that

$$
\|T\|_{W_p^m(a,b) \to W_p^m(-\infty,\infty)} \le c_5^l \left(1 + \frac{m^m}{(b-a)^{m-1/p'}} \right), \quad m \in \mathbb{N}_0\,,\ m \le l,
$$

where c_5 is a constant greater than 1.

Now we consider estimates from below for the minimal norm of an extension operator.

Lemma 4.11. *Let $l \in \mathbb{N}$, $1 \leq p \leq \infty$, $\infty < a < b < \infty$. Then for every extension operator* $T : W_p^l(a,b) \to W_p^l(-\infty,\infty)$,

$$\|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \ \geq \ \frac{1}{8\sqrt{l}}\left(\frac{4}{e}\right)^l l^l\,(b-a)^{-l+1/p'}. \qquad (35)$$

Remark 4.4. We shall discuss two proofs of Lemma 4.11. The first of them is a direct one: as in the proof of Lemma 4.6 it is based on the choice of a function $f \in W_p^l(a,b)$, which is the "worst" for extension. The second one is based on Lemma 3.2. In both proofs the polynomials $Q_{l-1;p}$ of degree $l-1$ closest to zero in $L_p(0,1)$ are involved, i.e., $Q_{l-1;p} = x^{l-1}+a_{l-2}x^{l-2}+\cdots+a_0$ and

$$\|Q_{l-1;p}\|_{L_p(0,1)} = \inf_{b_0,\ldots,b_{l-2}\in\mathbb{R}} \|x^{l-1}+b_{l-2}x^{l-2}+\cdots+b_0\|_{L_p(0,1)}.$$

We recall that $Q_{l-1;\infty}(x) = 2^{-l+1}R_{l-1}(2x-1)$, where R_m is the Chebyshev polynomial of the first type: $R_m(x) = 2^{-m+1}\cos(m\ \arccos x)$. Moreover,

$$\|Q_{l-1;p}\|_{L_p(0,1)} \leq \|Q_{l-1;\infty}\|_{L_p(0,1)} \leq \|Q_{l-1;\infty}\|_{L_\infty(0,1)} = 8\cdot 4^{-l}. \qquad (36)$$

Idea of the first proof of Lemma 4.11. In the inequality

$$\|T\| = \|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \geq \frac{\|Tf\|_{W_p(-\infty,\infty)}}{\|f\|_{W_p^l(a,b)}}$$

set

$$f(x) = \frac{(b-a)^{l-1}}{(l-1)!}Q_{l-1;p}\left(\frac{x-a}{b-a}\right), \qquad (37)$$

apply the following corollary of the Kolmogorov-Stein inequality

$$\|f_w^{(m)}\|_{L_p(-\infty,\infty)} \leq \frac{\pi}{2}\|f\|_{L_p(-\infty,\infty)}^{1-m/l}\|f_w^{(l)}\|_{L_p(-\infty,\infty)}^{m/l}, \qquad (38)$$

where $0 < m < l$ and $1 \leq p \leq \infty$, and the relation

$$\inf_{\substack{h\in W_p^1(-\infty,a)\\ h(a-)=1}} \|h\|_{W_p^1(-\infty,a)} \geq 1.$$

$\square$

The second proof of Lemma 4.11. By Lemma 3.2, for every extension operator $T : W_p^l(a,b) \to W_p^l(-\infty,\infty)$,

$$\|T\| \equiv \|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \geq \frac{C^*((a,b),p,\infty,l,l-1)}{C^*((-\infty,\infty),p,\infty,l,l-1)}.$$

It follows from (3), where $\Omega = (a,b), q = \infty, \beta = l-1$ and f is defined by (37), and from (36) that

$$C^*((a,b),p,\infty,l,l-1) \geq \frac{\|1\|_{L_\infty(a,b)}}{\frac{(b-a)^{l-1}}{(l-1)!}\left\|Q_{l-1;p}\left(\frac{x-a}{b-a}\right)\right\|_{L_p(a,b)}}$$
$$= \frac{(l-1)!\,(b-a)^{-l+1/p'}}{\|Q_{l-1;p}\|_{L_p(0,1)}} \geq \frac{1}{8}\,4^l(l-1)!\,(b-a)^{-l+1/p'}.$$

On the other hand, $C^*((-\infty,\infty),p,\infty,l,l-1) \leq \sqrt{2\pi}$. (This follows (see [19], p.134) by inequality (38).) Hence, applying Stirling's formula, we get

$$\|T\| \geq \frac{4^{l-1}(l-1)!}{2\sqrt{2\pi}}(b-a)^{-l+1/p'} \geq \frac{1}{8\sqrt{l}}\left(\frac{4}{e}\right)^l l^l\,(b-a)^{-l+1/p'}.$$

□

Finally, we give a formulation of the main result of Section 4.

Theorem 4.1. *There exist* $c_6, c_7 > 0$ *such that for all* $l \in \mathbb{N}$, $1 \leq p \leq \infty$ *and* $-\infty \leq a < b \leq \infty$,

$$c_6^l\left(1+\frac{l^l}{(b-a)^{l-1/p'}}\right) \leq \inf_T \|T\|_{W_p^l(a,b)\to W_p^l(-\infty,\infty)} \leq c_7^l\left(1+\frac{l^l}{(b-a)^{l-1/p'}}\right). \tag{39}$$

Proof. If $b-a=\infty$, then (39) follows from (11) and (28). If $b-a<\infty$, then (39) follows from (33), (34) and (35). □

5 Classes of open sets

We say that a domain $H \subset \mathbb{R}^n$ is an *elementary domain with a resolved boundary* with the parameters d, D, $0<d<\infty$, $0<D\leq\infty$ if

$$H = \{x \in \mathbb{R}^n : a_n < x_n < \varphi(\bar{x}),\ \bar{x} \in W\}, \tag{40}$$

where[8] $\operatorname{diam} H \le D$, $x = (\bar{x}, x_n)$, $\bar{x} = (x_1, \dots, x_{n-1})$, $W = \{\bar{x} \in \mathbb{R}^{n-1} : a_i < x_i < b_i,\ i = 1, \dots, n-1\}$, $-\infty \le a_i < b_i \le \infty$, and

$$a_n + d < \varphi(\bar{x}), \qquad \bar{x} \in W. \tag{41}$$

If, in addition, $\varphi \in C(\overline{W})$, or $\varphi \in C^l(\overline{W})$ for some $l \in \mathbb{N}$ and $\|D^\alpha \varphi\|_{C(\overline{W})} \le M$ if $1 \le |\alpha| \le l$ where $0 \le M < \infty$, or φ satisfies the Lipschitz condition

$$|\varphi(\bar{x}) - \varphi(\bar{y})| \le M\, |\bar{x} - \bar{y}|, \qquad \bar{x}, \bar{y} \in \overline{W}, \tag{42}$$

then we say that H is an *elementary domain with a continuous boundary* with the parameters d, D, *with a C^l-boundary* with the parameters d, D, M, or *with a Lipschitz boundary* with the parameters d, D, M respectively.

Moreover, we say that an open set $\Omega \subset \mathbb{R}^n$ has a *resolved boundary* with the parameters d, $0 < d < \infty$, D, $0 < D \le \infty$ and $\varkappa \in \mathbb{N}$ if there exist open parallelepipeds V_j, $j = \overline{1, s}$, where $s \in \mathbb{N}$ for bounded Ω and $s = \infty$ for unbounded Ω, such that
1) $(V_j)_d \cap \Omega \ne \emptyset$ and $\operatorname{diam} V_j \le D$,
2) $\Omega \subset \bigcup\limits_{j=1}^{s} (V_j)_d$,
3) for each ball B the number of V_j intersecting B is finite and the multiplicity of the covering $\{V_j\}_{j=1}^s$ does not exceed $\varkappa$,
4) there exist rotations λ_j, $j = \overline{1, s}$, such that

$$\lambda_j(V_j) = \{x \in \mathbb{R}^n : a_{ij} < x_i < b_{ij},\ i = 1, \dots, n\}$$

and

$$\lambda_j(\Omega \cap V_j) = \{x \in \mathbb{R}^n : a_{nj} < x_n < \varphi_j(\bar{x}),\ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, \dots, x_{n-1}), W_j = \{\bar{x} \in \mathbb{R}^{n-1} : a_{ij} < x_i < b_{ij}, i = 1, \dots, n-1\}$, and

$$a_{nj} + d < \varphi_j(\bar{x}) < b_{nj} - d, \qquad \bar{x} \in W_j, \tag{43}$$

if $V_j \cap \partial\Omega \ne \emptyset$. (If $V_j \subset \Omega$, then $\varphi_j(\bar{x}) \equiv b_{nj}$ and the left inequality is satisfied since $b_{nj} - a_{nj} \ge 2\,d$.)

We note that $\lambda_j(\Omega \cap V_j)$ and, if $V_j \cap \partial\Omega \ne \emptyset$, also $\lambda_j^-((^c\overline{\Omega}) \cap V_j)$ are elementary domains with a resolved boundary with the parameters d, D, where $\lambda_j^-(x) = (\lambda_{j,1}(x), \dots, \lambda_{j,n-1}(x), -\lambda_{j,n}(x))$.

[8] One can verify that the set H defined by (40) is a domain if, and only if, the function φ is lower semicontinuous on W.

If an open set $\Omega \subset \mathbb{R}^n$ has a resolved boundary with the parameters d, D, $\varkappa$ and, in addition, for some $l \in \mathbb{N}$ all functions $\varphi_j \in C^l(\overline{W}_j)$ and $\|D^\alpha \varphi_j\|_{C(\overline{W}_j)} \le M$ if $1 \le |\alpha| \le l$, where $0 \le M < \infty$ and is independent of j, or all functions φ_j satisfy the Lipschitz condition with the same constant M, then we say that Ω has a C^l*-boundary* (briefly $\partial\Omega \in C^l$) with the parameters d, D, $\varkappa$, M, or a *Lipschitz boundary* (briefly $\partial\Omega \in \mathrm{Lip}\,1$) with the parameters d, D, $\varkappa$, M respectively.

If all functions φ_j are continuous on $\overline{W}$, we say that Ω has a *continuous boundary* with the parameters d, D, $\varkappa$.

Finally, we say that an open set $\Omega \subset \mathbb{R}^n$ has a *quasi-resolved* (*quasi-continuous*) *boundary* with the parameters d, D, $\varkappa$ if $\Omega = \bigcup_{k=1}^s \Omega_k$, where $s \in \mathbb{N}$ or $s = \infty$, and Ω_k, $k = \overline{1,s}$, are open sets, which have a resolved (continuous) boundary with the parameters d, D, $\varkappa$, and the multiplicity of the covering $\{\Omega_k\}_{k=1}^s$ does not exceed $\varkappa$.

We call the set $V_x \equiv V_{x,B} = \bigcup_{y \in B}(x,y)$ a *conic body* with the vertex x constructed on the ball B (if $x \in B$, then $V_x = B$). A domain Ω star-shaped with respect to a ball B can be equivalently defined in the following way: $\forall x \in \Omega$ the conic body $V_x \subset \Omega$.

Let us consider now the cone

$$K \equiv K(r,h) = \left\{ x \in \mathbb{R}^n : \frac{h}{r}\left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} < x_n < h,\ 0 < \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} < r \right\}.$$

We say also that an open set Ω satisfies *the cone condition with the parameters* $r > 0$ *and* $h > 0$ if $\forall x \in \Omega$ there exists a cone $K_x \subset \Omega$ with the point x as vertex congruent to the cone K. Moreover, an open set Ω satisfies *the cone condition* if for some $r > 0$ and $h > 0$ it satisfies the cone condition with the parameters r and h.

Example 5.1. The domain $\Omega = \{x \in \mathbb{R}^n : -1 < x_n < 1 - |\overline{x}|^\gamma,\ |\overline{x}| < 1\}$, where $\overline{x} = (x_1, \dots, x_{n-1})$, for $\gamma \ge 1$ is an elementary bounded domain with a Lipschitz boundary and satisfies the cone condition. For $0 < \gamma < 1$ it does not have a Lipschitz boundary and does not satisfy the cone condition.

Example 5.2. The domain $\Omega = \{x \in \mathbb{R}^n : -1 < x_n < |\overline{x}|^\gamma,\ |\overline{x}| < 1\}$ satisfies the cone condition for each $\gamma > 0$. It has a Lipschitz boundary if, and only if, $\gamma \ge 1$.

Example 5.3. Let us suppose that $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_2 < 1$ if $-1 < x_1 < 0,\ -1 < x_2 < x_1^\gamma$ if $0 \le x_1 < 1\}$, where $0 < \gamma < 1$. Then Ω

is a bounded elementary domain with a resolved boundary, which is not a quasi-continuous boundary.

Example 5.4. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, x_1^\gamma < x_2 < 2x_1^\gamma\}$ where $0 < \gamma < \infty$, $\gamma \neq 1$. Then $\partial\Omega$ is not a quasi-resolved boundary while ${}^c\overline{\Omega}$ satisfies the cone condition.

Example 5.5. For the elementary domain Ω defined by (40) the Lipschitz condition (42) means geometrically that $\forall x \in \partial\Omega$ the cones

$$K_x^+ = \{y \in \mathbb{R}^n : y_n < \varphi(\bar{x}) - M|\bar{x} - \bar{y}|\},$$
$$K_x^- = \{y \in \mathbb{R}^n : \varphi(\bar{x}) + M|\bar{x} - \bar{y}| < y_n\}$$

are such that $K_x^+ \cap \widehat{W} \subset \Omega$, $K_x^- \cap \widehat{W} \subset {}^c\overline{\Omega}$, where $\widehat{W} = \{x \in \mathbb{R}^n : \bar{x} \in W, a_n < x_n < \infty\}$.

Example 5.6. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < \varphi(x_1)\}$, where $\varphi(x_1) = -|x_1|^\gamma$ if $x_1 \le 0$, $\varphi(x_1) = x_1^\gamma$ if $x_1 \ge 0$ and $\gamma > 0$. Then the function φ satisfies a Lipschitz condition on $\mathbb{R}$ if, and only if, $\gamma = 1$, while Ω has a Lipschitz boundary in the sense of the above definition for each $\gamma > 0$.

Lemma 5.1. *If an open set $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary with the parameters d, D, $\varkappa$ and M, then both Ω and ${}^c\overline{\Omega}$ satisfy the cone condition with the parameters r, h depending only on d, M and n.*

Lemma 5.2. 1. *A bounded open set $\Omega \subset \mathbb{R}^n$ satisfies the cone condition if, and only if, there exist $s \in \mathbb{N}$ and elementary bounded domains Ω_k, $k = 1, \ldots, s$, with Lipschitz boundaries with the same parameters such that $\Omega = \bigcup_{k=1}^s \Omega_k$.*

2. *An unbounded open set $\Omega \subset \mathbb{R}^n$ satisfies the cone condition if, and only if, there exist elementary bounded domains Ω_k, $k \in \mathbb{N}$, with Lipschitz boundaries with the same parameters such that*
1) $\Omega = \bigcup_{k=1}^\infty \Omega_k$,
and
2) *the multiplicity of the covering $\varkappa(\{\Omega_k\}_{k=1}^\infty)$ is finite.*

6 Pasting local extensions

We start by reducing the problem of extensions to the problem of local extensions.

Lemma 6.1. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Moreover, let $U_j \subset \mathbb{R}^n$, $j = \overline{1,s}$, where $s \in \mathbb{N}$ or $s = \infty$, be open sets such that*

$$\Omega \subset \bigcup_{j=1}^{s} (U_j)_\delta$$

for some $\delta > 0$. If $s = \infty$, suppose, in addition, that the multiplicity of the covering $\varkappa \equiv \varkappa(\{U_j\}_{j=1}^s)$ is finite.
1. *Suppose that for all $j = \overline{1,s}$ there exist bounded extension operators*

$$T_j : \widehat{V}_p^l(\Omega \cap U_j) \to V_p^l(U_j), \tag{44}$$

where $\widehat{V}_p^l(\Omega \cap U_j) = \{f \in V_p^l(\Omega \cap U_j) : \operatorname{supp} f \subset \overline{\Omega} \cap U_j\}$. If $s = \infty$, suppose also that $\sup_{j \in \mathbb{N}} \|T_j\| < \infty$. Then there exists a bounded extension operator

$$T : V_p^l(\Omega) \to V_p^l(\mathbb{R}^n). \tag{45}$$

Moreover,

$$\|T\| \le c_1 \sup_{j=\overline{1,s}} \|T_j\|, \tag{46}$$

where $c_1 > 0$ depends only on n, l, δ and $\varkappa$. If all the T_j are linear, then T is also linear.
2. *The statement 1 also holds if one replaces the space $V_p^l(\cdot)$ by the space $V_p^{l,\dots,l}(\cdot)$.*
3. *If Ω has a quasi-resolved boundary, then the statement 1 also holds if one replaces the space $V_p^l(\cdot)$ by the space $W_p^l(\cdot)$.*
4. *If Ω satisfies the cone condition, then the statement 1 also holds if one replaces the space $V_p^l(\cdot)$ by the space $W_p^{l,\dots,l}(\cdot)$.*

Idea of the proof. Assuming, without loss of generality, that $(U_j)_\delta \cap \Omega \ne \emptyset$ construct functions $\psi_j \in C^\infty(\mathbb{R}^n)$, $j = \overline{1,s}$, such that the collection $\{\psi_j^2\}_{j=1}^s$ is a partition of unity corresponding to the covering $\{U_j\}_{j=1}^s$, i.e., the following properties hold: $0 \le \psi_j \le 1$, $\operatorname{supp} \psi_j \subset U_j$, $\sum_{j=1}^s \psi_j^2 = 1$ on Ω and $\forall \alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \le l$, $\|D^\alpha \psi_j\|_{L_\infty(\mathbb{R}^n)} \le M_1$, where M_1 depends only on n, l and δ. For $f \in V_p^l(\Omega)$ set

$$Tf = \sum_{j=1}^{s} \psi_j \, T_j(f\psi_j) \qquad \text{on } \mathbb{R}^n. \tag{47}$$

(Assume that $\psi_j T_j(f\psi_j) = 0$ on ${}^c(U_j)$.) □

Proof. 1. Let $\eta_j \in C^\infty(\mathbb{R}^n)$ be "cap-shaped" functions satisfying $0 \le \eta_j \le 1$, $\eta_j = 1$ on $(U_j)_{\delta/2}$, $\eta_j = 0$ on ${}^c((U_j)_{\delta/4})$ and $|D^\alpha \eta_j(x)| \le M_2\, \delta^{-|\alpha|}$, $\alpha \in \mathbb{N}_0^n$, where M_2 depends only on n and α. Then $1 \le \sum\limits_{j=1}^{s} \eta_j^2 \le \varkappa$ on $\bigcup\limits_{j=1}^{s} (U_j)_{\delta/2}$. Further, let $\eta \in C_b^\infty(\mathbb{R}^n)$, $\eta = 1$ on Ω, $\eta = 0$ on ${}^c(\bigcup\limits_{j=1}^{s} (U_j)_{\delta/2})$. One can construct functions ψ_j by setting $\psi_j = \eta_j\, \eta \left(\sum\limits_{i=1}^{s} \eta_i^2 \right)^{-1/2}$ on $\bigcup\limits_{i=1}^{s} (U_i)_{\delta/2}$ assuming that $\psi_j = 0$ on ${}^c(\bigcup\limits_{i=1}^{s} (U_i)_{\delta/2})$.

2. The operator T defined by (47) is an extension operator. For, let $x \in \Omega$. If $x \in \operatorname{supp} \psi_j$ for some j, then $\psi_j(x)(T_j(f\psi_j))(x) = \psi_j^2(x)\, f(x)$. If $x \notin \operatorname{supp} \psi_j$, then $\psi_j(x)(T_j(f\,\psi_j))(x) = 0 = \psi_j^2(x)\, f(x)$. So $(Tf)(x) = \sum_{j=1}^{s} \psi_j^2(x)\, f(x) = f(x)$.

3. Let $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = l$. If $s \in \mathbb{N}$, then

$$D_w^\alpha(Tf) = \sum_{j=1}^{s} D_w^\alpha(\psi_j\, T_j(f\,\psi_j)) \qquad \text{on } \mathbb{R}^n. \tag{48}$$

If $s = \infty$, then (48) still holds, because on ${}^c(\bigcup\limits_{j=1}^{s} (U_j)_{\delta/2})$ both sides of (48) are equal to 0 and $\forall x \in \bigcup\limits_{j=1}^{s} (U_j)_{\delta/2}$ the number of sets $(U_j)_{\delta/2}$ intersecting the ball $B(x, \delta/2)$ is finite. Otherwise there exists a countable set of U_{j_s}, $s \in \mathbb{N}$, satisfying $(U_{j_s})_{\delta/2} \cap B(x, \delta/2) \neq \emptyset$. Hence $x \in U_{j_s}$, and we arrive to a contradiction since $\varkappa(\{U_j\}_{j=1}^\infty) < \infty$. Consequently, there exists $s_x \in \mathbb{N}$ such that $\operatorname{supp}(\psi_j T_j(f\psi_j)) \cap \overline{B(x, \delta/2)} \neq \emptyset$ for $j > s_x$. So

$$Tf = \sum_{j=1}^{s_x} \psi_j\, T_j(f\,\psi_j) \qquad \text{on } B(x, \delta/2).$$

Hence,

$$D_w^\alpha(Tf) = \sum_{j=1}^{s_x} D_w^\alpha(\psi_j T_j(f\,\psi_j)) = \sum_{j=1}^{\infty} D_w^\alpha(\psi_j T_j(f\,\psi_j)) \qquad \text{on } B(x, \delta/2).$$

Therefore, by the appropriate properties of weak derivatives, (48) with $s = \infty$ follows.

4. Let $\alpha \in \mathbb{N}_0^n$ and $\alpha = 0$ or $|\alpha| = l$. In (47) , for all $x \in \mathbb{R}^n$, and in (48), for almost all $x \in \mathbb{R}^n$, the number of nonzero summands does not exceed $\varkappa$. Hence, by Hölder's inequality for finite sums,

$$|D_w^\alpha(Tf)|^p \leq \varkappa^{p-1} \sum_{j=1}^{s} |D_w^\alpha(\psi_j\, T_j\,(f\psi_j))|^p$$

almost everywhere on $\mathbb{R}^n$ and consequently,

$$\int_{\mathbb{R}^n} |D_w^\alpha(Tf)|^p\, dx \leq \varkappa^{p-1} \sum_{j=1}^{s} \int_{\mathbb{R}^n} |D_w^\alpha(\psi_j\, T_j(f\psi_j))\,|^p\, dx.$$

5. Therefore,

$$\|Tf\|_{V_p^l(\mathbb{R}^n)} \leq M_3 \left(\sum_{j=1}^{s} \|\, \psi_j\, T_j(f\psi_j)\,\|^p_{V_p^l(\mathbb{R}^n)} \right)^{1/p},$$

where M_3 depends only on n, l and $\varkappa$. Since $\operatorname{supp} \psi_j \subset U_j$, by the Leibniz formula, we have

$$\begin{aligned} \|\psi_j\, T_j(f\psi_j)\,\|_{V_p^l(\Omega)} &\leq M_4 \,\|\, T_j(f\psi_j)\,\|_{V_p^l(U_j)} \\ &\leq M_4 \,\|\, T_j \,\|\, \|\, f\psi_j\,\|_{V_p^l(\Omega \cap U_j)} \\ &\leq M_5 \,\|T_j\|\, \|f\|_{V_p^l(\Omega \cap U_j)}, \end{aligned} \tag{49}$$

where M_4 and M_5 depend only on n, l and δ. Now, as in the proof of Lemma 3.3,

$$\|Tf\|_{V_p^l(\mathbb{R}^n)} \leq M_6 \sup_j \|T_j\|\, \|f\|_{W_p^l(\Omega)},$$

where M_6 depends only on n, l, δ and $\varkappa$. Hence (45) follows.
6. In case of the spaces $V_p^{l,\dots,l}(\cdot)$ we obtain (49) again by the Leibniz formula. Hence, statement 2 of the lemma follows.
7. In case of the spaces $W_p^l(\cdot)$ the application of the Leibniz formula requires also the inequality

$$\|D_w^\beta g\|_{L_p(\Omega)} \leq M_7 \,\|g\|_{W_p^l(\Omega)}, \tag{50}$$

where $|\beta| < l$ and M_7 is independent of g. (First for $g = \psi_j T_j(f\psi_j)$, then for $g = f\psi_j$.) If Ω has a quasi-resolved boundary, then this inequality holds (see [19], Chapter 4).

8. In case of the spaces $W_p^{l,\dots,l}(\cdot)$ one needs the inequality

$$\left\| \left(\frac{\partial^m g}{\partial x_k^m} \right)_w \right\|_{L_p(\Omega)} \leq M_8 \, \|g\|_{W_p^{l,\dots,l}(\Omega)}, \tag{51}$$

where $k = 1, \dots, n$, $m = 1, \dots, l-1$ and M_8 is independent of g. In a forthcoming paper we shall prove this inequality for open sets Ω satisfying the cone condition, using the extension theorem[9] for the spaces $W_p^{l,\dots,l}(\Omega)$ for open sets with a Lipschitz boundary. □

Remark 6.1. Suppose that in Lemma 6.1 the operators T_j satisfy the additional condition

$$f \in \widehat{V}_p^l(\Omega \cap U_j) \Longrightarrow \operatorname{supp} T_j f \subset U_j. \tag{52}$$

In this case the operator T may be constructed in a simpler way with the help of a standard partition of unity $\{\psi_j\}_{j=1}^s$, i.e., $\sum_{j=1}^s \psi_j = 1$ on Ω. We assume that $T_j(f\psi_j)(x) = 0$ if $x \in U_j$ and set

$$Tf = \sum_{j=1}^s T_j(f\psi_j) \qquad \text{on } \mathbb{R}^n.$$

The operator T is an extension operator. For, let $x \in \Omega$. If $x \in U_j$, then $(T_j(f\psi_j))(x) = \psi_j(x)\, f(x)$, and if $x \notin U_j$, then $(T_j(f\psi j))(x) = 0 = \psi_j(x)\, f(x)$. Thus $(Tf)(x) = \sum_{j=1}^s \psi_j(x)\, f(x) = f(x)$. Note also that for $f \in V_p^l(\Omega)$, because of (52), we have $T_j(f\psi_j) \in V_p^l(\mathbb{R}^n)$ and $\| T_j(f\psi_j) \|_{V_p^l(\mathbb{R}^n)} = \| T_j(f\psi_j) \|_{V_p^l(U_j)}$.

Remark 6.2. The method of pasting local extensions of Lemma 6.1 cannot be applied to the spaces $w_p^l(\Omega)$ and $w_p^{l,\dots,l}(\Omega)$ since estimates (50) and (51) do not hold if we replace $W_p^l(\Omega)$ by $w_p^l(\Omega)$, $W_p^{l,\dots,l}(\Omega)$ by $w_p^{l,\dots,l}(\Omega)$ respectively. In Lemmas 6.2–6.3 another method of pasting local extensions will be described, which is applicable also to the spaces $w_p^l(\Omega)$ and $w_p^{l,\dots,l}(\Omega)$.

Lemma 6.2. *Let $l \in \mathbb{N}$, $1 \leq p \leq \infty$ and let Ω be an open set. Moreover, let $\Omega_j \subset \mathbb{R}^n$, $j = \overline{1,s}$ where $s \in \mathbb{N}$, be open sets such that $\Omega \equiv \Omega_0 \subset \Omega_1 \subset \dots \subset \Omega_s$ and $\Omega^\delta \subset \Omega_s$ for some $\delta > 0$.*
1. *Suppose that for all $j = \overline{1,s}$ there exist bounded extension operators*

$$T_j : w_p^l(\Omega_{j-1}) \to w_p^l(\Omega_j).$$

[9] For this reason we shall not be able to apply Lemma 6.1 for the spaces $W_p^{l,\dots,l}(\Omega)$. (We shall apply Lemmas 6.2–6.3 instead.)

Then there exists a bounded extension operator

$$T : w_p^l(\Omega) \to w_p^l(\Omega^\delta). \tag{53}$$

If all T_j are linear, then T is also linear.
2. *The statement 1 also holds if one replaces the space $w_p^l(\cdot)$ by $w_p^{l,\dots,l}(\cdot)$.*
3. *The statement 1 also holds if one replaces the space $w_p^l(\cdot)$ by $W_p^l(\cdot)$, $W_p^{l,\dots,l}(\cdot)$, $V_p^l(\cdot)$ or $V_p^{l,\dots,l}(\cdot)$. Moreover, in this case in (53) Ω^δ could be replaced by $\mathbb{R}^n$.*

Proof. We set $T = T_s \cdots T_1$. Then T is an extension operator and

$$\|T\|_{w_p^l(\Omega)\to w_p^l(\Omega^\delta)} \le \prod_{j=1}^{s} \|T_j\|_{w_p^l(\Omega_{j-1})\to w_p^l(\Omega_j)}. \tag{54}$$

The second statement is proved in the same way.

In order to prove the third statement, let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta = 1$ on Ω and $\operatorname{supp}\eta \subset \Omega^{\delta/2}$, and set, for $f \in W_p^l(\Omega)$, $\widetilde{T}f = \eta T f$. Since $\forall g \in W_p^l(\Omega^\delta)$,

$$\|\eta g\|_{W_p^l(\Omega^\delta)} \le A\,\|g\|_{W_p^l(\Omega^\delta)},$$

where[10] A is independent of f, we have

$$\|\widetilde{T}\|_{W_p^l(\Omega)\to W_p^l(\mathbb{R}^n)} \le A \prod_{j=1}^{s} \|T_j\|_{W_p^l(\Omega_{j-1})\to W_p^l(\Omega_j)}. \tag{55}$$

The case of other spaces in statement 3 is similar. □

The proof being quite clear, the problem is in choosing appropriate Ω_j.

In the most simple case of the n-dimensional unit cube $\Omega = (0,1)^n$ one can take $\Omega_{j,1} = (0,2)^j \times (0,1)^{n-j}$, $j = 1,\dots,n$, and $\Omega_{j,2} = (-2,2)^j \times (0,2)^{n-j}$, $j = 1,\dots,n$. Then

$$\Omega \subset \Omega_{1,1} \subset \Omega_{2,1} \subset \cdots \subset \Omega_{n,1} \subset \Omega_{1,2} \subset \Omega_{2,2} \subset \cdots \subset \Omega_{n,2} \supset \Omega^\delta,$$

where $\delta = 1$.

For the general case of an open set with a Lipschitz boundary this can be done with the help of the following statement.

Lemma 6.3. *Let Ω be an open set with a Lipschitz boundary, satisfying the definition in Section 5 with the parallelepipeds $V_j, j = \overline{1,s}$, and with the*

[10] This follows by the Leibniz formula and inequality (2).

parameters $d, D, \varkappa$ *and* M. *Moreover, suppose*[11] *that if* $V_j \cap V_k \cap \Omega = \emptyset$, *then* $V_j \cap V_k = \emptyset$.

Given $0 < d_1 < d$, $M_1 > M$ *and* $\gamma > 0$, *there exists an open set* Ω_1 *with a Lipschitz boundary, satisfying the definition in Section* 5 *with the parallelepipeds* $V_j^{(1)}, j = \overline{1,s}$, *and with the parameters* $d_1, D, \varkappa$ *and* M_1, *for which*

$$V_1^{(1)} = V_1, \quad (V_j)_{d-d_1} \subset V_j^{(1)} \subset V_j, \qquad j = \overline{2,s},$$

(*if* $V_j \cap V_1 = \emptyset$, *then* $V_j^{(1)} = V_j$),

$$\Omega \subset \Omega_1 \subset \Omega \cup (\Omega^\gamma \cap V_1), \quad \partial\Omega \cap V_1 \subset \Omega_1,$$

and there exists ϱ, *depending only on* d, d_1, M, M_1 *and* γ, *such that*

$$\left(\Omega \cup (V_1)_{d/2}\right)^\varrho \subset \Omega_1.$$

For the case $s \in \mathbb{N}$ see the proof in [12].

Lemma 6.4. 1. *Let* $l \in \mathbb{N}$, $1 \le p \le \infty$. *Suppose that for each bounded elementary domain* $H \subset \mathbb{R}^n$, *defined by* (40), *with a Lipschitz boundary with the parameters* d, D *and* M *there exists a bounded linear extension operator*

$$T_H : \widetilde{W}_p^l(H) \to W_p^l(V),$$

where $V = \{x \in \mathbb{R}^n : a_i < x_i < b_i, i = 1, \ldots, n-1,\ a_n < x_n < \infty\}$, $\widetilde{W}_p^l(H) = \{f \in W_p^l(H) : \operatorname{supp} f \subset \overline{H} \cap V\}$ *and* $\|T_H\| \le c_2$, *where* $c_2 > 0$ *depends only on* n, l, p, d, D *and* M.

Then for each open set $\Omega \subset \mathbb{R}^n$ *with a Lipschitz boundary*[12] *there exists a bounded linear extension operator*

$$T : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n).$$

Moreover, $\|T\| \le c_3$, *where* $c_3 > 0$ *depends only on* $n, l, p, d, D, \varkappa$ *and* M.
2. *The statement* 1 *also holds if the space* $W_p^l(\cdot)$ *is replaced by* $V_p^l(\cdot)$.

[11] This requirement does not restrict the generality. One can prove that given the collection of parallelepipeds satisfying the conditions 1)–4) of Section 5 with the parameters $s, d, D, \varkappa$ and M, another collection of $\tilde{s}$ parallelepipeds could be constructed such that the conditions 1)–4) are satisfied with some other parameters $\tilde{d}, \widetilde{D}, \tilde{\varkappa}$ which depend only on d, D and $\varkappa$ and the same M, and this additional condition is also satisfied. (See [12].)

[12] If Ω is bounded, then it is enough to suppose that $\|T_H\| < \infty$ for each bounded elementary domain H.

Proof. By the assumptions of the lemma for all $j = \overline{1,s}$ there exist bounded extension operators

$$T_j : \widehat{W}_p^l(\lambda_j(\Omega \cap V_j)) \to W_p^l(\lambda_j(V_j)).$$

Let $(\Lambda_j f)(x) = f(\lambda_j(x))$ and define

$$T_j^{(1)} = \Lambda_j \, T_j \, \Lambda_j^{(-1)}.$$

We note that $\Lambda_j^{(-1)} : \widehat{W}_p^l(\Omega \cap V_j) \to \widehat{W}_p^l(\lambda_j(\Omega \cap V_j))$, $\Lambda_j : \widehat{W}_p^l(\lambda_j(V_j)) \to \widehat{W}_p^l(V_j)$ and $\| \Lambda_j \|$, $\| \Lambda_j^{(-1)} \|$ do not exceed some quantity depending only on n and l. Hence,

$$T_j^{(1)} : \widehat{W}_p^l(\Omega \cap V_j) \to \, W_p^l(V_j)$$

and

$$\| T_j^{(1)} \| \leq \| \Lambda_j \| \cdot \| T_j \| \cdot \| \Lambda_j^{(-1)} \| \leq \, M_1 \, \| T_j \|,$$

where M_1 depends only on n and l.

If Ω is bounded, then $s \in \mathbb{N}$ and by Lemma 6.1 there exists a bounded linear extension operator $T : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)$. If Ω is unbounded, then $s = \infty$ and by the definition of an open set with a C^l- or Lipschitz boundary each bounded elementary domain $\lambda_j(\Omega \cap V_j)$ has the same parameters d, D, M. Hence, by the assumptions of the lemma $\|T_j\| \leq c_2$. Moreover, in this case the multiplicity of the covering $\{V_j\}_{j=1}^{\infty}$ is finite. Thus, Lemma 6.1 is applicable, which ensures the existence of a bounded linear extension operator $T : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)$.

In case of the spaces $V_p^l(\cdot)$ the proof is similar. □

Remark 6.3. In the assumptions of Lemma 6.4 the condition $T_H : \widetilde{W}_p^l(H) \to W_p^l(V)$ can be replaced by $T_H : \widetilde{W}_p^l(H) \to W_p^l(H^\gamma \cap V)$ for some $\gamma > 0$, which depends only on n, d and M. This follows since one can construct another extension operator $T_H^{(1)} : \widetilde{W}_p^l(H) \to W_p^l(V)$ by setting $T_H^{(1)} f = \eta \, T_H f$, where $\eta \in C^\infty(\overline{V})$ is such that $\eta = 1$ on H, $\operatorname{supp} \eta \subset H^\gamma \cap V$ and, for $|\alpha| \leq l$, $\|D^\alpha \eta\|_{C(\overline{V})} \leq c_4$, where $c_4 > 0$ depends only on l, d and M. (To do this one can mollify the characteristic function of $H^{\varrho/2}$ with step of mollification equal to $\varrho/4$, where $\varrho = \operatorname{dist}(H, \partial(H^\gamma))$, and note that $\varrho \geq c_5$, where $c_5 > 0$ depends only on n, d and M.) Hence $\|T_H^{(1)}\| \leq c_6$, where $c_6 > 0$ depends only on n, l, p, d, D and M.

Lemma 6.5. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$. Suppose that for each bounded elementary domain $H \subset \mathbb{R}^n$, defined by (40), with a Lipschitz boundary with the parameters d, D and M, for some $\gamma > 0$, which depends only on n, d and M, there exists a bounded linear extension operator*

$$T_H : w_p^l(H) \to w_p^l(H^\gamma \cap V)$$

and $\|T_H\| \le c_7$, where $c_7 > 0$ depends only on n, l, p, d, D and M.

Then for each open set $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary[13] *there exists $\delta > 0$, which depends only on n, d and M, and a bounded linear extension operator*

$$T : w_p^l(\Omega) \to w_p^l(\Omega^\delta).$$

Moreover, $\|T\| \le c_8$, where $c_8 > 0$ depends only on $n, l, p, d, D, \varkappa$ and M.

Proof. 1. If $s \in \mathbb{N}$, then by Lemma 6.3 there exist open sets $\Omega_1, \dots, \Omega_s$ with Lipschitz boundaries such that $\Omega \subset \Omega_1 \subset \dots \subset \Omega_s$, further $\Omega_s \supset \Omega^\varrho$ for some $\varrho > 0$ and

$$\Omega_k \subset \Omega_{k-1} \cup (\Omega_{k-1}^\gamma \cap V_k), \quad k = 1, \dots, s \qquad (\Omega_0 \equiv \Omega).$$

Next we note that, for some rotation λ_k, $\lambda_k(\Omega_{k-1} \cap V_k)$ is a bounded elementary domain with a Lipschitz boundary. Hence, by the assumptions of the lemma, there exists a bounded linear extension operator

$$T_k^{(1)} : w_p^l(\lambda_k(\Omega_{k-1} \cap V_k)) \to w_p^l((\lambda_k(\Omega_{k-1}))^\gamma \cap \lambda_k(V_k)).$$

For a function g defined on $\lambda_k(\Omega_{k-1} \cap V_k)$, let $(\Lambda_k g)(x) = g(\lambda_k(x))$, $x \in \Omega_{k-1} \cap V_k$. For $f \in w_p^l(\Omega_{k-1} \cap V_k)$, we define $T_k^{(2)} f = \Lambda_k T_k^{(1)} \Lambda_k^{-1} f$. Since the operators

$$\begin{aligned} \Lambda_k &: w_p^l(\lambda_k(\Omega_{k-1} \cap V_k)) \to w_p^l(\Omega_{k-1} \cap V_k), \\ \Lambda_k^{-1} &: w_p^l(\Omega_{k-1} \cap V_k) \to w_p^l(\lambda_k(\Omega_{k-1} \cap V_k)) \end{aligned} \tag{56}$$

are bounded, the operator

$$T_k : w_p^l(\lambda_k(\Omega_{k-1} \cap V_k)) \to w_p^l(\Omega_{k-1}^\gamma \cap V_k)$$

is a bounded linear extension operator. Moreover, $\|T_k\| \le c_9$, where $c_9 > 0$ depends only on n, l, p, d, D, and M.

[13] If Ω is bounded, then it is enough to suppose that $\|T_H\| < \infty$ for each bounded elementary domain H .

Finally, for $f \in w_p^l(\Omega_{k-1})$, we set $(T_k f)(x) = f(x)$ if $x \in \Omega_{k-1}$ and $(T_k f)(x) = (T_k^{(2)} f)(x)$ if $x \in \Omega_k \setminus \Omega_{k-1}$. Clearly, the operator $T_k : w_p^l(\Omega_{k-1}) \to w_p^l(\Omega_k)$ is also bounded, since

$$\|T_k f\|_{w_p^l(\Omega_k)} \le \|f\|_{w_p^l(\Omega_{k-1})} + \|T_k^{(2)} f\|_{w_p^l(\Omega_{k-1}^\gamma \cap V_k)}$$
$$\le \left(1 + \|T_k^{(2)}\|_{w_p^l(\Omega_{k-1} \cap V_k) \to w_p^l(\Omega_{k-1}^\gamma \cap V_k)}\right) \|f\|_{w_p^l(\Omega_{k-1})}.$$

Hence, $T = T_s \cdots T_1 : w_p^l(\Omega) \to w_p^l(\Omega^\delta)$ is a bounded linear extension operator. However, by (54) it follows that $\|T\| \le \tilde{c}_8$, where $\tilde{c}_8 > 0$ depends only on n, p, d, D, M and on s instead of $\varkappa$.

2. If $s = \infty$, one should split the infinite collection of the parallelepipeds $\{V_j\}_{j=1}^\infty$ in a finite union of disjoint infinite subcollections, each of which consists of disjoint parallelepipeds: $\{V_j\}_{j=1}^\infty = \bigcup\limits_{\mu=1}^{m} \{V_{k\mu}\}_{k=1}^\infty$, where $m \in \mathbb{N}$, $V_{k\mu} \in \{V_j\}_{j=1}^\infty$, $\{V_{k\mu}\}_{k=1}^\infty \cap \{V_{k\tilde{\mu}}\}_{k=1}^\infty = \emptyset$ if $\mu \ne \tilde{\mu}$ and $V_{k\mu} \cap V_{\tilde{k}\mu} = \emptyset$ if $k \ne \tilde{k}$. Moreover, without loss of generality we assume that for each parallelepiped V_j either $V_{k\mu} \cap V_j = \emptyset$ or $V_{\tilde{k}\mu} \cap V_j = \emptyset$ $(k \ne \tilde{k})$. We also note that this can be done in such a way that m does not exceed some quantity, depending on n, d, D and $\varkappa$.

By Lemma 6.3, applied to Ω and V_{k1}, there exist open sets Ω_k, $k \in \mathbb{N}$, with Lipschitz boundaries with the parameters $d/2, D, \varkappa$ and $2M$ and with the parallelepipeds $V_j^{(k)}$ such that $\Omega \subset \Omega_{k1} \subset \Omega \cup (\Omega^\gamma \cap V_{k1})$, $\partial\Omega \cap V_{k1} \subset \Omega_{k1}$ and, for some $\varrho > 0$ depending only on d, M and γ, $(\Omega \cap (V_{k1})_{d/2})^\varrho \subset \Omega_{k1}$. We note that $V_{k1}^{(k)} = V_{k1}$. Moreover, if j is such that $V_j \cap V_{k1} = \emptyset$ (briefly $j \in J_{k1}$), then $V_j^{(k)} = V_j$, otherwise $(V_j)_{d/2} \subset V_j^{(k)} \subset V_j$. Next we consider the parallelepipeds $\widetilde{V}_j, j \in \mathbb{N}$, such that $\widetilde{V}_j = V_j^{(k)}$ for $j \in \widehat{J}_{k1} = \mathbb{N} \setminus J_{k1}$, $k \in \mathbb{N}$, and $\widetilde{V}_j = V_j$ for $j \in \bigcap_{k=1}^\infty J_{k1}$, and the set $\Omega_1 = \bigcup_{k=1}^\infty \Omega_{k1}$. (Since $\widehat{J}_{k1} \cap \widehat{J}_{\tilde{k}1} = \emptyset$, $k \ne \tilde{k}, \widetilde{V}_j$ are well-defined.) It follows that Ω_1 is an open set with a Lipschitz boundary with the parallelepipeds $\widetilde{V}_j$ and with the parameters $d/2, D, \varkappa$ and $2M$. In view of the properties of $V_j^{(k)}$ and Ω_{k1} we have that

$$\Omega \subset \Omega_1 \subset \Omega \cup \left(\Omega^\gamma \cap \left(\bigcup_{k=1}^\infty V_{k1}\right)\right), \quad \partial\Omega \cap \left(\bigcup_{k=1}^\infty V_{k1}\right) \subset \Omega_1$$

and

$$\left(\Omega \cap \left(\bigcup_{k=1}^\infty (V_{k1})_{d/2}\right)\right)^\varrho \subset \Omega_1.$$

(We have taken into account that $(A \cup B)^\varrho = A^\varrho \cup B^\varrho$.) As was noted in the first step of the proof, by assumptions of the lemma for each $k \in \mathbb{N}$ there exists a bounded linear extension operator $T_{k1} : w_p^l(\Omega \cap V_{k1}) \to w_p^l(\Omega^\gamma \cap V_{k1})$. Moreover, $\|T_{k1}\| \le c_9$, $k \in \mathbb{N}$. We define the extension operator T_1 for $f \in w_p^l(\Omega_1)$ by setting $T_1 f = T_{k1} f$ on Ω_{k1}, $k \in \mathbb{N}$. Since $(\Omega_{k1} \setminus \Omega) \cap (\Omega_{\tilde{k}1} \setminus \Omega) = \emptyset$ for $k \neq \tilde{k}$, the operator T_1 is well-defined. Moreover, $T_1 : w_p^l(\Omega) \to w_p^l(\Omega_1)$ and is bounded. Indeed, if $p < \infty$, then $\forall f \in w_p^l(\Omega)$,

$$
\begin{aligned}
\|T_1 f\|^p_{w_p^l(\Omega_1)} &= \Big(\sum_{|\alpha|=l} \|D_w^\alpha (T_1 f)\|_{L_p(\Omega_1)} \Big)^p \\
&\le l^{np} \sum_{|\alpha|=l} \int_{\Omega_1} |D_w^\alpha (T_1 f)|^p \, dx \\
&= l^{np} \sum_{|\alpha|=l} \Big(\int_{\Omega} |D_w^\alpha f|^p \, dx + \sum_{k=1}^{\infty} \int_{\Omega_{k1} \setminus \Omega} |D_w^\alpha (T_{k1} f)|^p \, dx \Big) \\
&\le l^{np} \Big(\|f\|^p_{w_p^l(\Omega)} + \sum_{k=1}^{\infty} \|T_{k1} f\|^p_{w_p^l(\Omega^\gamma \cap V_{k1})} \Big) \\
&\le l^{np} \Big(\|f\|^p_{w_p^l(\Omega)} + c_9^p \sum_{k=1}^{\infty} \|f\|^p_{w_p^l(\Omega \cap V_{k1})} \Big) \\
&\le l^{np} \Big(\|f\|^p_{w_p^l(\Omega)} + c_9^p l^{np} \sum_{k=1}^{\infty} \sum_{|\alpha|=l} \int_{\Omega \cap V_{k1}} |D_w^\alpha f|^p \, dx \Big) \\
&\le l^{np} \left(1 + c_9^p l^{np}\right) \|f\|^p_{w_p^l(\Omega)}.
\end{aligned}
$$

Hence, for $1 \le p < \infty$,

$$
\|T_1\|_{w_p^l(\Omega) \to w_p^l(\Omega_1)} \le l^n \left(1 + c_9 l^n\right).
$$

If $p = \infty$, the argument is similar.

In a similar way, starting from Ω_1, with the help of Lemma 6.3 we construct an open set $\Omega_2 \supset \Omega_1$ with a Lipschitz boundary and an appropriate extension operator T_2, and so on. Thus we obtain open sets $\Omega_1, \Omega_2, \ldots, \Omega_m$ with Lipschitz boundaries such that $\Omega \subset \Omega_1 \subset \cdots \subset \Omega_m$, and

$$
\Big(\Omega_{\mu-1} \cap \Big(\bigcup_{k=1}^{\infty} (V_{k\mu})_{d/2} \Big) \Big)^\varrho \subset \Omega_\mu,
$$

and bounded linear extension operators $T_k : w_p^l(\Omega_{k-1}) \to w_p^l(\Omega_k)$, $k = 1, \ldots, m$ $(\Omega_0 \equiv \Omega)$, satisfying the estimate

$$\|T_k\|_{w_p^l(\Omega_{k-1}) \to w_p^l(\Omega_k)} \le l^n \left(1 + c_9 l^n\right).$$

Finally, we note that

$$\Omega^\varrho = \left(\Omega \cap \left(\bigcup_{j=1}^{\infty} (V_j)_{d/2}\right)\right)^\varrho = \left(\Omega \cap \left(\bigcup_{\mu=1}^{m} \bigcup_{k=1}^{\infty} (V_{k\mu})_{d/2}\right)\right)^\varrho$$
$$\subset \bigcup_{\mu=1}^{m} \left(\Omega_{\mu-1} \cap \left(\bigcup_{k=1}^{\infty} (V_{k\mu})_{d/2}\right)\right) \subset \bigcup_{\mu=1}^{m} \Omega_\mu = \Omega_m.$$

(We have taken into account that $(A \cap B)^\varrho \subset A^\varrho \cap B^\varrho$.) Hence, $T = T_m \cdots T_1 : w_p^l(\Omega) \to w_p^l(\Omega^\delta)$ is a bounded linear extension operator. Moreover, by (54) $\|T\| \le c_8$, where $c_8 > 0$ depends only on n, p, d, D, M and $\varkappa$.

3. We also note that if $s \in \mathbb{N}$, then we can apply the same procedure as in the second step of the proof, and we shall obtain the desired estimate for $\|T\|$, thus, improving the estimate established in the first step of the proof. $\square$

Lemma 6.6. 1. *Let $l \in \mathbb{N}$, $1 \le p \le \infty$. Suppose that for each bounded elementary domain $H \subset \mathbb{R}^n$, defined by* (40)*, with a Lipschitz boundary with the parameters d, D and M and for each rotation λ there exists a bounded linear extension operator*

$$T_{H,\lambda} : w_p^{l,\ldots,l}(\lambda(H)) \to w_p^{l,\ldots,l}(\lambda(V))$$

and $\|T_{H,\lambda}\| \le c_{10}$, where $c_{10} > 0$ depends only on n, l, p, d, D and M.

Then for each open set $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary there exists $\delta > 0$, depending only on n, d and M, and a bounded linear extension operator

$$T : w_p^{l,\ldots,l}(\Omega) \to w_p^{l,\ldots,l}(\Omega^\delta).$$

Moreover, $\|T\| \le c_{11}$, where $c_{11} > 0$ depends only on $n, l, p, d, D, \varkappa$ and M.

2. *The statement* 1 *also holds if one replaces $w_p^{l,\ldots,l}(\cdot)$ by $W_p^{l,\ldots,l}(\cdot)$ or $V_p^{l,\ldots,l}(\cdot)$. In this case Ω^δ can be replaced by $\mathbb{R}^n$.*

Proof. The proof is actually the same as the proof of Lemma 6.5. The only distinction is that now we do not use the fact that the operators (56) where $w_p^l(\cdot)$ is replaced by $w_p^{l,\dots,l}(\cdot)$, $W_p^{l,\dots,l}(\cdot)$ or $V_p^{l,\dots,l}(\cdot)$, are bounded.[14]

In case of the spaces $W_p^{l,\dots,l}(\cdot)$ the same operator T is a bounded linear extension operator as $T : W_p^{l,\dots,l}(\Omega) \to W_p^{l,\dots,}(\Omega^\delta)$. Next let a function $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta = 1$ on Ω, $\operatorname{supp}\eta \subset \Omega^\delta$ and, for $|\alpha| \le l$, $\|D^\alpha \eta\|_{C(\mathbb{R}^n)} \le c_{12}$, where $c_{12} > 0$ depends only on n, l and δ.

We set $\widetilde{T}f = \eta T f$ and, as in the proof of Lemma 6.3, it follows that $\widetilde{T} : W_p^{l,\dots,l}(\Omega) \to W_p^{l,\dots,}(\mathbb{R}^n)$ is a bounded linear extension operator. Moreover, $\|\widetilde{T}\| \le c_{13}$, where $c_{13} > 0$ depends only on $n, l, p, d, D, \varkappa$ and M.

The case of the spaces $V_p^{l,\dots,l}(\Omega)$ is similar. □

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If $1 < p < \infty$, then it holds, but in case of the spaces $w_p^{l,\dots,l}(\cdot)$ these operators are not bounded. (In case of the spaces $W_p^{l,\dots,l}(\cdot)$ or $V_p^{l,\dots,l}(\cdot)$ they are bounded.) However, these statements, in the framework of this paper, follow from the extension theorems for the spaces $w_p^{l,\dots,l}(\Omega)$, $W_p^{l,\dots,l}(\Omega)$, $V_p^{l,\dots,l}(\Omega)$ respectively and could not be used in the proof of this lemma.

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