On Wiener's type regularity of a boundary point for higher order elliptic equations

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1 Introduction

In 1924 Wiener [1] gave his famous criterion for the so called regularity of a boundary point.

A point O at the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is called regular if solutions of the Dirichlet problem for the Laplace equation in Ω with the Dirichlet data, continuous at O, are continuous at this point. (I do not want to explain in which sense the solution is understood—this is not quite trivial and is also due to Wiener [2].)

Before Wiener's result only some special facts concerning the regularity were known. For example, since (by Riemann's theorem) an arbitrary Jordan domain in $\mathbb{R}^2$ is conformally homeomorphic to the unit disc, it follows that any point of its boundary is regular.

As for the n-dimensional case, it was known for years that a boundary point O is regular provided the complement of Ω near O is so thick that it contains an open cone with O as a vertex (Poincaré [3], Zaremba [4]). Lebesgue noticed that the vertex of a sufficiently thin cusp in $\mathbb{R}^3$ is irregular [5]. Therefore it became clear that, in order to characterize the regularity, one should find proper geometric or quasi-geometric terms describing the massiveness of $\mathbb{R}^n \setminus \Omega$ near O.

To this end Wiener introduced the harmonic capacity $\operatorname{cap}(K)$ of a compact set K in $\mathbb{R}^n$, which corresponds to the electrostatic capacity of a body when $n = 3$. Up to a constant factor, the harmonic capacity in the case $n > 2$ is equal to

$$\inf\left\{ \int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx \,:\, u \in C_0^\infty(\mathbb{R}^n),\ u > 1 \text{ on } K \right\}.$$

For $n = 2$ this definition of capacity needs to be altered.

The notion of capacity enabled Wiener to state and prove the following result.

Theorem (Wiener). *The point O at the boundary of the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is regular if and only if*

$$\sum_{k \geq 1} 2^{(n-2)k} \operatorname{cap}(B_{2^{-k}} \setminus \Omega) = \infty. \tag{1}$$

We assume that O is the origin of a coordinate system and use the notation $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$. It is straightforward that (1) can be rewritten in the integral form

$$\int_0 \frac{\operatorname{cap}(B_\sigma \setminus \Omega)}{\operatorname{cap}(B_\sigma)} \frac{d\sigma}{\sigma} = \infty. \tag{2}$$

Wiener's theorem was the first necessary and sufficient condition characterizing the dependence of properties of solutions on geometric properties of the boundary. The theorem strongly influenced potential theory, partial differential equations, and probability theory. Especially striking was the impact of the notion of the Wiener capacity, which gave an adequate language to answer many important questions. During the years many attempts have been made to extend the range of Wiener's result to different classes of linear equations of the second order, although some of them were successful only in the sufficiency part. I mention here three necessary and sufficient conditions.

First, for uniformly elliptic operators with measurable bounded coefficients in divergence form

$$u \mapsto \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j}, \tag{3}$$

Littman, Stampacchia and Weinberger [6] proved in 1963 that the regularity of a boundary point is equivalent to the Wiener condition (1).

Second, in 1982 Fabes, Jerison and Kenig [7] gave an interesting analog of the Wiener criterion for a class of degenerate elliptic operators of the form (3).

The third criterion for regularity, due to Dal Maso and Mosco [8], concerns the Schrödinger operator

$$u \mapsto -\Delta u + \mu u \quad \text{in } \Omega,$$

where μ is a measure. It characterizes both the geometry of Ω and the potential μ near the point O.

It seems worthwhile to mention a recently solved problem concerning a non-linear equation, which remained open for twenty five years. I mean the question of the regularity of a boundary point for the operator

$$u \mapsto \operatorname{div}(|\operatorname{grad} u|^{p-2} \operatorname{grad} u) \quad \text{in } \Omega, \tag{4}$$

where $p > 1$. This differential operator, often called the p-Laplacian, appears in some mechanical applications and is interesting from a pure mathematical point of view.

In 1970 I proved [9] that the following variant of the Wiener criterion is sufficient for the regularity with respect to (4)

$$\int_0 \left(\frac{p\text{-cap}(B_\sigma \setminus \Omega)}{p\text{-cap}(B_\sigma)} \right)^{1/(p-1)} \frac{d\sigma}{\sigma} = \infty. \tag{5}$$

Here $1 < p \le n$ and the p-capacity is a modification of the Wiener capacity generated by the p-Laplacian. This result was generalized by Gariepy and Ziemer [10] to a large class of elliptic quasilinear equations

$$\operatorname{div} A(x, u, \operatorname{grad} u) = B(x, u, \operatorname{grad} u).$$

Condition (5) and its generalizations also turned out to be relevant in studying the fine properties of elements in Sobolev spaces. See, e.g. the book [11].

For a long time it seemed probable that (5) is also necessary for the regularity with respect to (4), and indeed, for $p \ge n-1$, Lindqvist and Martio [12] proved this for the operator (4). Finally, Kilpeläinen and Malý gave a proof valid for arbitrary values of $p > 1$ [13]. A comprehensive exposition of the area surrounding these results can be found in the recent book by Malý and Ziemer [14].

So far I spoke only about the regularity of a boundary point for second order elliptic equations. However, the topic could be extended to include other equations, systems, boundary conditions and function spaces. In principle, the Wiener criterion suggests the possibility of the complete characterization of properties of domains, equivalent to various solvability and spectral properties of boundary value problems.

2 Regular points for arbitrary even order elliptic equations

Let $P_{2m}(D_x)$ be a strongly elliptic scalar or square matrix homogeneous differential operator of order $2m$ with constant coefficients, and let $D_x = (\partial/i\partial x_1, \dots, \partial/i\partial x_n)$.

Consider the Dirichlet problem

$$\begin{cases} P_{2m}(D_x)u = f, \quad f \in C_0^\infty(\Omega), \\ u \in \mathring{H}^m(\Omega), \end{cases} \tag{6}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $\mathring{H}^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm of the Sobolev space $H^m(\Omega)$.

Definition 1. We call a boundary point O regular with respect to $P_{2m}(D_x)$ if, for any $f \in C_0^\infty(\Omega)$,

$$u(x) \to 0 \text{ as } x \to O.$$

For $n = 2, 3, \dots, 2m-1$ the regularity is a consequence of the Sobolev imbedding theorem. Therefore, we assume $n \geq 2m$. One can show that in the case of the Laplacian this definition of the regularity corresponds to that given in the Introduction.

A general problem is to find more or less explicit conditions for regularity. Contrary to the case of the scalar second order equation with real coefficients, this problem is in a non-satisfactory state. Before mentioning a few known facts I introduce the m-harmonic capacity $\operatorname{cap}_m(K)$ of a compact set $K \subset \mathbb{R}^n, n > 2m$, defined as

$$\inf\Big\{ \int_{\mathbb{R}^n} \sum_{|\alpha|=m} |D^\alpha u(x)|^2 \, dx : \ u \in C_0^\infty(\mathbb{R}^n), \ u = 1 \text{ in a neighbourhood of } K \Big\}.$$

(This definition needs to be changed in the case $n = 2m$.) The extremal function U_m of this variational problem will be called the m-harmonic potential.

For $m = 1$ this capacity is proportional to Wiener's capacity. It is a deeper fact that for $m > 1$ the capacity $\operatorname{cap}_m(K)$ is equivalent to the potential theoretic Riesz capacity of order $2m$. In other words, replacing the condition $u = 1$ near K in the definition of $\mathrm{cap}_m(K)$ by $u \geq 1$ on K we arrive at an equivalent set function (see [15], Theorem 9.3.2/1).

In 1977 I proved that for $n = 4, 5, 6, 7$ the Wiener type condition

$$\int_0 \frac{\mathrm{cap}_2(B_\sigma \setminus \Omega)}{\mathrm{cap}_2(B_\sigma)} \frac{d\sigma}{\sigma} = \infty \tag{7}$$

garantees the regularity of O with respect to the biharmonic operator Δ^2 [16], [17].

The difference between the conditions (1) and (7) is that the harmonic capacity cap is replaced by the biharmonic capacity cap_2.

The restriction to dimensions $n < 8$ is dictated by the method of proof based upon the property of weighted positivity of the biharmonic operator:

$$\int_{\mathbb{R}^n} u(x)\Delta^2 u(x) \frac{dx}{|x|^{n-4}} \geq 0. \tag{8}$$

Unfortunately, this property fails for $n \geq 8$.

As a byproduct, in the same paper I proved that the Green function of the Dirichlet problem for Δ^2 satisfies

$$|G_2(x, y)| \leq \frac{c(n)}{|x - y|^{n-4}}, \quad n = 5, 6, 7, \tag{9}$$

where x, y are arbitrary points of Ω and $c(n)$ is a constant independent of Ω.

Extensions of the results obtained in [17] to the polyharmonic operator Δ^m, $m > 2$, were stated (without proofs) in my and Donchev's article [18], where the dimensions $n = 2m$, $2m + 1$, $2m + 2$ were considered. In what follows I prove all theorems formulated in [18] adding some new results. In particular, I show for the same dimensions as in [17], [18] that the regularity with respect to Δ^m is a local property, i.e. it does not depend on the geometry of Ω at any positive distance from the point O.

Now I formulate some open problems connected with the above mentioned results [19].

Problem 1. *Let $n \geq 8$ if $m = 2$ and $n \geq 2m + 3$ if $m > 2$. Prove or disprove that the point O is regular with respect to Δ^m provided*

$$\int_0 \frac{\mathrm{cap}_m(B_\sigma \setminus \Omega)}{\mathrm{cap}_m(B_\sigma)} \frac{d\sigma}{\sigma} = \infty. \tag{10}$$

Problem 2. *Let $n \geq 8$ if $m = 2$ and $n \geq 2m + 3$ if $m > 2$. Prove that the Green function G_m of the Dirichlet problem for Δ^m in an arbitrary domain $\Omega \subset \mathbb{R}^n$ satisfies*

$$|G_m(x,y)| \leq \frac{c(m,n)}{|x-y|^{n-2m}}, \tag{11}$$

where $c(m,n)$ is independent of Ω.

Clearly, (11) leads to the following estimate of the maximum modulus of the solution $u \in \mathring{H}^m(\Omega)$

$$||u||_{L_\infty(\Omega)} \leq c(m, n, \mathrm{mes}_n \Omega)||f||_{L_p(\Omega)},$$

where $p > n/2m$. However, the validity of this estimate for the same n and m as in Problem 2 is also an open question. Moreover, it has not been proved that $u \in L_\infty(\Omega)$ for any $f \in C_0^\infty(\Omega)$ without any assumptions about $\partial\Omega$.

It is unknown whether (10) is also a necessary condition which can be stated as

Problem 3. *Prove or disprove the following assertion: Let $n > 2m$. If O is regular with respect to Δ^m, then (10) is valid.*

Perhaps, in order to prove the necessity of (10) it would be helpful to verify or disprove the following estimate for the m-harmonic potential of a compact set $K \subset B_1$:

$$|U_m(x)| \leq c(m,n)\frac{\mathrm{cap}_m(K)}{|x|^{n-2m}} \quad \text{for } |x| > 2, \tag{12}$$

where $c(m,n)$ is independent of K and $n > 2m$.

It may happen that the condition (10) is sufficient (and even necessary and sufficient) for the regularity with respect to an arbitrary strongly elliptic scalar operator $P_{2m}(D_x)$ with real coefficients, provided this operator has a positive fundamental solution in $\mathbb{R}^n$, $n > 2m$. The importance of the last restriction will be commented on later in Sec. 4.

3 The Hölder regularity and the *k*-regularity

We say that O is Hölder regular with respect to $P_{2m}(D_x)$ if the solution of (6) with an arbitrary $f \in C_0^\infty(\Omega)$ satisfies

$$|u(x)| \leq c|x|^\alpha \tag{13}$$

with some positive α.

While discussing the Hölder regularity I shall restrict myself to the operator Δ^m, $2m < n$.

If $m = 2, n = 5, 6, 7$ or $m > 2$, $n = 2m + 1, 2m + 2$, one can prove that the solution of (6) satisfies the following estimate for small ρ and $r \in (0, \rho)$

$$\sup_{B_r \cap \Omega} |u| \leq c \sup_{B_\rho \cap \Omega} |u| \exp\Big(-C \int\limits_r^\rho \frac{\mathrm{cap}_m(B_\sigma \setminus \Omega)}{\mathrm{cap}_m(B_\sigma)} \frac{d\sigma}{\sigma} \Big), \tag{14}$$

where c and C do not depend on r and ρ. For the Laplacian and for the operator (3) with bounded measurable real coefficients this estimate was proved in [20], [21]. By (14) the Hölder condition (13) follows from the inequality

$$\liminf_{r \to 0} \frac{1}{|\log r|} \int_r^\rho \frac{\mathrm{cap}_m(B_\sigma \setminus \Omega)}{\mathrm{cap}_m(B_\sigma)} \frac{d\sigma}{\sigma} > 0. \tag{15}$$

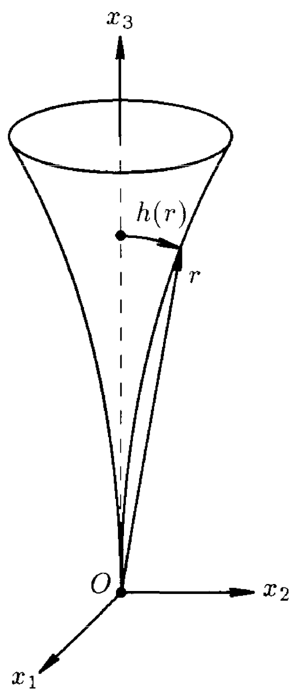
Although this condition for the Hölder regularity with respect to Δ^m is precise in a sense, a simple counterexample shows that it is not necessary and that it is impossible to give an equivalent description of Hölder regularity in terms of the Wiener integral. The sufficient condition (15) was improved in [22] for the case $m = 1$ (see also [23], where a similar result was obtained for second order elliptic differential operators in divergence form with measurable bounded coefficients).

Problem 4. *Find a necessary and sufficient condition for the Hölder regularity of a boundary point with respect to the Laplace operator.*

Without being precise, one can define the k-regularity of the point O with respect to $P_{2m}(D_x)$, $0 < k \leq m$, as follows. We say that O is k-regular if the k-th gradient of the solution to (6) with an arbitrary $f \in C_0^\infty(\Omega)$ vanishes at O in some sense.

The question of k-regularity was treated by Maz'ya and Tashchiyan [24], where a capacitary sufficient condition for the 1-regularity with respect to Δ^2 in a 3-dimensional domain was found. I prefer not to discuss this condition in its general form and restrict myself to an example.

Let a cusp be given in spherical coordinates (r, Θ, φ), $0 \leq \varphi < 2\pi$, $0 \leq \Theta \leq \pi$, by the inequalities $0 \leq \Theta \leq h(r)$, where h is a continuous increasing function on [0,1] such that $h(0) = 0$ and $h(2r) \leq \mathrm{const}\, h(r)$.



We consider the Dirichlet problem for the operator Δ^2 in the exterior of this cusp. In this special case the sufficient condition for the 1-regularity found in [24] is equivalent to

$$\int_0 h(\sigma)^2 \, \frac{d\sigma}{\sigma} = \infty. \tag{16}$$

Problem 5. *Prove or disprove that* (16) *is necessary for the 1-regularity of the above cuspidal point.*

4 Regularity of the vertex of a cone

For the time being there are no results on the regularity of a boundary point O with respect to the general operator $P_{2m}(D_x)$ similar to those in Sec. 1 for Δ^m. I have in mind sufficient conditions obtained without a priori assumptions about the structure of $\partial\Omega$ near O. It was recently discovered that the situation is indeed more complicated when we turn to the general operator. We shall see that even such a simple singularity of $\partial\Omega$ as the vertex of a cone gives rise to unexpected phenomena.

Speaking about a cone I shall always assume that its complement has a non-empty interior. Then, as I mentioned in the Introduction, the vertex is regular if $m = 1$, $P_{2m}(D_x)$ is scalar and has real coefficients. In [25] and [26] the regularity of the vertex of an arbitrary cone was proved for Δ^2 and for the Lamé operator of linear isotropic elasticity (as well as for the Stokes system although it does not satisfy the above condition on $P_{2m}(D_x)$).

A starting point for the derivation of these and similar results as well as for the construction of counterexamples is the well-known asymptotic formula for the solution of the problem (6) near the origin:

$$u(x) \sim \mathrm{const}|x|^{\lambda} \sum_{k=0}^{N} \big(\log|x|\big)^k \varphi_k\left(\frac{x}{|x|}\right). \tag{17}$$

Here λ is an eigenvalue of the Dirichlet problem for an elliptic polynomial operator pencil on the domain which is cut out by the cone on the unit sphere. The functions φ_k on this spherical domain form a Jordan chain of the pencil corresponding to λ.

By (17), information about λ and $\{\varphi_k\}$ leads to results on the continuity and differentiability properties of u. In particular, if there exist eigenvalues of the above mentioned operator pencil in the strip

$$\{\lambda \in \mathbb{C} : 0 > \mathrm{Re}\,\lambda > m - n/2\} \tag{18}$$

then there are solutions of (6) which are unbounded in an arbitrary neighbourhood of O and hence O is irregular.

4.1 Second order operators with complex coefficients

It turned out [27], Ch. 10, that the strip (18) may contain eigenvalues of the operator pencil corresponding to strongly elliptic operators

$$P_2(D_x) = \sum_{j,k=1}^{n} a_{jk}\partial^2/\partial x_j \partial x_k \tag{19}$$

provided $n > 4$ and some coefficients a_{jk} are non-real. This result was obtained by application of a singular perturbation technique developed in [26].

Consider the equation (or the system)

$$P_{2m}(D_x)u = 0 \tag{20}$$

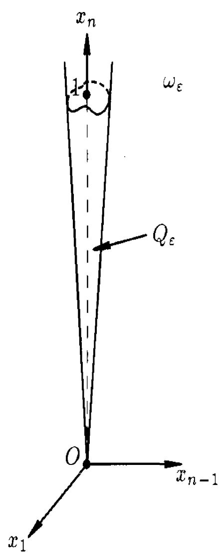
in the complement of a thin cone

$$Q_\varepsilon = \{x = (y, x_n) \in \mathbb{R}^n : x_n > 0,\ x_n^{-1}y \in \omega_\varepsilon\},$$

where ε is a small positive parameter and $\omega_\varepsilon = \{y \in \mathbb{R}^{n-1} : \varepsilon^{-1}y \in \omega\}$ with ω being a bounded domain in $\mathbb{R}^{n-1}$. We look for a solution of (20)

$$u(\varepsilon, x) = |x|^{\lambda(\varepsilon)}\varphi\left(\varepsilon, \frac{x}{|x|}\right) \tag{21}$$

satisfying the zero Dirichlet conditions on $\partial Q_\varepsilon \setminus \{O\}$. In [27] an asymptotic formula is given for small eigenvalues $\lambda(\varepsilon)$ of the above mentioned operator pencil on the spherical domain.



For the simplest case of the scalar equation (19) this formula has the explicit form

$$\begin{aligned}\lambda(\varepsilon) \sim \varepsilon^{n-3}\big((n-2)|S^{n-1}|\big)^{-1}\,\mathrm{cap}\big(\omega; P_2(D_y, 0)\big)& \\ \times \big(\det(a_{jk})_{k=1}^{n-1}\big)^{(2-n)/2}\big(\det(a_{jk})_{j,k=1}^{n}\big)^{(n-3)/2},&\end{aligned} \tag{22}$$

where $|S^{n-1}|$ is the area of the $(n-1)$-dimensional unit sphere and $\text{cap}(\omega; P_2(D_y, 0))$ is a complex valued function of the domain ω which is a generalization of the harmonic capacity. This set function is defined by

$$\text{cap}\big(\omega; P_2(D_y, 0)\big) = \int\limits_{\mathbb{R}^n \setminus \omega} \sum_{j,k=1}^{n-1} a_{jk} (\partial \overline{w}/\partial y_j)(\partial w/\partial y_k)\, dy,$$

where w is the solution of the Dirichlet problem

$$P_2(D_y, 0) w(y) = 0,\ y \in \mathbb{R}^{n-1} \setminus \overline{\omega}; \qquad w(y) = 1,\ y \in \partial\omega,$$

vanishing at infinity.

For $n > 4$ we can choose the coefficients a_{jk} in such a manner that the inequalities $0 > \text{Re}\,\lambda > 1 - n/2$ are fulfilled, which implies the irregularity of O.

This construction fails when $n = 3$ or $n = 4$ which gives rise to

Problem 6. *Prove or disprove that for $n = 3$ $(n = 4)$ a vertex of a cone is regular with respect to an arbitrary scalar elliptic operator $P_2(D_x)$ with complex coefficients.*

Using unbounded solutions of the form (21) to the Dirichlet problem for the equation $P_2(D_x)u = 0$, one can deduce, [27], Ch. 10, that H^1-solutions of the uniformly strongly elliptic equation

$$\sum_{j,k=1}^{n} \big(a_{jk}(x) u_{x_j}\big)_{x_k} = 0 \tag{23}$$

unbounded near an interior point of the domain may exist, provided $n > 4$ and some of the coefficients are non-real. In other words, for the equation (23) with complex coefficients the De Giorgi-Nash theorem on the local continuity of solutions is not valid if $n > 4$.

Problem 7. *Prove or disprove that for $n = 3$ $(n = 4)$ solutions of the uniformly strongly elliptic equation (23) with measurable bounded complex coefficients are Hölder-continuous in Ω.*

For $n = 2$ the Hölder continuity of solutions to (23) with complex coefficients follows from the well-known theorem of Morrey.

4.2 Higher order operators with real coefficients

I turn to the regularity of a conic point with respect to a higher order scalar elliptic operator $P_{2m}(D_x)$ with real coefficients. For the biharmonic operator the regularity was proved in [25] without restrictions to the cone. However, for other, even very simple, fourth order equations, the situation may be different.

Theorem (Maz'ya, Nazarov [28], see also [27], Ch. 10)**.** *Let $n \geq 8$, $a > 0$, and $(n-3)\arctan\sqrt{a} \in (2\pi, 4\pi)$. Then there exist an open cone $C \subset \mathbb{R}^n$ and a function $f \in C_0^\infty(\overline{C} \setminus \{O\})$ such that a solution $u \in H^2(\overline{C}, \mathrm{loc})$ of the Dirichlet problem*

$$\begin{aligned} \Delta^2 u(x) + a(\partial/\partial x_n)^4 u(x) &= f, \qquad x \in C; \\ u(x) = 0, \ \ \nabla u(x) &= 0, \qquad x \in \partial C, \end{aligned}$$

is unbounded near the vertex of C; the condition on the coefficient a is equivalent to

$$a > 5 + 2\sqrt{5} \text{ for } n = 8, \quad a > 3 \text{ for } n = 9.$$

The proof is based upon the following asymptotic formula for $\lambda(\varepsilon)$ in (21):

$$\lambda(\varepsilon) \sim \varepsilon^{n-5} k F(0, \ldots, 0, 1),$$

where $F(x)$ is a fundamental solution of the operator $\triangle^2 + a(\partial/\partial x_n)^4$ in $\mathbb{R}^n$ and k is a positive constant proportional to the biharmonic capacity of ω. It can be proved that this fundamental solution is negative at the point $(0, \ldots, 0, 1)$, which implies the theorem.

This argument shows that for higher order equations in the case $n > 2m + 1$ one cannot expect a theory of the regularity of a boundary point similar to that for second order equations without the complementary assumption of positivity of the fundamental solution.

Problem 8. *Let $n > 2m + 1$ and let $P_{2m}(D_x)$ be a scalar elliptic operator with real coefficients whose fundamental solution in $\mathbb{R}^n$ is positive. Prove or disprove that the vertex of an arbitrary cone is regular with respect to $P_{2m}(D_x)$.*

In the case $n = 2m + 1$ Kozlov and Maz'ya [29] verified the regularity of the vertex of a cone which can be explicitly represented in a Cartesian coordinate system. In this paper the operator $P_{2m}(D_x)$ has real coefficients.

It is unknown whether the restriction to the cone is important. This leads to the following

Problem 9. *Let $n = 2m + 1$ and let $P_{2m}(D_x)$ be a scalar elliptic operator with real coefficients. Prove or disprove that the vertex of an arbitrary cone is regular with respect to $P_{2m}(D_x)$.*

5 Weighted positivity of $(-\Delta)^m$

Henceforth, Ω is an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and O is a point of the closure $\overline{\Omega}$.

Let $B_\rho(p)$ be the ball $\{x \in \mathbb{R}^n : |x - p| < \rho\}$, where $p \in \mathbb{R}^n$ and let $B_\rho = B_\rho(O)$. We use the notation $\partial^\alpha = \partial^{|\alpha|}/\partial_1^\alpha \dots \partial x_n^{\alpha_n}$ and by ∇_ℓ we mean the gradient of order ℓ, i.e. $\nabla_\ell = \{\partial^\alpha\}$ with $|\alpha| = \ell$. In the sequel c is a positive constant, which depends only on m and n, and ω_{n-1} is the $(n-1)$-dimensonal measure of ∂B_1.

We shall deal with solutions of the Dirichlet problem

$$(-\Delta)^m u = f, \quad u \in \mathring{H}^m(\Omega). \tag{24}$$

By Γ we denote the fundamental solution of the operator $(-\Delta)^m$,

$$\Gamma(x) = \begin{cases} \gamma |x|^{2m-n} & \text{for } 2m < n, \\ \gamma \log \dfrac{\mathcal{D}}{|x|} & \text{for } 2m = n, \end{cases}$$

where $\mathcal{D}$ is a positive constant and

$$\gamma^{-1} = 2^{m-1}(m-1)!(n-2)(n-4)\dots(n-2m)\omega_{n-1}$$

for $n > 2m$, and

$$\gamma^{-1} = \left[2^{m-1}(m-1)!\right]^2 \omega_{n-1}$$

for $n = 2m$.

Proposition 1. *Let $n \geq 2m$ and let*

$$\int_\Omega u(x)(-\Delta)^m u(x) \Gamma(x - p)\, dx \geq 0 \tag{25}$$

for all $u \in C_0^\infty(\Omega)$ and for at least one point $p \in \Omega$. Then

$$n = 2m, 2m+1, 2m+2 \quad \textit{for } m > 2$$

and

$$n = 4, 5, 6, 7 \quad \textit{for } m = 2.$$

Proof. Assume that $n \geq 2m+3$ for $m > 2$ and $n \geq 8$ for $m = 2$. Denote by (r, ω), $r > 0$, $\omega \in \partial B_1(p)$, the spherical coordinates with center p, and by G the image of Ω under the mapping $x \mapsto (t, \omega)$, $t = -\log r$. Since

$$r^2 \Delta u = r^{2-n} \left(r\, \partial_r\right) \left(r^{n-2} \left(r\, \partial_r\right) u\right) + \delta_\omega u,$$

where δ_ω is the Beltrami operator on $\partial B_1(p)$, then

$$\Delta = e^{2t} \left(\partial_t^2 - (n-2)\partial_t + \delta_\omega\right) = e^{2t} \left\{ \left(\partial_t - \frac{n-2}{2}\right)^2 - A \right\},$$

where

$$A = -\delta_\omega + \frac{(n-2)^2}{4}. \tag{26}$$

Hence

$$r^{2m} \Delta^m = \prod_{j=0}^{m-1} \left\{ \left(\partial_t - \frac{n-2}{2} + 2j\right)^2 - A \right\}. \tag{27}$$

Let u be a function in $C_0^\infty(\Omega)$, which depends only on $|x-p|$. We set $w(t) = u(x)$. Clearly,

$$\int_\Omega (-\Delta)^m u(x) u(x) \Gamma(x-p)\, dx = \int_{\mathbb{R}^1} w(t) \mathcal{P}(d/dt) w(t)\, dt, \tag{28}$$

where

$$\begin{aligned}\mathcal{P}(\lambda) &= (-1)^m \gamma \omega_{n-1} \prod_{j=0}^{m-1} (\lambda + 2j)(\lambda - n + 2 + 2j) \\ &= (-1)^m \gamma \omega_{n-1} \lambda(\lambda - n + 2) \prod_{j=1}^{m-1} (\lambda + 2j)(\lambda - n - 2m + 2 + 2j).\end{aligned}$$

Let

$$\mathcal{P}(\lambda) = (-1)^m \gamma \omega_{n-1} \lambda^{2m} + \sum_{k=1}^{2m-1} a_k \lambda^k.$$

We have

$$a_2 = (\lambda^{-1}\mathcal{P}(\lambda))'\big|_{\lambda=0} = \frac{1}{2-n} + \sum_{j=1}^{m-1} \left(\frac{1}{2j} - \frac{1}{n-2-2m+2j} \right).$$

Hence and by $n \geq 2m+3$,

$$\begin{aligned} a_2 &= \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-2m} + \sum_{j=2}^{m-1} \frac{n-2-2m}{2j(n-2-2m+2j)} \\ &\geq \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-2m} > 0. \end{aligned}$$

We choose a real-valued function $\eta \in C_0^\infty(1,2)$ normalized by

$$\int_{\mathbb{R}^1} |\eta'(\sigma)|^2 \, d\sigma = 1$$

and we set $u(x) = \eta(\varepsilon t)$, where ε is so small that $\operatorname{supp} u \subset \Omega$. The quadratic form on the right hand side of (28) equals

$$\int_{\mathbb{R}^1} \Big(\varepsilon^{2m} \gamma \omega_{n-1} |\eta^{(m)}(\varepsilon t)|^2 + \sum_{k=1}^{m-1} a_{2k} (-1)^k \varepsilon^{2k} |\eta^{(k)}(\varepsilon t)|^2 \Big)\, dt$$
$$= -a_2 \varepsilon + O(\varepsilon^3) < 0,$$

which contradicts the assumption (25). $\square$

Now we prove the converse statement.

Proposition 2. *Let $\Gamma_p(x) = \Gamma(x-p)$, where $p \in \Omega$. If*

$$\begin{aligned} &n = 2m, 2m+1, 2m+2 && \textit{for } m > 2, \\ &n = 4, 5, 6, 7 && \textit{for } m = 2, \\ &n = 2, 3, 4, \ldots && \textit{for } m = 1, \end{aligned}$$

then for all $u \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega (-\Delta)^m u(x) \cdot u(x) \Gamma(x-p)\, dx \\ \geq 2^{-1} u^2(p) + c \sum_{k=1}^{m} \int_\Omega \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}}\, \Gamma(x-p)\, dx.\end{aligned} \tag{29}$$

(*In the case* $n = 2m$ *the constant* $\mathcal{D}$ *in the definition of* Γ *is greater than* $|x-p|$ *for all* $x \in \operatorname{supp} u$.)

Proof. We preserve the notation introduced in the proof of Proposition 1. We note first that (29) becomes identity when $m = 1$. The subsequent proof will be divided into four parts.

(i) *The case* $n = 2m + 2$. By (27),

$$r^{-2m} \Delta^m = \prod_{j=0}^{m-1} \left(\partial_t - m + 2j - A^{1/2}\right) \prod_{j=0}^{m-1} \left(\partial_t - m + 2j + A^{1/2}\right),$$

where $A = -\delta_\omega + m^2$ and $A^{1/2}$ is defined by using spherical harmonics. By setting $k = m - j$ in the second product, we rewrite the right-hand side as

$$\prod_{j=0}^{m-1} \left(\partial_t - m + 2j - A^{1/2}\right) \prod_{k=1}^{m} \left(\partial_t + m - 2k + A^{1/2}\right).$$

This can be represented in the form

$$\left(\partial_t - m - A^{1/2}\right)\left(\partial_t - m + A^{1/2}\right) \prod_{j=1}^{m-1} \left(\partial_t^2 - \mathcal{B}_j^2\right),$$

where $\mathcal{B}_j = A^{1/2} + m - 2j$. Therefore

$$\begin{aligned} 2^m \Delta^m &= \left(\partial_t^2 + \delta_\omega - 2m\, \frac{\partial}{\partial t}\right) \prod_{j=1}^{m-1} \left(\partial_t^2 - \mathcal{B}_j^2\right) \\ &= \left(\partial_t^2 + \delta_\omega\right) \prod_{j=1}^{m-1} \left(\partial_t^2 - \mathcal{B}_j^2\right) \\ &\quad + (-1)^m 2m\, \partial_t \sum_{\substack{0 \leq j \leq m-1 \\ k_1 < \cdots < k_j}} \left(-\partial_t^2\right)^{m-j-1} \mathcal{B}_{k_1}^2 \ldots \mathcal{B}_{k_j}^2. \end{aligned}$$

We extend u by zero outside Ω and introduce the function w defined by $w(t,\omega)=u(x)$. We write the left-hand side of (29) in the form $\gamma(I_1+I_2)$, where γ is the constant in the definition of Γ,

$$(2m^{-1}I_1)=\int_G \partial_t \sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}} \left(-\partial_t^2\right)^{m-j-1} \mathcal{B}_{k_1}^2\dots\mathcal{B}_{k_j}^2 w\cdot w\,dtd\omega,$$

and

$$I_2=(-1)^m\int_G \left(\partial_t^2+\delta_\omega\right)\prod_{j=1}^{m-1}\left(\partial_t^2-\mathcal{B}_j^2\right)w\cdot w\,dtd\omega.$$

Since the operators $\mathcal{B}_j$ are symmetric, it follows that

$$\begin{aligned}m^{-1}I_1&=\sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}\int_{\mathbb{R}^1}\partial_t\int_{\partial B_1}\left(\partial_t^{m-j-1}\,\mathcal{B}_{k_1}\dots\mathcal{B}_{k_j}w\right)^2\,d\omega dt\\&=\sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}\int_{\partial B_1}\left|\left(\partial_t^{m-j-1}\mathcal{B}_{k_1}\dots\mathcal{B}_{k_j}w\right)(+\infty,\omega)\right|^2\,d\omega.\end{aligned}$$

Since $u\in C^\infty(\Omega)$, we have $w(t,\omega)=u(p)+O(e^{-t})$ as $t\to+\infty$, and this can be differentiated. Therefore, all terms with $j<m-1$ are equal to zero and we find

$$\begin{aligned}I_1&=m\int_{\partial B_1}\left|\left(\mathcal{B}_1\dots\mathcal{B}_{m-1}w\right)(+\infty,\omega)\right|^2 d\omega\\&=mu^2(p)\int_{\partial B_1}\left|\mathcal{B}_1\dots\mathcal{B}_{m-1}1\right|^2 d\omega.\end{aligned}$$

By $\mathcal{B}_j=(-\delta_\omega+m^2)^{1/2}+m-2j$, we have

$$I_1=4^{m-1}m[(m-1)!]^2\omega_{2m+1}u^2(p).$$

Since in the case $n=2m+2$

$$\gamma^{-1}=2^{2m-1}m[(m-1)!]^2\omega_{2m+1},$$

we conclude that

$$I_1=(2\gamma)^{-1}u^2(p). \tag{30}$$

We now wish to obtain the lower bound for I_2. Let $\widetilde{w}$ denote the Fourier transform of w with respect to t. Then

$$I_2 = \int_{\partial B_1} \int_{\mathbb{R}^1} (\lambda^2 - \delta_\omega) \prod_{j=1}^{m-1} (\lambda^2 + \mathcal{B}_j^2) \widetilde{w}(\lambda, \omega) \overline{\widetilde{w}(\lambda, \omega)} \, d\lambda d\omega.$$

Clearly,

$$\mathcal{B}_j \geq (m^2 - \delta_\omega)^{1/2} - m + 2 \geq 2m^{-1}(m^2 - \delta_\omega)^{1/2},$$

and

$$\lambda^2 + \mathcal{B}_j^2 \geq 4m^{-2}(\lambda^2 + 1 - \delta_\omega),$$

the operators being compared with respect to their quadratic forms. Thus

$$\begin{aligned}\left(\frac{m}{2}\right)^{2m-2} I_2 &\geq \int_{\partial B_1 \times \mathbb{R}^1} (\lambda^2 - \delta_\omega)(\lambda^2 + 1 - \delta_\omega)^{m-1} \widetilde{w}(\lambda, \omega) \cdot \overline{\widetilde{w}(\lambda, \omega)} \, d\lambda d\omega \\ &\geq c \left(\|\partial_t w\|^2_{H^{m-1}(G)} + \|\nabla_\omega w\|^2_{H^{m-1}(G)} \right),\end{aligned}$$

where H^{m-1} is the Sobolev space. This is equivalent to the inequality

$$I_2 \geq c \int_\Omega \sum_{k=1}^m \frac{|\nabla_k u(x)|^2}{|x-p|^{n-2k}} \, dx,$$

which along with (30) completes the proof for $n = 2m + 2$.

(ii) *The case* $n = 2m+1$. We shall treat this case by descent from $n = 2m+2$ to $n = 2m + 1$. Let $z = (x, s)$, where $x \in \Omega$, $s \in \mathbb{R}^1$, and let $q = (p, 0)$, where $p \in \Omega$, $0 \in \mathbb{R}^1$. We introduce a cut-off function $\eta \in C_0^\infty(-2, 2)$ which satisfies $\eta(s) = 1$ for $|s| \leq 1$ and $0 \leq \eta \leq 1$ on $\mathbb{R}^1$. Let

$$U_\varepsilon(z) = u(x)\eta(\varepsilon s)$$

and let $\Gamma^{(n)}$ denote the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$.

By integrating

$$(-\Delta_z)^m \Gamma^{(n+1)}(z, q) = \delta(z - q),$$

with respect to $s \in \mathbb{R}^1$ we have

$$\Gamma^{(n)}(x, y) = \int_{\mathbb{R}^1} \Gamma^{(k+1)}(z, q) \, ds. \tag{31}$$

From part (i) of the present proof we obtain

$$\int_{\Omega\times\mathbb{R}^1} (-\Delta_z)^m U_\varepsilon(z)\cdot U_\varepsilon(z)\Gamma^{(k+1)}(z-q)\,dz$$
$$\geq \frac{1}{2}U_\varepsilon^2(q) + c\int_{\Omega\times\mathbb{R}^1} \sum_{k=1}^m \frac{|\nabla_k U_\varepsilon(z)|^2}{|z-q|^{2(m+1-k)}}\,dz.$$

By letting $\varepsilon \to 0$, we find

$$\int_{\Omega\times\mathbb{R}^1} (-\Delta_x)^m u(x)\cdot u(x)\Gamma^{(n+1)}(z-q)\,dsdx$$
$$\geq \frac{1}{2}u^2(p) + c\int_{\Omega\times\mathbb{R}^1} \sum_{k=1}^m \frac{|\nabla_k u(x)|^2}{|z-q|^{2(m+1-k)}}\,dsdx.$$

The result follows from (31).

(iii) *The case* $m=2, n=7$. By (27),

$$30\omega_6 \int_\Omega \Delta^2 u(x)\cdot u(x)\Gamma(x-p)\,dx$$
$$= \int_G (w_{tt} - 5w_t + \delta_\omega w)(w_{tt} + w_t - 6w + \delta_\omega w)\,dtd\omega.$$

Since $w(t,\omega) = u(p) + O(e^{-t})$ as $t\to+\infty$, the last integral equals

$$\int_G \Big(w_{tt}^2 - 5w_t^2 - 6w_{tt}w + 2w_{tt}\delta_\omega w + (\delta_\omega w)^2 - 6w\delta_\omega w\Big)\,dtd\omega + 15\omega_6 u^2(p).$$

After integrating by parts we rewrite this in the form

$$\int_G \Big(w_{tt}^2 + (\delta_\omega w)^2 + 2(\nabla_\omega w_t)^2 + 6(\nabla_\omega w)^2 + w_t^2\Big)\,dtd\omega + 15\omega_6 u^2(p).$$

Using the variables (r,ω), we obtain that the left-hand side exceeds

$$c\int_\Omega \left(\frac{(\Delta u(x))^2}{|x-p|^3} + \frac{|\nabla u(x)|^2}{|x-p|}\right)\,dx + 15\omega_6 u^2(p).$$

Since

$$|\nabla_2 u|^2 - (\Delta u)^2 = \Delta((\nabla u)^2) - \frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right),$$

it follows that

$$\int_\Omega \frac{(\nabla_2 u(x))^2}{|x-p|^3}\,dx \le \int_\Omega \frac{(\Delta u(x))^2}{|x-p|}\,dx + c\int_\Omega \frac{(\nabla u(x))^2}{|x-p|}\,dx,$$

which completes the proof.

(iv) *The case* $n = 2m$. By (27),

$$\begin{aligned} r^{2m}\Delta^m &= \prod_{j=0}^{m-1}\Big\{(\partial_t - m + 1 + 2j)^2 - (m-1)^2 + \delta_\omega\Big\} \\ &= \prod_{j=0}^{m-1}\Big(\partial_t - m + 1 + 2j - \mathcal{E}^{1/2}\Big)\prod_{j=0}^{m-1}\Big(\partial_t - m + 1 + 2j + \mathcal{E}^{1/2}\Big), \end{aligned}$$

where $\mathcal{E} = -\delta_\omega + (m-1)^2$. We introduce $k = m - 1 - j$ in the second product and obtain

$$r^{2m}\Delta^m = \prod_{j=0}^{m-1}\left(\partial_t^2 - \mathcal{F}_j^2\right),$$

where $\mathcal{F}_j = m - 1 - 2j + \mathcal{E}^{1/2}$. Hence

$$\begin{aligned} &\int_\Omega (-\Delta)^m u(x)\cdot u(x)\Gamma(x-p)\,dx \\ &\qquad = \gamma\int_G \prod_{j=0}^{m-1}\left(-\partial_t^2 + \mathcal{F}_j^2\right)w\cdot(\ell+t)w\,dtd\omega \end{aligned} \tag{32}$$

where $\ell = \log\mathcal{D}$. Since $w(t,\omega) = u(p) + O(e^{-t})$ and

$$\prod_{j=0}^{m-1}\left(-\partial_t^2 + \mathcal{F}_j^2\right) = \sum_{j=0}^{m}\left(-\partial_t^2\right)^{m-j}\sum_{k_1<\dots<k_j}\mathcal{F}_{k_1}^2\dots\mathcal{F}_{k_j}^2,$$

the right-hand side in (32) can be rewritten as

$$\gamma \int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} \partial_t^{m-j} \mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w \partial_t^{m-j} \left((\ell+t)\mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w\right) dt d\omega$$

$$= \gamma \int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} \left(\partial_t^{m-j} \mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w\right)^2 (\ell+t)\, dt d\omega$$

$$+ \frac{\gamma}{2} \int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} (m-j)\partial_t \left(\partial_t^{m-1-j} \mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w\right)^2 dt d\omega.$$

The second integral in the right-hand side equals

$$\lim_{t\to+\infty} \int_{\partial B_1(p)} \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} (m-j) \left|\partial_t^{m-1-j} \mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w\right|^2 d\omega$$

$$= \lim_{t\to+\infty} \int_{\partial B_1(p)} \sum_{k_1<\cdots<k_{m-1}} \left(\mathcal{F}_{k_1}\dots\mathcal{F}_{k_{m-1}} e\right)^2 d\omega$$

and since $(\mathcal{F}_{m-1}w)(t,\omega) = O(e^{-t})$ the last expression is equal to

$$\lim_{t\to+\infty} \int_{\partial B_1(p)} (\mathcal{F}_0\dots\mathcal{F}_{m-2} w)^2 d\omega = \left(2^{m-1}(m-1)!\right)^2 \omega_{n-1} u^2(p).$$

Hence

$$\int_\Omega (-\Delta)^m u(x)\cdot u(x)\Gamma(x-p)\, dx$$

$$= \frac{1}{2} u^2(p) + \gamma \int_G (\ell+t) \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} \left(\partial_t^{m-1-j} \mathcal{F}_{k_1}\dots\mathcal{F}_{k_j} w\right)^2 dt d\omega.$$

Since $\mathcal{F}_{m-1} \ge c(-\delta)^{1/2}$ and $\mathcal{F}_k \ge c(-\delta+1)^{1/2}$ for $k < m-1$, the last integral majorizes

$$c\int_G (\ell+t) \sum_{1\le \mu+\nu\le m-1} \left(\partial_t^{\mu} (-\delta)^{\nu/2} w\right)^2 dt d\omega$$

$$\ge c \int_\Omega \log\frac{\mathcal{D}}{|x-p|} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}}\, dx,$$

which completes the proof. □

We introduce the bilinear form

$$Q(V,W) = \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} \partial^\alpha V \partial^\alpha W \, dx$$

for V and W in $\mathring{H}^m(\Omega)$. Clearly, for $u \in C_0^\infty(\Omega)$, we have

$$Q(u, u\Gamma_p) = \int_\Omega (-\Delta)^m u(x) \cdot u(x) \Gamma(x-p) \, dx,$$

where $\Gamma_p(x) = \Gamma(x-p)$ and $p \in \Omega$. The quadratic form $Q(u, u\Gamma_p)$ is positive by Proposition 2.

Let u be an arbitrary function in the space $\mathring{H}^m(\Omega) \cap C^\infty(\Omega)$. We approximate u in the norm of $\mathring{H}^m(\Omega)$ by a sequence $\{u_\nu\}$ of functions in $C_0^\infty(\Omega)$ in such a way that $u_\nu(x) = u(x)$ in a neighbourhood of p. Then $u_\nu \to u$ in the norm $v \mapsto (Q(v, v\Gamma_p))^{1/2}$ and Proposition 2 implies the following assertion.

Corollary 1. *Let n and m be the same as in Proposition 2. Then for all $u \in \mathring{H}^m(\Omega) \cap C^\infty(\Omega)$ and $p \in \Omega$*

$$Q(u, u\Gamma_p) \geq 2^{-1} u^2(p) + c \sum_{k=1}^m \int_\Omega \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}} \, \Gamma(x-p) \, dx. \tag{33}$$

6 Regularity of a boundary point as a local property

Proposition 3. *In the case $m = 1$ the regularity in the sense of Definition 1 is equivalent to Wiener's regularity.*

Proof. Let O be regular in the Wiener sense and let u be the solution of (4) with $m = 1$. We introduce the Newton potential u_f with the density f and we note that u_f is smooth in a neighbourhood of $\partial\Omega$. Since $v = u - u_f$ is the $H^1(\Omega)$-solution of the Dirichlet problem

$$-\Delta v = 0 \quad \text{on } \Omega,$$
$$v = -u_f \quad \text{on } \partial\Omega,$$

it follows from Wiener's regularity that u is continuous at O (see [6], Sec. 3). Hence O is regular in the sense of Definition 1.

In order to prove the converse assertion consider the Dirichlet problem

$$-\Delta w = 0 \quad \text{on } \Omega, \quad w \in H^1(\Omega),$$
$$w(x) = (2n)^{-1}|x|^2 \quad \text{on } \partial\Omega.$$

We show that w is continuous at O provided O is regular in the sense of Definition 1. In fact, since the function

$$z(x) = w(x) - (2n)^{-1}|x|^2$$

satisfies

$$-\Delta z = 1 \quad \text{on } \Omega, \quad z \in \mathring{H}^1(\Omega),$$

we have

$$z(x) = \int_\Omega G(x,s)\, ds,$$

where G is Green's function of the Dirichlet problem. Therefore,

$$z(x) = \int_\Omega G(x,s)h(s)\, ds + \int_\Omega G(x,s)(1-h(s))\, ds,$$

where $h \in C_0^\infty(\Omega)$, $0 \le h \le 1$ and $h = 1$ on a domain ω, $\overline{\omega} \subset \Omega$.

The first integral tends to zero as $x \to 0$ by the regularity assumption. Hence

$$\limsup_{x \to O} |z(x)| \le c \int_{\Omega \setminus \omega} \frac{ds}{|x-s|^{n-2}} = O\left((\mathrm{mes}_n(\Omega \setminus \omega))^{2/n} \right).$$

Since $\mathrm{mes}_n(\Omega \setminus \omega)$ can be made arbitrarily small, $z(x) \to 0$ as $x \to O$. As a result we obtain that z satisfies the definition of barrier (see [30], Ch. 4, Sec. 2) and by Theorem 4.8 in [30] the regularity of O in the Wiener sense follows. □

Lemma 1. *Let n and m be the same as in Proposition 2. If O is regular in the sense of Definition 1, then the solution $u \in \mathring{H}^m(\Omega)$ of*

$$(-\Delta)^m u = \sum_{\{\alpha:\, |\alpha| \le m\}} \partial^\alpha f_\alpha \quad \text{on } \Omega,$$

with $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighbourhood of O, satisfies

$$u(x) \to 0 \text{ as } x \to O. \tag{34}$$

Proof. Let $\zeta \in C_0^\infty(\Omega)$. We represent u as the sum $v+w$, where $w \in \mathring{H}^m(\Omega)$ and

$$(-\Delta)^m v = \sum_{\{\alpha:\, |\alpha| \le m\}} \partial^\alpha (\zeta f_\alpha).$$

By the regularity of O the term v satisfies (34). We shall verify that w can be made arbitrarily small by making the Lebesgue measure of the support of $1-\zeta$ sufficiently small. Let $f_\alpha = 0$ on B_δ and let $p \in \Omega, |p| < \delta/2$. By definition of w and by Corollary 1,

$$\sum_{\{\alpha \colon |\alpha| \le m\}} \int_\Omega (1-\zeta) f_\alpha (-\partial)^\alpha (w \Gamma_p)\, dx$$
$$\ge 2^{-1} w^2(p) + c \sum_{k=1}^{m} \int_\Omega \frac{|\nabla_k w(x)|^2}{|x-p|^{2(m-k)}}\, \Gamma(x-p)\, dx.$$

The result follows. $\square$

Theorem 1. *Let O be a regular point for the operator $(-\Delta)^m$ on Ω and let Ω' be a domain such that*

$$\Omega' \cap B_{2\rho} = \Omega \cap B_{2\rho}$$

for some $\rho > 0$. Then O is regular for the operator $(-\Delta)^m$ on Ω'.

Proof. Let $u \in \mathring{H}^m(\Omega')$ satisfy (24) on Ω' with $f \in C_0^\infty(\Omega')$ and introduce $\eta_\rho(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on $B_{3/2}$. Then $\eta_\rho u \in \mathring{H}^m(\Omega)$ and

$$(-\Delta)^m (\eta_\rho u) = \eta_\rho f + [(-\Delta)^m, \eta_\rho] u \quad \text{on } \Omega.$$

Since the commutator $[(-\Delta)^m, \eta_\rho]$ is a differential operator of order $2m-1$, with smooth coefficients supported by $B_{2\rho} \setminus \overline{B_{3\rho/2}}$, it follows that

$$(-\Delta)^m (\eta_\rho u) = \sum_{\{\alpha \colon |\alpha| \le m\}} \partial^\alpha f_\alpha \quad \text{on } \Omega,$$

where $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighbourhood of O. Therefore, $(\eta_\rho u)(x) = o(1)$ as x tends to O by Lemma 1 and by the regularity of O with respect to $(-\Delta)^m$ on Ω. $\square$

7 A local estimate

In the next lemma and henceforth we use the notation

$$M_\rho(u) = \rho^{-n} \int_{\Omega \cap S_\rho} u^2(x)\, dx,$$

where, as before, $S_\rho = \{x : \rho < |x| < 2\rho\}$.

Lemma 2. *Let m and n be the same as in Proposition 2, $u \in \mathring{H}^m(\Omega)$ and*

$$\Delta^m u = 0 \quad on\ \Omega \cap B_{2\rho}. \tag{35}$$

Then, for an arbitrary point $p \in B_\rho$,

$$Q(u\eta_\rho, u\eta_\rho \Gamma_p) \leq c M_\rho(u),$$

where $\eta_\rho(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on $B_{3/2}$. (In the case $n = 2m$, here and in what follows we set $\mathcal{D} = 4\rho$ in the definition of the fundamental solution Γ.)

Proof. By definition of the quadratic form Q,

$$\begin{aligned} &Q\left(u\eta_\rho, u\eta_\rho\Gamma_p\right) - Q\left(u, u\eta_\rho^2\Gamma_p\right) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega \Big([\partial^\alpha, \eta_\rho]u \cdot \partial^\alpha(u\eta_\rho\Gamma_p) - \partial^\alpha u \cdot [\partial^\alpha, \eta_\rho](u\eta_\rho\Gamma_p) \Big)\, dx, \end{aligned} \tag{36}$$

where $[A, B] = AB - BA$. Since u is m-harmonic on $\Omega \cap B_{2\rho}$ it follows that

$$Q\left(u, u\eta_\rho^2\Gamma_p\right) = 0.$$

The right-hand side in (36) is majorized by

$$c \sum_{j=0}^{m} \rho^{2j-n} \int_\Omega \zeta_\rho |\nabla_j u|^2\, dx,$$

where $\zeta_\rho(x) = \zeta(x/\rho)$, $\zeta \in C_0^\infty(S)$ and $\eta\zeta = \eta$. By the well-known local energy estimate

$$\int_\Omega \zeta_\rho |\nabla_j u|^2\, dx \leq c\rho^{-2j} \int_{\Omega\cap S_\rho} u^2\, dx$$

the result follows. □

Combining Corollary 1 and Lemma 2 we arrive at the following local estimate.

Corollary 2. *Let m and n be the same as in Proposition 2 and let us suppose that $u \in \mathring{H}^m(\Omega)$ satisfy (35). Then, for an arbitrary point $p \in \Omega \cap B_\rho$,*

$$u^2(p) + \int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}}\, \Gamma(x-p)\, dx \leq c M_\rho(u) \tag{37}$$

8 Local estimates stated in terms of the m-harmonic capacity

We say that a compact subset of the ball $\overline{B}_\rho = \{x : |x| \le \rho\}$ is m-small, $2m \le n$, if

$$\mathrm{cap}_m\,(e, B_{2\rho}) \le 16^{-n} \rho^{n-2m}.$$

In the case $2m > n$ only the empty subset of $\overline{B}_\rho$ will be called m-small.

Let $\overline{u}_\rho$ denote the mean value of u on the ball B_ρ, i.e.

$$\overline{u}_\rho = (\mathrm{mes}_n B_\rho)^{-1} \int_{B_\rho} u(x)\,dx.$$

We introduce the seminorm

$$|||u|||_{m,B_\rho} = \left(\sum_{j=1}^{m} \rho^{2(j-m)}\, \|\nabla_j u\|^2_{L_2(B_\rho)} \right)^{1/2}.$$

Proposition 4 ([15], 10.1.2)**.** *Let e be a closed subset of the ball $\overline{B}_\rho$.*
1) *For all $u \in C^\infty\left(\overline{B}_\rho\right)$ with* $\mathrm{dist}(\mathrm{supp}\, u, e) > 0$ *the inequality*

$$\|u\|_{L_2(B_\rho)} \le C|||u|||_{m,B_\rho} \tag{38}$$

is valid, where

$$C^{-2} \ge c\rho^{-n} \mathrm{cap}_m\,(e, B_\rho)$$

and c depends only on m and n.
2) *If e is m-small and if the inequality* (38) *holds for all $u \in C^\infty\left(\overline{B}_\rho\right)$ with* $\mathrm{dist}\,(\mathrm{supp}\, u, e) > 0$ *then the best constant C in* (1) *satisfies*

$$C^{-2} \le c\rho^{-n} \mathrm{cap}\,(e, B_\rho)$$

The second assertion of this Proposition will not be used in the sequel and therefore will not be proved here. Its proof can be found in [15], pp. 405–406. In order to check the first assertion we need the following auxiliary result.

Lemma 3. *Let e be a compact set in $\overline{B}_1$. There exists a constant c depending on n and m and such that*

$$\begin{aligned} &c^{-1} \mathrm{cap}_m\,(e, B_2) \\ &\qquad \le \inf \left\{ \|1-u\|^2_{H^m_{(B_1)}} : u \in C^\infty\left(\overline{B}_1\right),\ \mathrm{dist}\,(\mathrm{supp}\, u, e) > 0 \right\} \\ &\qquad \le c\ \mathrm{cap}_m\,(e, B_2)\,. \end{aligned} \tag{39}$$

Proof. To obtain the left estimate we need the following well-known assertion.

There exists a linear continuous mapping $A\colon C^{k-1,1}\left(\overline{B}_1\right) \to C^{k-1,1}\left(\overline{B}_2\right)$, $k = 1, 2, \ldots$, such that

(i) $Av = v$ on $\overline{B}_1$;
(ii) if $\operatorname{dist}(\operatorname{supp} v, e) > 0$, then $\operatorname{dist}(\operatorname{supp} Av, e) > 0$;
(iii) the inequality

$$\|\nabla_i(Av)\|_{L_2(B_2)} \le c\,\|\nabla_i v\|_{L_2(B_1)} \tag{40}$$

is valid with $i = 0, 1, \ldots, l$ and c independent of v.

Let $v = A(1-u)$ and let η denote a function in $C_0^\infty(B_2)$ which is equal to 1 in a neighbourhood of the ball B_1. Then

$$\operatorname{cap}(e, B_2) \le c\,\|\nabla_l(\eta v)\|^2_{L_2(B_2)} \le c\,\|v\|^2_{H^m(B_2)}\,. \tag{41}$$

Now the left estimate in (39) follows from (40) and (41).

Next we derive the right estimate in (39). Let $w \in C_0^\infty(B_2)$, $w = 1$ on a neighbourhood of e.

Then

$$\|w\|_{H^m(B_1)} \le c\,\|\nabla_m w\|_{L_2(B_2)}\,.$$

Minimizing the last norm we obtain

$$\inf_u \|1-u\|^2_{H^m(B_1)} \le \inf \|w\|^2_{H^m(B_1)} \le c \operatorname{cap}(e, B_2)\,.$$

The proof is complete. □

Proof of the first assertion of Proposition 4. It suffices to consider only the case $d = 1$ and then use a dilation.

1) Let

$$N = \left(\frac{1}{\operatorname{mes}_n B_1}\int_{B_1} u^2(x)\,dx\right)^{1/2}.$$

Since $\operatorname{dist}(\operatorname{supp} u, e) > 0$, it follows from Lemma 3 that

$$\operatorname{cap}_m(e, B_2) \le c\,\left\|1 - N^{-1}u\right\|^2_{H^m(B_1)} = cN^{-2}|||u|||^2_{m,B_1} + c\,\left\|1 - N^{-1}u\right\|^2_{L_2(B_1)}$$

i.e.

$$N^2 \mathrm{cap}_m\,(e, B_2) \le c|||u|||^2_{m,B_1} + c\,\|N-u\|^2_{L_2(B_1)}\,. \tag{42}$$

Without loss of generality we assume that $\overline{u}_1 \ge 0$. Then

$$\sqrt{\mathrm{mes}_n B_1}\,|N-\overline{u}_1| \;= \|u\|_{L_2(B_1)} - \|\overline{u}_1\|_{L_2(B_1)} \le \|u-\overline{u}_1\|_{L_2(B_1)}\,.$$

Consequently,

$$\|N-u\|_{L_2(B_1)} \le \|N-\overline{u}_1\| + \|u-\overline{u}_1\|_{L_2(B_1)} \le 2\,\|u-\overline{u}_1\|_{L_2(B_1)}\,.$$

Hence, by (42) and the Poincaré inequality

$$\|u-\overline{u}_1\|_{L_2(B_1)} \le c\,\|\nabla u\|_{L_2(B_1)}$$

we obtain

$$\mathrm{cap}\,(e, B_2)\,\|u\|^2_{L_2(B_1)} \le c|||u|||^2_{m,(B_1)}\,,$$

which completes the proof. □

Lemma 4. *Let m and n be as in Proposition 2 and let the function $u \in \mathring{H}^m(\Omega)$ satisfy $\Delta^m u = 0$ on $\Omega \cap B_{2\rho}$. Then for all points $p \in \Omega \cap B_\rho$ there holds the estimate*

$$\begin{aligned} u^2(p) + \int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}}\,\Gamma(x-p)\,dx \\ \le \frac{c}{\gamma_m(\rho)} \int_{\Omega\cap S_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}}\,dx, \end{aligned} \tag{43}$$

where

$$\gamma_m(\rho) = \begin{cases} \rho^{2m-n}\mathrm{cap}_m(S_\rho \setminus \Omega) & \text{for } n > 2m, \\ \mathrm{cap}_m(S_\rho \setminus \Omega, B_{4\rho}) & \text{for } n = 2m. \end{cases}$$

Proof. We combine Corollary 2 with the inequality

$$\int_{\Omega\cap S_\rho} u^2(x)\,dx \le \frac{c}{\gamma_m(\rho)} \int_{\Omega\cap S_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}}\,dx$$

proved in Proposition 4. □

We are in a position to obtain a growth estimate for the solution formulated in terms of a Wiener type m-capacitary integral. Before stating the result we note that the function $\gamma_m(\rho)$ is measurable not only for $n > 2m$ when it is monotonous but also for $n = 2m$. In fact, one can easily show that the function

$$(\rho/2, \infty) \ni r \mapsto \mathrm{cap}_m(S_\rho \setminus \Omega, B_{4r})$$

is continuous. Hence, being monotonous in ρ, the function of two variables $(\rho, r) \mapsto \mathrm{cap}_m(S_\rho \setminus \Omega, B_{4r})$ satisfies the so-called Carathéodory conditions which imply the measurability of $\gamma_m(\rho)$ in the case $n = 2m$ (see [31], [32], p. 152).

Theorem 2. *Let m and n be as in Proposition 2 and let the function $u \in \mathring{H}^m(\Omega)$ satisfy $\Delta^m u = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,*

$$\begin{aligned} \sup\left\{|u(p)|^2 : p \in \Omega \cap B_\rho\right\} + \int_{\Omega \cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}}\, dx& \\ \le c M_R(u) \exp\left(-c \int_\rho^R \gamma_m(\tau)\, \frac{d\tau}{\tau}\right)&. \end{aligned} \tag{44}$$

Proof. It is sufficient to assume that $2\rho \le R$, since in the opposite case the result follows from Corollary 2. Denote the first and the second terms on the left by φ_ρ and ψ_ρ respectively. From Lemma 3 it follows that for $r \le R$

$$\varphi_r + \psi_r \le \frac{c}{\gamma_m(r)} \left(\psi_{2r} - \psi_r\right) \le \frac{c}{\gamma_m(r)} \left(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r\right).$$

This along with the obvious inequality $\gamma_m(r) \le c$ implies

$$\varphi_r + \psi_r \le c e^{-c_0 \gamma_m(r)} \left(\varphi_{2r} + \psi_{2r}\right).$$

By setting $r = 2^{-j} R,\ j = 1, \dots$ we arrive at the estimate

$$\varphi_{2^{-\ell} R} + \psi_{2^{-\ell} R} \le c\ \exp\Big(-c \sum_{j=1}^{\ell} \gamma_m(2^{-j} R)\Big)(\varphi_R + \psi_R).$$

We choose ℓ so that

$$\ell < \log_2 \frac{R}{\rho} \le \ell + 1$$

in order to obtain

$$\varphi_\rho + \psi_\rho \le c\ \exp\Big(-c_0 \sum_{j=1}^{\ell} \gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

Now we notice that by Corollary 2

$$\varphi_R + \psi_R \le c M_R.$$

It remains to use the inequality

$$\sum_{j=1}^{\ell} \gamma_m(2^{-j}R) \ge c_1 \int_\rho^R \gamma_m(\tau)\,\frac{d\tau}{\tau} - c_2,$$

which follows from the subadditivity of the Riesz capacity. □

We formulate a sufficient condition for the regularity of O, which directly follows from Theorem 2.

Corollary 3 ([17], [18])**.** *Let* $n = 2, 3, \dots$ *for* $m = 1$, $n \le 7$ *for* $m = 2$, *and* $n \le 2m + 2$ *for* $m > 2$. *If*

$$\int_0 \gamma_m(\rho)\,\frac{d\rho}{\rho} = \infty$$

or, which is the same,

$$\sum_{j\ge 1} \gamma_m(2^{-j}) = \infty,$$

the point O *is regular with respect to* $(-\Delta)^m$.

Remark. One can see that the assertions and proofs of Theorems 1 and 2 can be extended to the following class of differential operators. Let

$$L(x,\partial)u(x) = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha(a_{\alpha\beta}(x)\partial^\beta u),$$

where $2m < n$ and $a_{\alpha\beta}$ are complex-valued measurable bounded functions, and let $\Phi(x,p)$ be a complex-valued function defined for x and p in a neighbourhood of the point $O \in \partial\Omega$ and subject to the inequality

$$|\nabla_{k,x}\Phi(x,p)| \le c\,|x-p|^{2m-n-k}$$

for $k = 0, 1, \dots, m$. The operator L satisfies the inequality

$$\Re \int_\Omega L(x,\partial)u(x) \cdot \overline{u(x)}\Phi(x,p)\,dx \geq C\Big(|u(p)|^2 + \int_\Omega \sum_{j=1}^{m} \frac{|\nabla_j u(x)|^2}{|x-p|^{n-2j}}\,dx\Big), \tag{45}$$

where u is an arbitrary function in $C_0^m(\Omega)$, supported by a neighbourhood of O, p is an arbitrary point of Ω situated in this neighbourhood and C is a positive constant independent of u and p. The left-hand side in (45) is understood as

$$\Re \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\partial^\beta u(x)\partial^\alpha(\overline{u(x)}\Phi(x,p))\,dx. \tag{46}$$

If, in particular, the operator L has constant coefficients and Φ is its fundamental solution, then the positivity of (46) follows from the inequality

$$\Re \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{L(i\xi)}{L(i(\xi-\eta))} f(\xi)\overline{f(\eta)}\,d\xi d\eta > 0$$

valid for all non-zero $f \in C_0^\infty(\mathbb{R}^n)$. The last inequality was studied recently by S. Eilertsen [33].

9 A pointwise estimate for a function, m-harmonic in $\Omega \setminus B_\rho$

Theorem 3. *Let m and n be the same as in Proposition 2 and let $u \in \mathring{H}(\Omega)$ satisfy*

$$\Delta^m u = 0 \quad on\ \Omega \setminus B_\rho.$$

Then for an arbitrary $p \in \Omega \setminus B_\rho$,

$$|u(p)| \leq c\,(M_\rho(u))^{1/2}\left(\frac{\rho}{|p|}\right)^{n-2m} \exp\Big(-c\int_\rho^{|p|} \gamma_m(\tau)\,\frac{d\tau}{\tau}\Big). \tag{47}$$

Proof. Let w denote the Kelvin transform of u, i.e. the function

$$w(y) = |y|^{2m-n} u\left(\frac{y}{|y|^2}\right)$$

defined on the image $I\Omega$ of Ω under the inversion $x \mapsto y = x|x|^{-2}$. It is well known that

$$\Delta_y^m \left(|y|^{2m-n} u \left(\frac{y}{|y|^2} \right) \right) = |y|^{-n-2m} (\Delta^m u) \left(\frac{y}{|y|^2} \right).$$

(A simple way to check this formula is to introduce the variables (t, ω), $t = \log r^{-1}$, and to use (27).) Consequently,

$$\int_{I\Omega} w(y) \Delta_y^m w(y)\, dy = \int_{\Omega} u(x) \Delta_x^m u(x)\, dx \tag{48}$$

and therefore $w \in \mathring{H}^m(I\Omega)$ and $u \in \mathring{H}^m(\Omega)$ simultaneously.

By Corollary 2,

$$|w(q)| \leq c \left(\rho^n \int_{B_{2/\rho} \setminus B_{1/\rho}} w^2(y)\, dy \right)^{1/2} \exp\left(-c \int_{1/|q|}^{1/\rho} \gamma_m(\tau) \frac{d\tau}{\tau} \right)$$

for all $q \in I\Omega \cap B_{1/\rho}$, which is equivalent to the inequality

$$|q|^{2m-n} \left| u \left(\frac{q}{|q|^2} \right) \right| \leq c \left(\rho^n \int_{B_{2/\rho} \setminus B_{1/\rho}} |y|^{2(2m-n)} u^2 \left(\frac{y}{|y|^2} \right) dy \right)^{1/2} \times \exp\left(-c \int_{|p|}^{\rho} \gamma_m(\tau) \frac{d\tau}{\tau} \right).$$

By putting $p = q|q|^{-2}$, $x = y|y|^{-2}$ we complete the proof. □

By (48) and Theorem 9.3.2/1 in [15] mentioned in the beginning of Sec. 2, one can obtain that $\mathrm{cap}_m(IK, B_{4/\rho})$ is equivalent to $\rho^{2(2m-n)} \mathrm{cap}_m(K, B_{4\rho})$ for $K \subset S_\rho$. Hence the function

$$\gamma_m^*(\rho) = \rho^{2m-n} \mathrm{cap}_m(S_\rho \setminus I\Omega, B_{4\rho})$$

satisfies the equivalence relation

$$\gamma_m^*(\rho) \sim \rho^{n-2m} \mathrm{cap}_m(S_{1/\rho} \setminus \Omega, B_{4/\rho}),$$

which, together with the easily checked property of the capacity

$$\mathrm{cap}_m(S_\rho \setminus \Omega, B_{4\rho}) \sim \mathrm{cap}_m(S_\rho \setminus \Omega),$$

valid for $n > 2m$ (see [15], Proposition 9.1.1/3), implies

$$\int_{1/|p|}^{1/\rho} \gamma_m^*(\tau)\,\frac{d\tau}{\tau} \sim \int_\rho^{|p|} \gamma_m(\tau)\,\frac{d\tau}{\tau}.$$

Here $|p| > \rho$ and c_1, c_2 are positive constants depending on n and m. Furthermore, by definition of w,

$$M_{1/\rho}(w) \sim \rho^{n-2m} M_\rho(u)$$

and the result follows from (44) applied to w.

By a standard argument Theorems 2 and 3 yield the following variant of the Phragmén-Lindelöf principle.

Corollary 4. *Let m and n be the same as in Proposition 2 and let $\zeta u \in \mathring{H}^m(\Omega)$ for all $\zeta \in C^\infty(\mathbb{R}^n)$, $\zeta = 0$ near O. If*

$$\Delta^m u = 0 \quad \text{on } \Omega \cap B_1,$$

then either $u \in \mathring{H}(\Omega)$ and

$$\limsup_{\rho\to 0} \sup_{B_\rho\cap\Omega} |u(x)| \exp\left(c\int_\rho^1 \gamma_m(\tau)\frac{d\tau}{\tau}\right) < \infty \tag{49}$$

or

$$\liminf_{\rho\to 0}\ \rho^{n-2m} M_\rho(u) \exp\left(-c\int_\rho^1 \gamma_m(\tau)\,\frac{d\tau}{\tau}\right) > 0. \tag{50}$$

10 Estimates for the Green function

Let G_m be the Green function of the Dirichlet problem for $(-\Delta)^m$, i.e. the solution of the equation

$$(-\Delta_x)^m G_m(x,y) = \delta(x-y), \quad y \in \Omega,$$

with zero Dirichlet data understood in the sense of the space $\mathring{H}^m$.

Theorem 4. *Let $n = 5, 6, 7$ for $m = 2$ and $n = 2m+1$, $2m+2$ for $m > 2$. There exists a constant c, which depends only on m, such that*

$$\big|G_m(x,y) - \gamma|x-y|^{2m-n}\big| \le c\, d_y^{2m-n} \quad \text{if } |x-y| \le d_y,$$
$$|G_m(x,y)| \le c\,|x-y|^{2m-n} \quad \text{if } |x-y| > d_y,$$

where $d_y = \operatorname{dist}(y, \partial\Omega)$.

Proof. Let $\Omega_y = \{x \in \Omega : |x-y| < d_y\}$ and $a\Omega_y = \{x \in \Omega : |x-y| < ad_y\}$. We introduce the cut-off function $\eta \in C_0^\infty[0,1)$ equal to 1 on the segment $[0,1/2]$. Put

$$H(x,y) = G_m(x,y) - \eta\left(\frac{|x-y|}{d_y}\right)\Gamma(x-y).$$

Clearly, the function $x \mapsto (-\Delta_x)^m H(x,y)$ is supported by $\Omega_y \setminus 2^{-1}\Omega_y$ and the inequality

$$|\Delta_x^m H(x,y)| \le c\, d_y^{-n}$$

holds.

By Corollary 2 applied to the function $x \mapsto H(x,y)$, we have

$$H(p,y)^2 \le 2\int_{\Omega_y} (-\Delta_x)^m H(x,y) \cdot H(x,y)\Gamma(x-p)\,dx.$$

Therefore,

$$\sup_{p\in\Omega_y} H(p,y)^2 \le 2 \sup_{x\in\Omega_y} |H(x,y)| \sup_{p\in 2\Omega_y} \int_{\Omega_y} |\Delta_x^m H(x,y)|\Gamma(x-p)\,dx, \quad (51)$$

and hence,

$$\sup_{p\in 2\Omega_y} |H(p,y)| \le c d_y^{-n} \sup_{p\in 2\Omega_y} \int_{\Omega_y} \Gamma(x-p)\,dx \le c d_y^{2m-n}. \quad (52)$$

Since $\Delta_p^m H(p,y) = 0$ for $p \notin \Omega_y$, we obtain from (52) and Corollary 4, where O is replaced by p, that for $p \notin 2\Omega_y$,

$$|H(p,y)| \le c\left(\frac{d_y}{|p-y|}\right)^{n-2m} \sup_{x\in 2\Omega_y} |H(x,y)| \le c|p-y|^{2m-n}.$$

The result follows. □

Theorem just proved along with Corollary 2 yields

Corollary 5. *Let m and n be the same as in Theorem 4. The Green function G_m satisfies*

$$|G_m(x,y)| \le \frac{c}{|y|^{n-2m}} \exp\left(-c\int_{|x|}^{|y|} \gamma_m(\tau)\,\frac{d\tau}{\tau}\right)$$

for $2|x| < |y|$.

We conclude with the following analogue of Theorem 4 in the case $n = 2m$.

Theorem 5. *Let $n = 2m$ and let Ω be a domain with diameter D. Let also*

$$\Gamma(x-y) = \gamma \log \frac{D}{|x-y|}.$$

Then

$$|G_m(x-y) - \Gamma(x-y)| \le c_1 \log \frac{D}{d_y} + c_2 \quad \textit{if } |x-y| \le d_y,$$

$$|G_m(x,y)| \le c_3 \log \frac{D}{d_y} + c_4 \quad \textit{if } |x-y| > d_y.$$

Proof. Proceeding in the same way as in the proof of Theorem 4 we arrive at (51). Therefore,

$$\sup_{p \in 2\Omega_y} |H(p,y)| \le c d_y^{-2m} \sup_{p \in 2\Omega_y} \int_{\Omega_y} \Gamma(x-p)\, dx \le c_1 \log \frac{D}{d_y} + c_2.$$

Hence and by Corollary 2 we obtain for $p \notin \Omega_y$

$$|H(p,y)| \le c \sup_{x \in 2\Omega_y} |H(x,y)| \le c \left(c_1 \log \frac{D}{d_y} + c_2 \right).$$

Since $G_m(p,y) = H(p,y)$ for $p \notin 2\Omega_y$, the result follows. $\square$

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