Functions of least gradient and BV functions

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In these lectures I will present work concerning applications of BV theory to a variety of problems including the problem of least gradient. In Sections 1 through 3 I will discuss the problem of least gradient whose work is based on [SWZ1], [SWZ2], [SWZ3], [SZ] and [ISZ].

1 Functions of least gradient

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, and for $g\colon \partial\Omega \to \mathbb{R}^1$ continuous, we consider the problem

$$\inf\{\|\nabla u\|(\Omega) : u \in BV(\Omega) \cap C^0(\overline{\Omega}), u = g \text{ on } \partial\Omega\}. \tag{1.1}$$

Here, $\|\nabla u\|(\Omega)$ denotes the total variation of the vector-valued measure ∇u evaluated on Ω. In (1.1), the direct method can be easily seen to provide a minimizer, utilizing the compactness ensured by the constraint. However, in (1.1), the compactness in $L^1(\Omega)$ of a sequence whose BV-norms are bounded does not ensure, *a priori*, continuity of the limiting function or that it will assume the boundary values g, thus making the question of existence more subtle. We will show (Theorem 3.6) that a solution $u \in BV(\Omega) \cap C^0(\overline{\Omega})$ exists provided $\partial\Omega$ satisfies two conditions, namely, that $\partial\Omega$ has non-negative mean curvature (in a weak sense) and that $\partial\Omega$ is not locally area-minimizing. Furthermore, if either condition fails, it can be shown that there exists boundary data g for which the corresponding problem (1.1) has no solution. It should be noted that the question of existence was treated by Parks [P1], [P2] where it was shown that for a strictly convex domain with boundary values satisfying the bounded slope condition, a unique Lipschitz solution exists. Other authors also investigated properties of least gradient, including [M], [PZ] and [BDG]. In [BDG] it was shown, among other things, that the superlevel sets of a function of least gradient are area-minimizing. This result provides the major motivation for the techniques employed here. Indeed, this fact, along with the co-area formula (see (2.12) below), suggests that the existence of a function of least gradient can be established by actually constructing each of its superlevel sets in such a way that they reflect the appropriate boundary condition and that they are area-minimizing. The main thrust of this work is to show that this is possible. Thus, we show that there exists a solution to (1.1) and we also show (Theorem 3.9) that it is unique. Both existence and uniqueness are developed by extensive use of BV theory and sets of finite perimeter as well as certain maximum principles.

Finally, concerning regularity, in two dimensions, functions of the form $u(x, y) = f(y/x)$ are functions of least gradient, thus showing that in general, functions of least gradient have regularity in the interior no better than that at the boundary. However, we show that for boundary data of class $C^{0,\alpha}$, the solution is of class $C^{0,\alpha/2}$. Examples are given which demonstrate that this result is optimal.

2 Notation and preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ will be denoted by $|E|$ and $H^\alpha(E)$, $\alpha > 0$, will denote α-dimensional Hausdorff measure of E. Throughout, we almost exclusively employ H^{n-1}. The Euclidean distance between points $x, y \in \mathbb{R}^n$ will be denoted by $|x - y|$. If $\Omega \subset \mathbb{R}^n$ is an open set, the class of functions $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is denoted by $BV(\Omega)$ and is called the space of *functions of bounded variation in Ω*. The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{1;\Omega} + \|\nabla u\|(\Omega) \tag{2.1}$$

where $\|u\|_{1;\Omega}$ denotes the L^1-norm of u on Ω and where $\|\nabla u\|$ is the total variation of the vector-valued measure ∇u.

The following compactness result for $BV(\Omega)$ will be needed later, cf. [G2] or [Z].

Theorem 2.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then*

$$BV(\Omega) \cap \{u : \|u\|_{BV(\Omega)} \leq 1\}$$

is compact in $L^1(\Omega)$. Moreover, if $u_i \to u$ in $L^1(\Omega)$, and $U \subset \Omega$ is open, then

$$\liminf_{i\to\infty} \|\nabla u_i\|(U) \geq \|\nabla u\|(U).$$

A Borel set $E \subset \mathbb{R}^n$ is said to have *finite perimeter in Ω* provided the characteristic function of E, χ_E, is a function of bounded variation in Ω. Thus, the partial derivatives of χ_E are Radon measures on Ω and the perimeter of E in Ω is defined as

$$P(E,\Omega) = \|\nabla\chi_E\|(\Omega). \tag{2.2}$$

A set E is said to be of *locally finite perimeter* if $P(E,\Omega) < \infty$ for every bounded open set $\Omega \subset \mathbb{R}^n$.

One of the fundamental results of the theory of sets of finite perimeter is that they possess a measure-theoretic exterior normal which is suitably general to ensure the validity of the Gauss-Green theorem. A unit vector ν is defined as the measure-theoretic exterior normal to E at x provided

$$\lim_{r\to 0} r^{-n} \left|B(x,r) \cap \{y : (y-x)\cdot\nu < 0, y \notin E\}\right| = 0$$

and

$$\lim_{r\to 0} r^{-n} \left|B(x,r) \cap \{y : (y-x)\cdot\nu > 0, y \in E\}\right| = 0, \tag{2.3}$$

where $B(x,r)$ denotes the open ball of radius r centered at x. The measure-theoretic normal of E at x will be denoted by $\nu(x,E)$ and we define

$$\partial^* E = \{x : \nu(x,E) \text{ exists}\}. \tag{2.4}$$

Clearly, $\partial^* E \subset \partial E$, where ∂E denotes the topological boundary of E. Also, the *topological interior of* E is denoted by $E^i = (\mathbb{R}^n - \partial E) \cap E$ and the *topological exterior* by $E^e = (\mathbb{R}^n - \partial E) \cap (\mathbb{R}^n - E)$. We employ E^c to denote $\mathbb{R}^n - E$. The notation $E \subset\subset F$ means that the closure of E is a subset of F^i. If $E \subset \mathbb{R}^n$ is a Borel set, we define the *measure-theoretic boundary of* E as

$$\partial_M E = \left\{x : 0 < \limsup_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}\right\} \cap \left\{x : \liminf_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} < 1\right\}. \tag{2.5}$$

In other words, the measure-theoretic boundary of E is all points at which the metric density of E is neither 1 nor 0. Clearly, $\partial^* E \subset \partial_M E \subset \partial E$. Moreover, it is well known that

$$E \text{ is of finite perimeter if and only if } H^{n-1}(\partial_M E) < \infty \tag{2.6}$$

and that

$$\begin{aligned} P(E,\Omega) &= H^{n-1}(\Omega \cap \partial_M E) \\ &= H^{n-1}(\Omega \cap \partial^* E) \qquad \text{whenever } P(E,\Omega) < \infty, \end{aligned} \tag{2.7}$$

cf. [F2], § 4.5. From this it easily follows that

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E, \Omega) + P(F, \Omega), \tag{2.8}$$

thus implying that sets of finite perimeter are closed under finite unions and intersections.

The definition implies that sets of finite perimeter are defined only up to sets of measure 0. In other words, each set determines an equivalence class of sets of finite perimeter. In order to avoid this ambiguity, whenever a set of finite perimeter, E, is considered we shall always employ the measure theoretic closure as the set to represent E. Thus, with this convention, we have

$$x \in E \text{ if and only if } \limsup_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0. \tag{2.9}$$

Also, it can be shown with convention (2.9) that,

$$\overline{\partial^* E} = \partial E, \tag{2.10}$$

cf. [G2], Theorem 4.4. Here, $\overline{A}$ denotes the topological closure of A. This convention will apply, in particular, to all competitors of the variational problems (2.21) and (2.22) below as well as to the sets defined by (2.18).

Of particular importance to us are sets of finite perimeter whose boundaries are area-minimizing. If E is a set of locally finite perimeter and U a bounded, open set, let

$$\psi(E, U) = \|\nabla \chi_E\|(U) - \inf\{\|\nabla \chi_F\|(U) : E \Delta F \subset\subset U\}, \tag{2.11}$$

where $E \Delta F$ denotes the symmetric difference of E and F. The set ∂E is said to be *area-minimizing in* U if $\psi(E, U) = 0$ and *locally area-minimizing* if $\psi(E, U) = 0$ whenever U is bounded.

Another tool that will play a significant role in this paper is the co-area formula. It states that if $u \in BV(\Omega)$, then

$$\|\nabla u\|(\Omega) = \int_{-\infty}^{\infty} P(E_t, \Omega)\, dt \tag{2.12}$$

where $E_t = \{u \geq t\}$. In case u is Lipschitz, we have

$$\int_\Omega |\nabla u|\, dx = \int_{-\infty}^{\infty} H^{n-1}(u^{-1}(t) \cap \Omega)\, dt.$$

Conversely, if u is integrable on Ω then

$$\int_{-\infty}^{\infty} P(E_t, \Omega)\, dt < \infty \text{ implies } u \in BV(\Omega). \tag{2.13}$$

See [F1], [FR].

The regularity of ∂E will play a crucial role in our development. In particular, we will employ the notion of tangent cone. Suppose ∂E is area-minimizing in U and for convenience of notation, suppose $0 \in U \cap \partial E$. For each $r > 0$, let $E_r = \mathbb{R}^n \cap \{x : rx \in E\}$. It is known (cf. [S1], § 35, [MM], § 2.6) that for each sequence $\{r_i\} \to 0$ there exists a subsequence (denoted by the full sequence) such that $\chi_{E_{r_i}}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to χ_C, where C is a set of locally finite perimeter. In fact, ∂C is area-minimizing and is called the tangent cone to E at 0. Although it is not immediate, C is a cone and therefore the union of half-lines issuing from 0. It follows from [S1], § 37.6, that if $\overline{C}$ is contained in $\overline{H}$ where H is any half-space in $\mathbb{R}^n$ with $0 \in \partial H$, then ∂E is regular at 0. That is, there exists $r > 0$ such that

$$B(0, r) \cap \partial E \text{ is a real analytic hypersurface.} \tag{2.14}$$

Furthermore, ∂E is regular at all points of $\partial^* E$ and

$$H^\alpha((\partial E - \partial^* E) \cap U) = 0 \quad \text{for all } \alpha > n - 8, \tag{2.15}$$

cf. [G2], Theorem 11.8.

Finally, we conclude with a result which is a direct consequence of a maximum principle for area-minimizing hypersurfaces which was established independently in [Mo] and [S2].

Theorem 2.2. *Let $E_1 \subset E_2$ and suppose ∂E_2 and ∂E_1 are area-minimizing in an open set $U \subset \mathbb{R}^n$. Further, suppose $x \in (\partial E_1) \cap (\partial E_2) \cap U$. Then ∂E_1 and ∂E_2 agree in some neighborhood of x.*

Let

$$[a, b] = \{\cap I : I \text{ an interval containing } g(\partial \Omega)\}. \tag{2.16}$$

The boundary data g admits a continuous extension $G \in BV(\mathbb{R}^n - \overline{\Omega}) \cap C^0(\mathbb{R}^n - \Omega)$, [G2], Theorem 2.16; in fact, $G \in C^\infty(\mathbb{R}^n - \overline{\Omega})$, but we will only need that G is continuous on the complement of Ω. Clearly, we can require that the support of G is contained in $B(0, R)$ where R is chosen so that $\Omega \subset\subset B(0, R)$. We have

$$G \in BV(\mathbb{R}^n - \overline{\Omega}) \cap C^0(\mathbb{R}^n - \Omega) \text{ with } G = g \text{ on } \partial\Omega. \tag{2.17}$$

We now introduce sets that will ensure that our constructed solution satisfies the required Dirichlet condition $u = g$ on $\partial\Omega$. Thus, for each $t \in [a, b]$, let

$$\mathcal{L}_{\mathrm{t}} = (\mathbb{R}^n - \Omega) \cap \{x : G(x) \geq t\}. \tag{2.18}$$

Note that the co-area formula (2.12) and the fact that $G \in BV(\mathbb{R}^n - \overline{\Omega})$ imply that $P(\mathcal{L}_{\mathrm{t}}, \mathbb{R}^n - \overline{\Omega}) < \infty$ for almost all t. For all such t, we remind the reader that we employ our convention (2.9) in defining $\mathcal{L}_{\mathrm{t}}$.

We let

$$T = [a, b] \cap \{t : P(\mathcal{L}_t, \mathbb{R}^n - \overline{\Omega}) < \infty\}. \tag{2.19}$$

Thus, by (2.7) and the fact that $H^{n-1}(\partial\Omega) < \infty$, we obtain

$$H^{n-1}(\partial_m \mathcal{L}_{\mathrm{t}}) = P(\mathcal{L}_{\mathrm{t}}, \mathbb{R}^n - \overline{\Omega}) + H^{n-1}[(\partial_m \mathcal{L}_{\mathrm{t}}) \cap (\partial\Omega)] < \infty. \tag{2.20}$$

For each $t \in T$, the variational problems

$$\min\{P(E, \mathbb{R}^n) : E - \overline{\Omega} = \mathcal{L}_{\mathrm{t}} - \overline{\Omega}\} \tag{2.21}$$
$$\max\{|E| : E \text{ is a solution of } (2.21)\} \tag{2.22}$$

will play a central role in our development. In light of Theorem 2.1, a solution to both problems can be obtained from the direct method. (2.20) is also used to obtain existence in (2.21). We will denote by E_t the solution to (2.22). In this regard, note that our convention (2.9) ensures $E_t - \overline{\Omega} = \mathcal{L}_{\mathrm{t}} - \overline{\Omega}$; furthermore, because of our convention, $\mathcal{L}_{\mathrm{t}}$ need not be a closed set.

3 Construction of a function of least gradient

In this section we will construct a solution u of (1.1) by using $E_t \cap \overline{\Omega}$ to define the set $\{u \geq t\}$ up to a set of measure zero for almost all t. This construction will be possible for bounded Lipschitz domains Ω whose boundaries satisfy the following two conditions.

(i) For every $x \in \partial\Omega$ there exists $\varepsilon_0 > 0$ such that for every set of finite perimeter $A \subset\subset B(x, \varepsilon_0)$

$$P(\Omega, \mathbb{R}^n) \leq P(\Omega \cup A, \mathbb{R}^n). \tag{3.1}$$

(ii) For every $x \in \partial\Omega$, and every $\varepsilon > 0$ there exists a set of finite perimeter $A \subset\subset B(x, \varepsilon)$ such that

$$P(\Omega, B(x, \varepsilon)) > P(\Omega - A, B(x, \varepsilon)). \tag{3.2}$$

Clearly, we may assume that $x \in A$.

The first condition states that $\partial\Omega$ has non-negative mean curvature (in the weak sense) while the second states that $\partial\Omega$ is not locally area-minimizing with respect to interior variations. Also, it can be easily verified that if $\partial\Omega$ is smooth, then both conditions together are equivalent to the condition that the mean curvature of $\partial\Omega$ is positive on a dense set of $\partial\Omega$.

Since Ω is a Lipschitz domain, for each $x_0 \in \partial\Omega$, $\partial\Omega$ can be represented as the graph of a non-negative Lipschitz function h defined on some ball $B'(x_0', r) \subset \mathbb{R}^{n-1}$ where $x_0' \in \mathbb{R}^{n-1}$. That is, $\{(x', h(x')) : x' \in B'(x_0', r)\} \subset \partial\Omega$. Throughout we will use the notation $B'(x_0', r)$ and x' to denote elements in $\mathbb{R}^{n-1}$ and thus they will be distinguished from their n-dimensional counterparts $B(x_0, r)$ and x.

We assume our configuration is oriented in such a way that $\{(x', x'') : 0 < x'' < h(x')\} \subset \Omega$. Using the fact that Ω is a Lipschitz domain, we have that Ω is a set of finite perimeter and $P(\Omega, U) = H^{n-1}(\partial^*\Omega \cap U) = H^{n-1}(\partial\Omega \cap U)$, whenever $U \subset \mathbb{R}^n$ is an open set, cf. [F2]. Also, with $S = \{(x', h(x')) : x' \in B'(x_0', r)\}$ we have that

$$H^{n-1}(S) = \int_{B'(x_0', r)} \sqrt{1 + |\nabla h|^2}\, dH^{n-1}(x').$$

These facts lead almost immediately to the following result.

Lemma 3.1. *If Ω is a Lipschitz domain with non-negative mean curvature in the sense of* (3.1)*, then the function h, whose graph represents $\partial\Omega$ locally, is a weak supersolution of the minimal surface equation. That is, for r sufficiently small,*

$$\int_{B'(x_0', r)} \frac{\nabla h \cdot \nabla \varphi}{\sqrt{1 + |\nabla h|^2}}\, dx' \geq 0$$

whenever $\varphi \in C_0^{1,1}(B'(x_0', r))$, $\varphi \geq 0$.

Proof. For $t > 0$ and $\varphi \in C_0^{1,1}(B'(x_0', r))$, $\varphi \geq 0$ let

$$f(t) = \int_{B'(x_0', r)} \sqrt{1 + |\nabla h|^2 + 2t\nabla h \cdot \nabla\varphi + t^2 |\nabla\varphi|^2}\, dx',$$
$$A = \{(x', x'') : h(x') \leq x'' \leq h(x') + t\varphi(x'), x' \in B'(x_0', r)\}.$$

Assuming that r has been chosen sufficiently small so that condition (3.1) can be invoked, we have $P(\Omega) \leq P(A \cup \Omega)$ and hence

$$\begin{aligned} 0 \leq P(A \cup \Omega) - P(\Omega) &= H^{n-1}(\partial(A \cup \Omega)) - H^{n-1}(\partial\Omega) \\ &= f(t) - f(0). \end{aligned}$$

Hence, $f'(0) \geq 0$, which establishes our conclusion. $\square$

We will also need the following result from [SWZ3], Lemma 4.2, whose proof is an easy consequence of the weak Harnack inequality.

Lemma 3.2. *Suppose W is an open subset of $\mathbb{R}^{n-1}$. If $v_1, v_2 \in C^{0,1}(W)$ are respectively weak super and subsolutions of the minimal surface equation in W and if $v_1(x_0') = v_2(x_0')$ for some $x_0' \in W$ while $v_1(x') \geq v_2(x')$ for all $x' \in W$, then*

$$v_1(x') = v_2(x')$$

for all x' in some closed ball contained in W centered at x_0'.

An important step in our development is the following lemma.

Lemma 3.3. *For almost all $t \in [a,b]$, $\partial E_t \cap \partial \Omega \subset g^{-1}(t)$.*

Proof. We will prove the lemma for all $t \in T$, where T is defined by (2.19). The proof will proceed by contradiction and we first show that ∂E_t is locally area-minimizing in a neighborhood of each point $x_0 \in \partial E_t \cap \partial \Omega - g^{-1}(t)$; that is, we claim there exists $\varepsilon > 0$ such that for every set F with the property that $F \Delta E_t \subset\subset B(x_0, \varepsilon)$, we have

$$P(E_t, B(x_0,\varepsilon)) \leq P(F, B(x_0,\varepsilon)) \tag{3.3}$$

or equivalently,

$$P(E_t, \mathbb{R}^n) \leq P(F, \mathbb{R}^n).$$

By our assumption, either $g(x_0) < t$ or $g(x_0) > t$. First consider the case $g(x_0) < t$. Since $G(x_0) = g(x_0) < t$ and G is continuous on $\mathbb{R}^n - \Omega$, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \cap \mathcal{L}_t = \emptyset$. We will assume that $\varepsilon < \varepsilon_0$, where ε_0 appears in condition (3.1). We proceed by taking a variation F satisfying $F \Delta E_t \subset\subset B(x_0, \varepsilon)$. Note that because of (3.1) and (2.8) we have for every $A \subset\subset B(x_0, \varepsilon_0)$,

$$\begin{aligned} P(A \cap \Omega, \mathbb{R}^n) + P(A \cup \Omega, \mathbb{R}^n) &\leq P(A, \mathbb{R}^n) + P(\Omega, \mathbb{R}^n) \\ &\leq P(A, \mathbb{R}^n) + P(A \cup \Omega, \mathbb{R}^n). \end{aligned}$$

Hence,

$$P(A \cap \Omega, \mathbb{R}^n) \leq P(A, \mathbb{R}^n). \tag{3.4}$$

Define $F' = (F - B(x_0,\varepsilon)) \cup (F \cap \overline{\Omega})$. Clearly,

$$\begin{aligned} F' - \overline{\Omega} &= (F - B(x_0,\varepsilon)) - \overline{\Omega} \\ &= (F - \overline{\Omega}) - B(x_0,\varepsilon) = (E_t - \overline{\Omega}) - B(x_0,\varepsilon) \\ &= \mathcal{L}_{\mathrm{t}} - \overline{\Omega} - B(x_0,\varepsilon) = \mathcal{L}_{\mathrm{t}} - \overline{\Omega}. \end{aligned}$$

Thus F' is admissible in (2.21) and therefore

$$P(E_t,\mathbb{R}^n) \le P(F',\mathbb{R}^n).$$

It remains to show $P(F',\mathbb{R}^n) \le P(F,\mathbb{R}^n)$. First observe from $E_t \Delta F \subset\subset B(x_0,\varepsilon)$ and $(E_t - \overline{\Omega}) \cap B(x_0,\varepsilon) = (\mathcal{L}_t - \overline{\Omega}) \cap B(x_0,\varepsilon) = \emptyset$ that $F' \cap B(x_0,\varepsilon) = F \cap B(x,\varepsilon) \cap \overline{\Omega}$ and $F' \Delta F \subset\subset B(x_0,\varepsilon)$. Hence, we obtain by (3.4),

$$\begin{aligned} P(F,\mathbb{R}^n) - P(F',\mathbb{R}^n) &= P(F, B(x_0,\varepsilon)) - P(F', B(x_0,\varepsilon)) \\ &= P(F \cap B(x_0,\varepsilon), B(x_0,\varepsilon)) \\ &\quad - P(F \cap B(x_0,\varepsilon) \cap \Omega, B(x_0,\varepsilon)) \\ &= P(F \cap B(x_0,\varepsilon), \mathbb{R}^n) - P(F \cap B(x_0,\varepsilon) \cap \Omega, \mathbb{R}^n) \\ &\ge 0. \end{aligned} \tag{3.5}$$

This establishes (3.3) when $g(x_0) < t$.

The argument to establish (3.3) in case $g(x_0) > t$ is analogous to the first case, but we present it for the sake of completeness. Since $G(x_0) = g(x_0) > t$, the continuity of G in Ω^c implies that $\overline{B}(x_0,\varepsilon) - \Omega \subset \mathcal{L}_{\mathrm{t}}$, provided ε is sufficiently small. We also require that $\varepsilon < \varepsilon_0$. Let F be a variation such that $F \Delta E_t \subset\subset B(x_0,\varepsilon)$ and now define $F' = F \cup (B(x_0,\varepsilon) - \overline{\Omega})$. Then

$$\begin{aligned} F' - \overline{\Omega} &= (F - \overline{\Omega}) \cup (B(x_0,\varepsilon) - \overline{\Omega}) \\ &= [(F - B(x_0,\varepsilon)) - \overline{\Omega}] \cup (B(x_0,\varepsilon) - \overline{\Omega}) \\ &= [(E_t - B(x_0,\varepsilon)) - \overline{\Omega}] \cup (B(x_0,\varepsilon) - \overline{\Omega}) \\ &= (\mathcal{L}_{\mathrm{t}} - B(x_0,\varepsilon) - \overline{\Omega}) \cup (B(x_0,\varepsilon) - \overline{\Omega}) \\ &= (\mathcal{L}_{\mathrm{t}} - B(x_0,\varepsilon) - \overline{\Omega}) \cup (\mathcal{L}_{\mathrm{t}} - \overline{\Omega} \cap B(x_0,\varepsilon)) = \mathcal{L}_{\mathrm{t}} - \overline{\Omega}. \end{aligned}$$

Thus, since F' is a competitor for (2.21), it follows that $P(E_t,\mathbb{R}^n) \le P(F',\mathbb{R}^n)$. Then it remains to show

$$P(F', R^n) \le P(F,\mathbb{R}^n). \tag{3.6}$$

For this, note that $E_t \Delta F \subset\subset B(x_0,\varepsilon)$ and $B(x_0,\varepsilon) - \Omega = B(x_0,\varepsilon) \cap \mathcal{L}_t \subset E_t$ imply $(F')^c \cap B(x_0,\varepsilon) = F^c \cap B(x_0,\varepsilon) \cap \Omega$ and $(F')^c \Delta F^c \subset\subset B(x_0,\varepsilon)$. In light of $P(F,\mathbb{R}^n)-P(F',\mathbb{R}^n) = P(F^c,\mathbb{R}^n)-P((F')^c,\mathbb{R}^n)$, (3.6) follows from (3.5) with F and F' replaced by F^c and $(F')^c$.

We thus have demonstrated that ∂E_t is area-minimizing in $B(x_0,\varepsilon)$. We will show that this leads to a contradiction. Assume first that $g(x_0) < t$ so that $G < t$ on $(\mathbb{R}^n - \Omega) \cap B(x_0,\varepsilon)$ provided ε has been chosen sufficiently small. Consequently,

$$E_t \cap B(x_0,\varepsilon) \subset \overline{\Omega} \cap B(x_0,\varepsilon). \tag{3.7}$$

We recall the notation concerning the representation of $\partial\Omega$ as the graph of a Lipschitz function that preceded Lemma 3.1. Thus, with $x_0 \in \partial E_t \cap \partial\Omega - g^{-1}(t)$, we express $\partial\Omega$ locally about x_0 as $\{(x',h(x')) : x' \in B'(x_0',\varepsilon')\}$ where $x_0 = (x_0',x_0'')$ and $x_0'' = h(x_0') > 0$. For simplicity of notation, we take $x_0' = 0$. The number ε' is chosen so that $\varepsilon' < \varepsilon$ and that

$$\{(x',h(x')) : |x'| \le \varepsilon'\} \subset B(x_0,\varepsilon). \tag{3.8}$$

We define the half-infinite cylinder above $B'(0,\varepsilon')$ as $C = B'(0,\varepsilon') \times [0,\infty)$. Because of the local nature of the argument, we may assume that $\Omega \cap C = \{(x',x'') : |x'| < \varepsilon, 0 \le x'' < h(x')\}$.

Now consider the solution to the minimal surface equation on $B'(0,\varepsilon')$ relative to the boundary data $f = h|_{\partial B'(0,\varepsilon')}$, [MM], Chapter 3. Thus we let v be the unique solution of

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0 \text{ on } B'(0,\varepsilon'),$$
$$v = f \text{ on } \partial B'(0,\varepsilon').$$

Since h is a weak supersolution of the minimal surface equation by Lemma 3.1, we have that $h \ge v$ on $\overline{B'}(0,\varepsilon')$, cf. [GT], Theorem 10.7. In fact $h > v$ on $B'(0,\varepsilon')$ because the set $\{h = v\}$ is obviously closed in $B'(0,\varepsilon')$ and it is also open in $B'(0,\varepsilon')$ because of Lemma 3.2. Hence, if this set is non-empty, $h = v$ in $B'(0,\varepsilon')$ which would contradict (3.2). Consequently, with $\delta = h(0) - v(0)$, we have $\delta > 0$. Now consider a 1–parameter family of graphs $v_\tau(x') = v(x') + \tau$ and let

$$\tau^* = \max\{\tau : \text{there exists } x' \in \overline{B'}(0,\varepsilon') \text{ such that } (x',v_\tau(x')) \in \partial E_t \cap \overline{\Omega}\}.$$

Note that $\tau^* \ge \delta$ since $v(0) + \delta = h(0)$ and $(0,h(0)) \in \partial E_t \cap \overline{\Omega}$. Let

$$V_{\tau^*} = \{(x',x'') : |x'| < \varepsilon', x'' \le v(x') + \tau^*\}$$

and, in view of our choice of ε', observe that

$$E_t \cap \{x : |x'| < \varepsilon'\} \subset V_{\tau^*}.$$

Observe also that if a point $(x', v_{\tau^*}(x'))$ is an element of $(\partial E_t) \cap \overline{\Omega}$, then $|x'| < \varepsilon'$ for otherwise we would have $v(x') + \tau^* \leq h(x')$, which would imply that $\tau^* \leq 0$. Thus, the set $\partial[E_t \cap \{x : |x'| < \varepsilon'\}] \cap \{(x', v_{\tau^*}(x')) : |x'| < \varepsilon'\}$ is non-empty and according to Theorem 2.2, it is open as well as closed in the connected set $\{(x', v_{\tau^*}(x')) : |x'| < \varepsilon'\}$. This implies that

$$\partial[E_t \cap \{x : |x'| < \varepsilon'\}] \supset \{(x', v_{\tau^*}(x')) : |x'| < \varepsilon'\}. \tag{3.9}$$

Since $\tau^* > 0$, it follows that $v_{\tau^*}(x') > h(x')$ whenever $|x'| = \varepsilon'$. Consequently, using the continuity of v_{τ^*}, the graph $\{(x', v_{\tau^*}(x')) : |x'| < \varepsilon'\}$ contains points in $\mathbb{R}^n - \overline{\Omega}$, say $(y', v_{\tau^*}(y'))$, $|y'| < \varepsilon'$, as well as points in $\overline{\Omega} \cap B(x_0, \varepsilon)$, say $(z', v_{\tau^*}(z'))$, $|z'| < \varepsilon'$. The point $(y', v_{\tau^*}(y'))$, $|y'| < \varepsilon'$, could possibly be an element of $\mathbb{R}^n - B(x_0, \varepsilon)$. Consider the line segment, λ, in $B'(x_0, \varepsilon')$ that joins y' and z'. Let a' be that point on λ closest to y' with the property that $(a', v_{\tau^*}(a')) \in \partial\Omega$. Then, all points a on λ that are closer to y' than a' and that are sufficiently near a' have the property that $(a, v_{\tau^*}(a)) \in \mathbb{R}^n - \overline{\Omega} \cap B(x_0, \varepsilon)$. Here we have used (3.8) and the continuity of v_{τ^*}. In view of (3.9), this implies that $E_t \cap B(x_0, \varepsilon) \cap \mathbb{R}^n - \overline{\Omega} \neq \emptyset$, contradicting (3.7). This contradiction was reached under the assumption that $g(x_0) < t$ and the fact that ∂E_t is area minimizing in $B(x_0, \varepsilon)$. A similar proof is employed in case $g(x_0) > t$. $\square$

In order to ultimately identify $E_t \cap \overline{\Omega}$ as the set $\{u \geq t\}$ (up to a set of measure zero) for almost all t, we will need the following result.

Lemma 3.4. *If $s, t \in T$ with $s < t$, then $E_t \subset\subset E_s$.*

Proof. We first show $E_t \subset E_s$. Note that

$$\begin{aligned}(E_s \cap E_t) - \overline{\Omega} &= (E_s - \overline{\Omega}) \cap (E_t - \overline{\Omega}) \\ &= (\mathcal{L}_\mathrm{s} - \overline{\Omega}) \cap (\mathcal{L}_\mathrm{t} - \overline{\Omega}) = \mathcal{L}_\mathrm{t} - \overline{\Omega}\end{aligned}$$

and

$$\begin{aligned}(E_s \cup E_t) - \overline{\Omega} &= (E_s - \overline{\Omega}) \cup (E_t - \overline{\Omega}) \\ &= (\mathcal{L}_\mathrm{s} - \overline{\Omega}) \cup (\mathcal{L}_\mathrm{t} - \overline{\Omega}) = \mathcal{L}_\mathrm{s} - \overline{\Omega}.\end{aligned}$$

Thus $E_s \cap E_t$ is a competitor with E_t, and $E_s \cup E_t$ is a competitor with E_s in (2.21). Thus $P(E_s \cap E_t, \mathbb{R}^n) \geq P(E_t, \mathbb{R}^n)$, and $P(E_s \cup E_t, \mathbb{R}^n) \geq P(E_s, \mathbb{R}^n)$. Then employing (2.8), we obtain

$$P(E_s \cup E_t, \mathbb{R}^n) = P(E_s, \mathbb{R}^n).$$

Reference to (2.22) yields $|E_s \cup E_t| = |E_s|$, which in turn implies $|E_t - E_s| = 0$. In view of (2.9), $E_t \subset E_s$.

It remains to show that this containment is in fact strict. This will follow from a general maximum principle, Theorem 2.2. For this purpose, first note that

$$E_t - \overline{\Omega} = \mathcal{L}_\mathrm{t} - \overline{\Omega} \subset\subset \mathcal{L}_\mathrm{s} - \overline{\Omega} = E_s - \overline{\Omega}. \tag{3.10}$$

relative to the topology on Ω^c. Now observe that Lemma 3.3 implies

$$\partial E_t \cap \partial E_s \cap \partial \Omega = \emptyset. \tag{3.11}$$

In view of (3.10) and (3.11), it remains to show that

$$\partial E_t \cap \partial E_s \cap \Omega = \emptyset \tag{3.12}$$

in order to establish the Lemma. For this purpose, let $S \equiv \partial E_s \cap \partial E_t \cap \Omega$. Assume by contradiction that $S \neq \emptyset$. Observe that S is open relative to ∂E_s for if $x \in S$, then since $E_t \subset E_s$ and since both ∂E_t and ∂E_s are area minimizing in Ω, we can apply Theorem 2.2 to conclude that ∂E_t and ∂E_s agree on a neighborhood in ∂E_s containing x. Since S is obviously closed relative to ∂E_s, it follows from (3.11) that S consists only of components of ∂E_s that do not intersect $\partial \Omega$. We now could appeal to the proof of Theorem 4.4 (parts 2,3, and 3) of [SWZ3] to conclude that S is empty. This method uses only topological arguments along with (2.22). Alternatively, we will use the area-minimizing property of S and proceed as in the proof of Theorem 2.2. Thus, suppose S' is a component of the set of regular points of S. We first show that S' is a cycle in the sense of currents; that if, we wish to show that

$$\int_{S'} \mathrm{d}\varphi = 0 \tag{3.13}$$

whenever φ is a smooth $(n-2)$-form supported in $B(0,R)$ where $B(0,R)$ is the ball introduced earlier having the property that $\Omega \subset\subset B(0,R)$. Since S' is area-minimizing in Ω, we appeal to the monotonicity formula [S1], § 17.6, to conclude that only a finite number of components of (∂E_s) can intersect any given compact subset of Ω, in particular, $\operatorname{spt}\varphi \cap \overline{S'}$. Thus, there exists a smooth function ζ that is 1 on $\operatorname{spt}\varphi \cap \overline{S'}$ and 0 in a neighborhood of $\partial E_s - \overline{S'}$. Then, (3.13) is established by

$$\int_{S'} \mathrm{d}\varphi = \int_{S'} \mathrm{d}(\zeta\varphi) = \int_{\partial E_s} \mathrm{d}(\zeta\varphi) = 0.$$

Thus, S' is an $(n-1)$-rectifiable cycle in the sense of currents; that is, $\partial S' = 0$. Now appeal to [S1], 27.6, to conclude that there is a measurable set $F \subset B(0,R)$ such that $\partial F = S'$. It follows from elementary considerations that for a given vector $\nu \in \mathbb{R}^n$, there is a hyperplane, P, with normal ν such that $P \cap \overline{S'} \neq \emptyset$ and

$$F \subset \{x : (x - x_0) \cdot \nu \leq 0\}$$

where $x_0 \in P \cap \overline{S'}$. Theorem 2.2 implies $P \cap \overline{S'}$ is open as well as closed in P, thus yielding $P = S'$, a contradiction. □

We now are in a position to construct the solution u to problem (1.1). For this purpose, we first define for $t \in T$,

$$A_t = \overline{E_t \cap \Omega}.$$

Observe that E_t is closed relative to Ω since each point of ∂E_t is either a regular point of ∂E_t or a point at which a tangent cone exists. This implies that each point of ∂E_t is an element of $\partial_M E_t$. From our convention (2.9), we therefore have $A_t \cap \Omega = E_t \cap \Omega$. Also, with the help of Lemma 3.3, observe that for $t \in T$,

$$\{g > t\} \subset (E_t)^i \cap \partial\Omega \subset A_t \cap \partial\Omega, \tag{3.14}$$

$$\overline{\{g > t\}} \subset A_t \cap \partial\Omega \subset \overline{E_t} \cap \partial\Omega = [(E_t)^i \cup \partial E_t] \cap \partial\Omega \subset \{g \geq t\}. \tag{3.15}$$

Finally, note that (3.15) and Lemma 3.4 imply

$$A_t \subset\subset A_s \tag{3.16}$$

relative to the topology on $\overline{\Omega}$ whenever $s, t \in T$ with $s < t$. We now define our solution u by

$$u(x) = \sup\{t : x \in A_t\}. \tag{3.17}$$

Theorem 3.5. *The function u defined by* (3.17) *satisfies the following:*
(i) $u = g$ *on* $\partial\Omega$,
(ii) u *is continuous on* $\overline{\Omega}$,
(iii) $A_t \subset \{u \geq t\}$ *for all* $t \in T$ *and* $|\{u \geq t\} - A_t| = 0$ *for almost all* $t \in T$.

Proof. To show that $u = g$ on $\partial\Omega$, let $x_0 \in \partial\Omega$ and suppose $g(x_0) = t$. If $s < t$, then $G(x) > s$ for all $x \in \Omega^c$ near x_0. Hence, $x_0 \in (E_s)^i \cap \partial\Omega$ by (3.14) and consequently, $x_0 \in A_s$ for all $s \in T$ such that $s < t$. By (3.17), this implies $u(x) \geq t$. To show that $u(x) = t$ suppose by contradiction that $u(x) = \tau > t$. Select $r \in (t, \tau) \cap T$. Then $x \in A_r$. But $A_r \cap \partial\Omega \subset \{g \geq r\}$ by (3.15), a contradiction since $g(x) = t < r$.

For the proof of (ii), it is easy to verify that

$$\{u \geq t\} = \{\bigcap A_s : s \in T, s < t\} \text{ and } \{u > t\} = \{\bigcup A_s : s \in T, s > t\}.$$

The first set is obviously closed while the second is open relative to $\overline{\Omega}$ by (3.16). Hence, u is continuous on $\overline{\Omega}$.

For the proof of (iii), it is clear that $\{u \geq t\} \supset A_t$. Now, $\{u \geq t\} - A_t \subset u^{-1}(t)$. But $\left|u^{-1}(t)\right| = 0$ for almost all t because $|\Omega| < \infty$. □

Theorem 3.6. *If Ω is a bounded Lipschitz domain that satisfies* (3.1) *and* (3.2), *then the function u defined by* (3.17) *is a solution to* (1.1).

Proof. Let $v \in BV(\Omega)$, $v = g$ on $\partial\Omega$ be a competitor in problem (1.1). We recall the extension $G \in BV(\mathbb{R}^n - \overline{\Omega})$ of g, (2.17). Now define an extension $\overline{v} \in BV(\mathbb{R}^n)$ of v by $\overline{v} = G$ in $\mathbb{R}^n - \overline{\Omega}$. Let $F_t = \{\overline{v} \geq t\}$. It is sufficient to show that

$$P(E_t, \Omega) \leq P(F_t, \Omega) \tag{3.18}$$

for almost every $t \in T$ (see (2.19)), because then $v \in BV(\Omega)$ and (2.12) would imply

$$\int_a^b P(E_t, \Omega)\, dt \leq \int_{-\infty}^{\infty} P(F_t, \Omega)\, dt = \|\nabla v\|(\Omega) < \infty.$$

Hence, by (2.13), $u \in BV(\Omega)$; furthermore, $\|\nabla u\|(\Omega) \leq \|\nabla v\|(\Omega)$ by (2.12).

We know that E_t is a solution of

$$\min\{P(E, \mathbb{R}^n) : E - \overline{\Omega} = \mathcal{L}_t - \overline{\Omega}\},$$

while $F_t - \overline{\Omega} = \mathcal{L}_t - \overline{\Omega}$ almost everywhere. Hence,

$$P(E_t, \mathbb{R}^n) \leq P(F_t, \mathbb{R}^n). \tag{3.19}$$

Next, note that

$$\begin{aligned} P(E_t, \mathbb{R}^n) &= H^{n-1}(\partial^* E_t - \overline{\Omega}) \\ &\qquad + H^{n-1}(\partial^* E_t \cap \partial\Omega) + H^{n-1}(\partial^* E_t \cap \Omega) \\ &\geq H^{n-1}(\partial^* \mathcal{L}_t - \overline{\Omega}) + P(E_t, \Omega). \end{aligned} \tag{3.20}$$

We will now show that

$$\begin{aligned} P(F_t, \mathbb{R}^n) &= H^{n-1}(\partial^* \mathcal{L}_t - \overline{\Omega}) + H^{n-1}(\partial^* F_t \cap \Omega) \\ &= H^{n-1}(\partial^* \mathcal{L}_t - \overline{\Omega}) + P(F_t, \Omega), \end{aligned} \tag{3.21}$$

which will establish (3.18) in light of (3.19) and (3.20).

Observe

$$P(F_t, \mathbb{R}^n) = H^{n-1}(\partial^* \mathcal{L}_t - \overline{\Omega}) + H^{n-1}(\partial^* F_t \cap \partial\Omega) + H^{n-1}(\partial^* F_t \cap \Omega).$$

We claim that $H^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$ for almost all t because $\partial^* F_t \subset \partial F_t \subset \overline{v}^{-1}(t)$ since $\overline{v} \in C^0(\mathbb{R}^n)$. But $H^{n-1}(\overline{v}^{-1}(t) \cap \partial\Omega) = 0$ for all but countably many t since $H^{n-1}(\partial\Omega) < \infty$. □

In [BDG], the least gradient problem was posed in terms of minimizing the total variation among (possibly discontinuous) functions in $BV(\Omega)$. The following result shows that the function u defined by (3.17) is a solution to this problem as well.

Theorem 3.7. *If Ω is a bounded Lipschitz domain that satisfies* (3.1) *and* (3.2)*, then the function u defined by* (3.17) *is a solution to*

$$\inf \{\|\nabla v\|(\Omega) : v \in BV(\Omega), v = g \text{ on } \partial\Omega\}, \tag{3.22}$$

where $g : \partial\Omega \to \mathbb{R}^1$ is continuous. Here, $v = g$ on $\partial\Omega$ is understood in the sense of trace theory in BV.

Proof. Obviously, the infimum defined by (3.22) is no greater than that defined by (1.1). To show they are equal, the proof proceeds as in Theorem 3.6. Thus, we consider a competitor $v \in BV(\Omega)$ in problem (3.22) and let $\overline{v} \in BV(\mathbb{R}^n)$ be the extension as defined above. Note that since g is continuous on $\partial\Omega$, we have $\overline{v} \in BV(\mathbb{R}^n) \cap C^0(\mathbb{R}^n - \Omega)$. As in the proof of

Theorem 3.6, we need only establish (3.18) for almost all $t \in T$. For this, we argue as follows. Since g is the trace on $\partial\Omega$ of $\overline{v} \in BV(\Omega)$ in the sense of BV theory, we know (cf. [Z], § 5.14) for H^{n-1}-almost all $x \in \partial\Omega$

$$\lim_{r\to 0} \fint_{B(x,r)\cap\Omega} |\overline{v}(y) - g(x)|\, dy = 0. \tag{3.23}$$

Consider such an x that is also an element of $\partial^* F_t \cap \partial\Omega$. For such an x, observe that $g(x) = t$. Indeed, if $g(x) < t$, say $g(x) = t - \varepsilon$, then

$$\begin{aligned}
0 &= \lim_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \left(\int_{B(x,r)\cap\Omega\cap\{\overline{v}<t\}} |\overline{v}(y) - g(x)|\, dy \right. \\
&\qquad \left. + \int_{B(x,r)\cap\Omega\cap\{\overline{v}\ge t\}} |\overline{v}(y) - g(x)|\, dy \right) \\
&\ge \limsup_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \int_{B(x,r)\cap\Omega\cap\{\overline{v}\ge t\}} |\overline{v}(y) - g(x)|\, dy \\
&\ge \varepsilon \limsup_{r\to 0} \frac{|B(x,r)\cap\Omega\cap\{\overline{v}\ge t\}|}{|B(x,r)\cap\Omega|}.
\end{aligned}$$

Using also the fact that g is the trace of $\overline{v} \in BV(\mathbb{R}^n - \Omega)$, we can employ a similar argument to show that

$$0 = \limsup_{r\to 0} \frac{|B(x,r)\cap(\mathbb{R}^n - \Omega)\cap\{\overline{v}\ge t\}|}{|B(x,r)\cap(\mathbb{R}^n - \Omega)|}.$$

Hence, we conclude that

$$\lim_{r\to 0} \frac{|B(x,r)\cap\{\overline{v}\ge t\}|}{|B(x,r)|} = 0.$$

This implies that $x \notin \partial^* F_t$, a contradiction. Similarly, a contradiction is reached if $g(x) > t$. In view of (3.19), it follows that $H^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$. Thus,

$$P(F_t, \mathbb{R}^n) = H^{n-1}(\partial^* \mathcal{L}_{\mathrm{t}} - \overline{\Omega}) + P(F_t, \Omega)$$

and as in Theorem 3.6, this is sufficient to establish (3.18). □

We conclude this section with the observation that conditions (3.1) and (3.2) are necessary to ensure existence of solutions to (1.1) with arbitrary boundary data g. To support this claim, we state the following without proof.

Theorem 3.8. *Suppose Ω is a bounded Lipschitz domain which fails to satisfy* (3.2)*. Then there exists continuous boundary data g for which the problem* (1.1) *has no solution.*

Thus having demonstrated the existence of our solution, the questions of uniqueness and regularity become important. We quote the following results without proof.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Lipschitz domain satisfying* (3.1) *and* (3.2)*. If $u_1, u_2 \in C^0(\overline{\Omega}) \cap BV(\Omega)$ are solutions of* (1.1) *relative to their own boundary data, then*

$$\sup_{\Omega} |u_1 - u_2| = \sup_{\partial\Omega} |u_1 - u_2| \,.$$

In particular, the solution to (1.1) *is unique.*

Concerning regularity, note that in two dimensions, one can readily construct functions of least gradient by ensuring the level sets are straight line segments. Thus, functions of the form $u(x, y) = f(y/x)$ have least gradient. It is then easy to construct functions of least gradient having regularity properties in the interior no better than the regularity of the boundary data.

It can be shown that if the Dirichlet boundary data is of class $C^{0,\alpha}$, then the solution will be of class $C^{0,\alpha/2}$. In fact, a similar result can be obtained in terms of the modulus of continuity of the boundary data. Our result on regularity is as follows.

Theorem 3.10. *Suppose Ω is a bounded, open subset of $\mathbb{R}^n$ with C^2 boundary having strictly positive mean curvature. Suppose $g \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha \leq 1$ and $u \in C^0(\overline{\Omega}) \cap BV(\Omega)$ is a function of least gradient in Ω relative to its boundary data, g. Then $u \in C^{0,\alpha/2}(\overline{\Omega})$.*

4 Area minimizing sets subject to a volume constraint

The work in this and the next section is based on [StZ] and is concerned with the problem of minimizing area subject to a volume constraint in a given convex set. In precise terms we have the following. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set. Thus, $|\Omega| < \infty$ where $|\Omega|$ denotes Lebesgue measure. For a number $0 < v < |\Omega|$, let $E \subset \Omega$ denote a set with $|E| = v$ such that

$$P(E) \leq P(F)$$

for all sets $F \subset \Omega$ with $|F| = v$, where $P(E)$ denotes the perimeter of E. The main question we investigate is whether E is convex.

It should be emphasized that the perimeter of a competitor F is taken relative to $\mathbb{R}^n$, or what is the same, the perimeter is taken relative to the closure of Ω since F is assumed to be a subset of Ω. This problem is considerably different from minimizing perimeter relative to the interior of Ω. This was considered in [Gr] where it was shown that a minimizer is regular and intersects $\partial\Omega$ orthogonally.

The question of existence of a solution to our problem is resolved immediately in the context of sets of finite perimeter. Regularity questions have been considered by other authors. Tamanini [T] has shown that an area minimizing set E subject to a volume constraint has the property that $\partial E \cap \Omega$ is real analytic except for a closed set whose Hausdorff dimension does not exceed $n-8$. Also, under the assumption that $\partial\Omega \in C^1$, it was shown in [GMT2] that ∂E is an $(n-1)$-manifold of class C^1 in some neighborhood of each point in $\partial E \cap \partial\Omega$. In $\mathbb{R}^2$, and in $\mathbb{R}^n$, $n > 2$ under an additional condition on Ω, we are able to obtain regularity results and ultimately establish that a minimizer E is convex. Assuming only that Ω is bounded and convex, the convexity of E is an open question in $\mathbb{R}^n$, $n > 2$.

The additional condition we impose on Ω if $n > 2$ is the following.

> We assume that a largest closed ball, B_Ω, contained in Ω has a great circle that is a subset of $\partial\Omega$. A great circle of B_Ω is defined as the intersection of ∂B_Ω with a hyperplane, T_{B_Ω}, passing through the center of B_Ω. The equatorial "disk" is defined as $D_{B_\Omega} = T_{B_\Omega} \cap B_\Omega$. (4.1)

Also, assuming initially that $\partial\Omega \in C^2$ and strictly convex, we invoke a result of [BK] to conclude that $\partial E \in C^{1,1}$ at points near $\partial\Omega$. We then show, Theorem 5.9, that E is convex. Finally, through an approximation procedure, we show that E is convex with $C^{1,1}$ boundary assuming only that Ω satisfies a great circle condition. Clearly, there is no uniqueness if v is too small. However, with H_Ω denoting the union of all largest balls in Ω, if $|H_\Omega| \le v < |\Omega|$, then E is unique. In addition for such v we show that perimeter minimizers E are nested as a function of v. In general for nonconvex Ω one can expect neither uniqueness nor nestedness as indicated by examples in [GMT1].

Definition 4.1. Let M denote a k-dimensional C^1 submanifold of $\mathbb{R}^n$, $0 < k < n$, and let $f\colon M \to \mathbb{R}$ be an arbitrary function. We will say that f is differentiable at $x_0 \in M$ if f is the restriction to M of a function $\bar{f}\colon U \to \mathbb{R}$

where is $U \subset \mathbb{R}^n$ is some open set containing x_0 and where $\bar{f}$ is differentiable at x_0. We leave the proof of the following to the reader.

Lemma 4.2. *Let M be an $(n-1)$-dimensional C^1 submanifold of $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable at H^{n-1} almost all points of M.*

In view of the preceding Lemma, we can define the directional derivative of f relative to M at H^{n-1}-almost all $x \in M$ in the usual manner. Given a vector τ in the tangent space to M at x, let $\gamma\colon (-1,1) \to M$ be any C^1 curve with $\gamma(0) = x$ and $\gamma'(0) = \tau$. Define

$$D_\tau f(x) = (\bar{f} \circ \gamma)'(0)$$

where it is understood that $\bar{f}$ is differentiable at x. Observe that this definition is independent of the extension $\bar{f}$.

If we are given a Lipschitz vector field $X\colon M \to \mathbb{R}^n$, by using usual methods, it now becomes clear how to define the divergence of X relative to M, denoted by $\operatorname{div}_M X$.

If the closure $\overline{M}$ of M is a C^1-manifold with boundary $\partial M = \overline{M} - M$ and if $X\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field with the property that for each $x \in M$, $X(x)$ is an element of the tangent space to M at x, then the classical divergence theorem states

$$\int_M \operatorname{div}_M X \, dH^{n-1} = \int_{\partial M} X \cdot \eta \, dH^{n-2} \tag{4.2}$$

where η is the outward pointing unit co-normal of ∂M. That is, $|\eta| = 1$, η is normal to ∂M, and tangent to M.

Definition 4.3. Let M be an oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ of class $C^{1,1}$; that is, M is of class C^1 and its unit normal ν is Lipschitz. From Lemma 4.2, we have that the components of ν are differentiable at H^{n-1}-almost all points of M. Thus, $\operatorname{div}_M \nu$ is defined H^{n-1}-almost everywhere on M. At such points, we define the *mean curvature* of M at x as

$$\mathcal{H}_M(x) = \operatorname{div}_M \nu(x)$$

If $X\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, consider its decomposition into its tangent and normal parts relative to M,

$$X = X^\top + X^\perp$$

where

$$X^{\perp} = (X \cdot \nu)\nu.$$

Then, at H^{n-1}-almost all points in M, it follows that

$$\operatorname{div}_M X^{\perp} = (X \cdot \nu) \operatorname{div}_M \nu.$$

Hence,

$$\operatorname{div}_M X^{\perp} = \mathcal{H}_M X \cdot \nu.$$

On the other hand, from (4.2) we have

$$\int_M \operatorname{div}_M X^{\top} \, dH^{n-1} = \int_{\partial M} X \cdot \eta \, dH^{n-2}.$$

Since $\operatorname{div}_M X = \operatorname{div}_M X^{\top} + \operatorname{div}_M X^{\perp}$, we obtain

$$\int_M \operatorname{div}_M X \, dH^{n-1} = \int_M \mathcal{H}_M X \cdot \nu \, dH^{n-1} + \int_{\partial M} X \cdot \eta \, dH^{n-2}. \tag{4.3}$$

5 Main results

In this section we consider the following situation.

$$\left.\begin{array}{l}\text{Let } \Omega \text{ be a bounded, convex domain in } \mathbb{R}^n, n \geq 2. \text{ Let } E \subset \overline{\Omega} \\ \text{denote a set which minimizes perimeter in the closure of } \Omega \\ \text{subject to a volume constraint } |E| = v < |\Omega|. \text{ Thus} \\ \qquad\qquad P(E, \mathbb{R}^n) \leq P(F, \mathbb{R}^n) \\ \text{for all sets } F \subset \overline{\Omega} \text{ with } |F| = v.\end{array}\right\} \tag{5.1}$$

We will first establish boundary regularity and curvature properties for such perimeter minimizers under the assumption that Ω is *strictly convex and that* $\partial\Omega \in C^2$. Convexity, nestedness and uniqueness results will then be established under the further assumption that

$$n = 2 \quad \text{or} \quad \Omega \text{ satisfies a great circle condition.}$$

The assumption of strict convexity and C^2 regularity will then be dispensed with in part through an approximation argument.

Associated with (5.1) is some further notation. We let H denote the convex hull of a minimizer E of (5.1), and we denote by H^+ that part of H that lies "above" the equatorial disk D_{B_Ω} of B_Ω as defined in (4.1). Since P divides H into two parts, we arbitrarily call one of them the part that lies "above" P.

Next, we recall some facts concerning area minimizing sets with a volume constraint. The main result of [GMT1] is that if E is area minimizing with a volume constraint, then

$$\psi(x, r) \leq Cr^n \tag{5.2}$$

for each $x \in \partial E$ and for all sufficiently small $r > 0$. Consequently, it follows from work of Tamanini [T] that an area minimizing set E with a volume constraint possesses an area minimizing tangent cone at each point of $(\partial E) \cap \Omega$. From this it follows that $(\partial E) \cap \Omega$ enjoys the same regularity properties as an area minimizing set; that is, $(\partial E) \cap \Omega$ is real analytic except for a closed singular set S whose Hausdorff dimension does not exceed $n - 8$. Furthermore, it was established in [GMT2], Theorem 3, that ∂E is an $(n-1)$-manifold of class C^1 in some neighborhood of each point $x \in \partial E \cap \partial \Omega$.

The object of this section is to prove that E is convex and we begin by proving $C^{1,1,}$ regularity of ∂E near $\partial \Omega$. For this we will need the following result of Brézis and Kinderlehrer, [BK].

Theorem 5.1. *Let $a\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a C^2 vector field satisfying the condition that for each compact $C \subset \mathbb{R}^{n-1}$, there exists a constant $\nu = \nu(C) > 0$ such that*

$$(a(p) - a(q)) \cdot (p - q) \geq \nu \, |p - q|^2$$

for all $p, q \in C$. Let $U \subset \mathbb{R}^{n-1}$ be an open connected set and let $\beta \in C^2(U)$ satisfy $\beta \leq 0$ on ∂U. Let $f \in C^1(U)$. With $\mathbf{K} = \mathbf{K}_\beta$ denoting the convex set of Lipschitz functions v satisfying $v \geq \beta$ in U and $v = 0$ on ∂U, let $u \in \mathbf{K}$ be a solution of

$$\int_U a(\nabla u) \cdot \nabla (v - u) \, dx \geq \int_U f(v - u) \, dx$$

for all $v \in \mathbf{K}$. Then $u \in C^{1,1}(V)$ on any domain V with $\overline{V} \subset U$.

We now apply this result to obtain $C^{1,1}$ regularity of the boundary of a minimizer E of the variational problem (5.1) near $\partial \Omega$. Since ∂E is an $(n-1)$-manifold of class C^1 in some neighborhood of each point $x \in \partial E \cap \partial \Omega$, it follows that near such a point x, we may represent both ∂E and $\partial\Omega$ as graphs of functions u and β, respectively, defined on an open set $U' \in \mathbb{R}^{n-1}$ containing x' where $x = (x', y'')$, $y'' \in \mathbb{R}$. We will assume u and β chosen in such a way that $u \geq \beta$, $u = 0$ on $\partial U'$ and $\beta \leq 0$ on $\partial U'$. Using the convexity of Ω, this can be accomplished by considering a hyperplane P_0 passing through E and parallel to the tangent plane to ∂E at x. By taking P_0 sufficiently close to the tangent plane, U' can be defined as $P_0 \cap E$. Now select $v \in \mathbf{K}$ and for $0 < \varepsilon < 1$, define u_ε on U' as $u_\varepsilon = u + \varepsilon(v - u)$. We will assume ε chosen small enough so that the graph of u_ε remains in $\overline{\Omega}$. Note that $u_\varepsilon \in \mathbf{K}$. Select a point $z \in (\partial E) \cap \Omega$ at which ∂E is regular. Thus, ∂E is real analytic near z and its mean curvature is a constant K there. In a neighborhood of z, we can represent ∂E as the graph of a function w defined on some open set $V' \subset \mathbb{R}^{n-1}$ containing z' where $z = (z', z'')$. The neighborhoods about x and z where ∂E is represented as a graph are taken to be disjoint. Let $\varphi \in C_0^\infty(V')$ denote a function with the property that

$$\int_{V'} \varphi \, dH^{n-1} = \int_{U'} (v - u) \, dH^{n-1}, \tag{5.3}$$

and define $w_\varepsilon = w - \varepsilon\varphi$. The graphs of the functions u_ε and w_ε produce a perturbation of the set E, say E_ε. Because of (5.3), we have that $|E| = |E_\varepsilon|$. With

$$F(\varepsilon) = \int_{U'} \sqrt{1 + |\nabla u_\varepsilon|^2} + \int_{V'} \sqrt{1 + |\nabla w_\varepsilon|^2},$$

the minimizing property of ∂E implies that $F(0) \leq F(\varepsilon)$ for all small ε and therefore that $F'(0) \geq 0$. Thus,

$$\int_{U'} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla(v - u) - \int_{V'} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \cdot \nabla\varphi \geq 0.$$

Since w has constant mean curvature K, we obtain

$$\int_{V'} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \cdot \nabla\varphi = -\int_{V'} K\varphi = -K \int_{V'} \varphi = -K \int_{U'} (v - u),$$

and therefore

$$\int_{U'} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla(v - u) \geq -K \int_{U'} (v - u). \tag{5.4}$$

□

If $\eta \in C_0^\infty(U')$ denotes an arbitrary non-negative test function, then with $v - u = \eta$, (5.4) states that u is a weak solution of $\mathcal{H}_{\partial E} \leq K$. This combined with the $C^{1,1}$- regularity of u implies that $\mathcal{H}_{\partial E} \leq K$ pointwise almost everywhere in a neighborhood of $\partial\Omega$. Since $\mathcal{H}_{\partial E} = K$ in $\partial E \cap (\Omega \setminus S)$ with $H^{n-1}(S) = 0$ we have the following result.

Theorem 5.2. *Assume that Ω is bounded, convex and has a C^2 boundary. If E is a minimizer of* (5.1)*, then $\partial E \in C^{1,1}$ in some neighborhood of $\partial\Omega$ and $\mathcal{H}_{\partial E} \leq K$ H^{n-1}-almost everywhere on ∂E.*

We now will exploit Theorem 5.2 to establish both regularity and a mean curvature estimate for the boundary of the convex hull of E.

Theorem 5.3. *Assume that Ω is bounded, strictly convex and has a C^2 boundary. If E is a minimizer of* (5.1) *with convex hull H then $\partial H \in C^{1,1}$ and $\mathcal{H}_{\partial H} \leq K$ H^{n-1}-almost everywhere on ∂H.*

Proof. Note that the singular set S in ∂E is a closed subset of Ω and thus separated from $\partial\Omega$, in fact it is contained in the interior of H, for if $x \in \partial E \cap \partial H \cap \Omega$, then the tangent cone to ∂E at x must be a hyperplane because $E \subset H$ and H is convex. Consequently ∂E is regular at x. Let N be an open neighborhood of S with compact closure in the interior of H. Thus by Theorem 5.2 and the analyticity of ∂E in $\Omega \setminus S$ we see that ∂E is $C^{1,1}$ at points in $G := \partial E \setminus N$. Therefore for some C we have

$$|\nu(x) - \nu(z)| \leq C|x - z|, \qquad x, z \in G \tag{5.5}$$

where $\nu(x)$ is the outward unit normal to ∂E at x. Also since ∂E is C^1 at points in G there exists an ε such that for all $x \in G$ and $z \in \partial E \cap B(x, \varepsilon)$ we have

$$|\nu(x) \cdot (x - z)| \leq \frac{1}{2}|x - z|. \tag{5.6}$$

Choose $x \in \partial E \cap \partial H \subset G$ and let $0 < \alpha < 1/2$. Then define

$$d = \alpha \min\{\varepsilon, \operatorname{dist}(\partial H, N), (2C)^{-1}, \operatorname{diam} E\}.$$

Let $y = x - d\nu(x)$ and observe that y is in the interior of E since ∂E cannot intersect the line segment $\overline{xy}$ at a point $z \neq x$ due to (5.6). Let $r = \operatorname{dist}(y, \partial E)$ and note that $0 < r \leq d$. Now choose any $z \in \partial E$ such that $|y - z| = r$. Note that $z \in G$, for otherwise we would have $z \in N$ and since $|x - z| \leq |x - y| + |y - z|$, it would follow that

$$2d \geq |x - z| \geq \operatorname{dist}(\partial H, N) \geq \frac{d}{\alpha} > 2d,$$

a contradiction. Then, $|x-z| \le |x-y| + |y-z| \le 2d < \varepsilon$ and both (5.5) and (5.6) hold. Thus, since $x = y + d\nu(x)$ and $z = y + r\nu(z)$, we have $|d-r| \le |\nu(x)\cdot(x-z)|$ and

$$|x-z| = |(d-r)\nu(x) + r(\nu(x)-\nu(z))| \le (1/2 + Cr)|x-z| \le 3/4|x-z|,$$

(since $r \le d \le \alpha/(2c) \le 1/(4c)$) which implies that $x = z$ and therefore $r = d$. This implies that for every $x \in \partial E \cap \partial H$ there exists a ball $B_x \subset E$ of radius d containing x.

Given any $p \in \partial H$ we claim that p is a convex combination of points $\{x_i\}$ in $\partial E \cap \partial H$. To see this note that if C is a convex set with $E \subset C$ then $\overline{E} \subset C$ since if $x \in \overline{E}$ then either $x \in C$ or $x \in \partial C$; in the later case x lies in a support plane of C so if $x \in \Omega$, regularity theory implies that $x \in E \subset C$, and if $x \in \partial\Omega$ then x is not in the singular set S of E (since S is a compact subset of Ω) so again $x \in E \subset C$. Consequently from the definition of convex hull H of E as the intersection of all convex sets containing E, we see that $\overline{E} \subset H$. Moreover H is the convex hull of $\overline{E}$ from which we conclude by a well known result that H is closed since $\overline{E}$ is a compact subset of $\mathbb{R}^n$. Note that the set of finite convex combinations of points from E is convex, contains E, and is contained in any convex set which contains E and so equals H. Thus if $p \in \partial H$ we have $p \in H$, since H is closed, and consequently $p = \sum_{i=1}^k \lambda_i x_i$ for $x_i \in E$ and $\sum_{i=1}^k \lambda_i = 1$, $\lambda_i \ge 0, i = 1, \ldots, k$. If we take k to be as small as possible then either $k = 1$ and $p \in E$ and the claim is trivially true, or p lies in the k dimensional interior of the convex hull M of $\{x_i\}$ in which case no x_i can lie in the interior of H since then the same would be true of p. Consequently $x_i \in \partial E \cap \partial H$, $i = 1, \ldots, k$, as claimed.

Taking the convex hull of $\bigcup_{i=1}^k B_{x_i}$ we see that there exists a ball $B_p \subset H$ of radius d containing p, i.e. H satisfies a uniform interior sphere condition. We claim that this implies ∂H is $C^{1,1}$. To see this, consider the problem of prescribing unit vectors $\nu_1, \nu_2 \in \mathbb{R}^n$, and finding a convex set $\widetilde{H}$, satisfying the interior sphere condition noted above, and points $x, y \in \partial\widetilde{H}$ with $\nu(x) = \nu_1$, $\nu(y) = \nu_2$, such that $|x-y|$ is minimized. It is clear that x, y must lie in a two dimensional plane orthogonal to the intersection of two hyperplanes having ν_1, ν_2 as normals, i.e. one needs only consider the two dimensional case where it is easy to see that one must have $B_x = B_y$. Taking the center of this ball to be the origin then $\nu(x) = x/d, \nu(y) = y/d$ and we trivially have

$$|\nu(x) - \nu(y)| \le \frac{1}{d}|x-y|.$$

Since this is the case when $|x-y|$ is smallest for fixed $\nu(x), \nu(y)$ we have established that $\nu(x)$ is Lipschitz in general.

We now prove that $\mathcal{H}_{\partial H} \le K$ H^{n-1}-almost everywhere in ∂H. Note that $\mathcal{H}_{\partial H} = \mathcal{H}_{\partial E}$ H^{n-1}-almost everywhere on $\partial E \cap \partial H$ by Theorem 5.2. Thus we need only consider points $p \in \partial H \setminus \partial E$. In fact since ∂H is $C^{1,1}$ we need only consider $p \in \partial H \setminus \partial E$ at which ∂H is classically twice differentiable. As above, any such p lies in the k dimensional interior of the convex hull M of certain points $p_i \in \partial E$, $i = 1, \ldots, k$. Note that $k \neq 1$ due to $p \notin \partial E$. Choose a coordinate system such that points in $\mathbb{R}^n$ are represented as (x, y, z), $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k-1}$, $z \in \mathbb{R}$, with $z = 0$ the tangent plane to ∂H at p, $p_i = (x_i, 0, 0)$, $i = 1, \ldots, k$, and $z \ge 0$ in H. We will construct an analytic function g whose graph does not lie below ∂H, contains M, and has mean curvature bounded above by $K + \varepsilon$ (for any $\varepsilon > 0$) in a small neighborhood of p. This will lead to the conclusion that $\mathcal{H}_{\partial H} \le K$ at p.

Let ∂E be represented as $z = f(x, y)$ for f defined in a neighborhood in $\mathbb{R}^k \times \mathbb{R}^{n-k-1}$ of $\cup(x_i, 0)$. Thus

$$(x_i, y, f(x_i, y)) \in \partial E \subset H$$

for small $|y|$, and consequently

$$\sum_{i=1}^{k} \lambda_i (x_i, y, f(x_i, y)) \in H \quad \text{if} \quad \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \ge 0 \tag{5.7}$$

for small $|y|$. For any given x in N, where N is the convex hull of the points x_i, $i = 1, \ldots, k$, let $\lambda = \lambda(x) = (\lambda_1(x), \ldots, \lambda_k(x))$ be the unique vector such that

$$x = \sum_{i=1}^{k} \lambda_i(x) x_i, \quad \sum_{i=1}^{k} \lambda_i(x) = 1, \quad \lambda_i(x) \ge 0.$$

Thus if we define

$$g(x, y) = \sum_{i=1}^{k} \lambda_i(x) f(x_i, y)$$

we see from (5.7) for $x \in N$ and small $|y|$ that

$$(x, y, g(x, y)) \in H,$$

and so the surface $z = g(x, y)$ does not lie below ∂H at such (x, y).

Note that $M \cap \partial\Omega = \emptyset$, for otherwise the plane $z = 0$, which contains M, would be a tangent plane to $\partial\Omega$, thus contradicting the strict convexity of $\partial\Omega$. Also M does not intersect the singular set of ∂E since $M \subset \partial H$. Thus ∂E is analytic at each p_i and therefore both $f(x_i, y)$ and $g(x, y)$ are smooth for small $|y|$. Furthermore,

$$0 \leq \Delta_y f(x_i, 0) \leq \Delta f(x_i, 0) \leq K$$

since $\nabla f(x_i, 0) = 0$, $\mathcal{H}_{\partial E}$ equals Δf at points where the gradient is zero, and the second derivatives of f are non-negative at $(x_i, 0)$ due to the fact that $f \geq 0$, $f(x_i, 0) = 0$ for all i. Hence, for any $\varepsilon > 0$, $\Delta_y f(x_i, y) \leq (K + \varepsilon)$ for small enough $|y|$, so $\Delta_y g(x, y) \leq (K + \varepsilon)$ as well. However $\Delta_x g = 0$ and so $\Delta g \leq (K + \varepsilon)$ for small $|y|$. Recall that ∂H is trapped between $\{z = 0\}$ and the graph of g over a region which contains p in its interior. Since $g(p) = 0$ and ∂H is twice differentiable at p we conclude that $\mathcal{H}_{\partial H}(p) \leq K$ as required. □

Theorem 5.4. *Assume that Ω is bounded, strictly convex and satisfies a great circle condition. If E is a minimizer of (5.1) with $|B_\Omega| \leq |E|$ then*

$$B_\Omega \subset E$$

where B_Ω is the largest ball in Ω.

Proof. If $|E| = |B_\Omega|$ then clearly E must be a ball. Since there is only one largest ball in Ω due to strict convexity, we have $E = B_\Omega$. Otherwise $|B_\Omega| < |E|$. In this case translate the upper and lower hemispheres of B_Ω by a distance d in opposite directions orthogonal to T_{B_Ω} until H, the convex hull of the two translated hemispheres, intersects E in a set of measure $|B_\Omega|$, i.e.

$$|H \cap E| = |B_\Omega|. \tag{5.8}$$

This is possible because of the great circle condition and because Ω is bounded and convex. Now translate the hemispheres back to their original positions while rigidly carrying along the parts of E lying in the exterior of H. Let $\widetilde{E}$ be the union of the translated parts of E with B_Ω. Note that

$$|\widetilde{E}| = |E| \quad \text{and therefore} \quad P(\widetilde{E}) \geq P(E). \tag{5.9}$$

Using a standard inequality, cf. [MM], we have

$$P(E) + P(H) \geq P(E \cap H) + P(E \cup H)$$

where $P(S)$ denotes $P(S, \mathbb{R}^n)$. For brevity, write $D = D_{B_\Omega}$. Observe that

$$P(H) = 2dH^{n-2}(\partial D) + P(B_\Omega), \quad P(E \cup H) = P(\widetilde{E}) + 2dH^{n-2}(\partial D)$$

and thus

$$P(E) + P(B_\Omega) \geq P(E \cap H) + P(\widetilde{E}).$$

In view of (5.9) it follows that $P(E \cap H) \leq P(B_\Omega)$. But then the isoperimetric inequality and (5.8) imply that $E \cap H$ is a ball. However Ω contains only one largest ball and so we must have $E \cap H = B_\Omega$, i.e. $B_\Omega \subset E$. □

Suppose M is an oriented $(n-1)$-dimensional C^1-submanifold of $\mathbb{R}^n$ and $f\colon M \to \mathbb{R}^{n-1}$ a C^1 mapping. Let $Jf(x)$ denote the Jacobian of f at x and note that the sign of the Jacobian depends on the orientation of M. We recall the following result, cf. [F2], Theorem 3.2.20: For any H^{n-1}-measurable set $E \subset M$ and any H^{n-1}-measurable function φ,

$$\int_E \varphi[f(x)]\,|Jf(x)|\; dH^{n-1}(x) = \int \varphi(y) N(f, E, y)\, dy \tag{5.10}$$

where $N(f, E, y)$ denotes the number (possibly infinite) of points in $f^{-1}(y) \cap E$. Here equality is understood in the sense that if one side is finite, then so is the other. In our application (5.11) below, we will know the left side is finite, therefore ensuring that $N(f, E, y)$ is finite for almost all y.

Lemma 5.5. *There is a constant $C = C(n)$ such that for each $x \in (\partial E) \cap \Omega$ we have*

$$\frac{H^{n-1}((\partial E) \cap B(x,r))}{r^{n-1}} \leq C$$

for almost all sufficiently small $r > 0$.

Proof. It follows from (5.2) that we may as well assume ∂E is area minimizing. In this case the result follows immediately from the fact that

$$\frac{H^{n-1}((\partial E) \cap B(x,r))}{r^{n-1}}$$

is non-decreasing in r, for $r > 0$ sufficiently small, cf. [F2], Theorem 3.4.3. □

Lemma 5.6. *For every $\varepsilon > 0$ and any open set $V \subset \mathbb{R}^n$ containing the singular set S of ∂E, there exists an open set W and a Lipschitz function f such that*

$$S \subset W \subset \{f = 1\},$$
$$\operatorname{spt} f \subset V,$$
$$\int_{\partial E} |\nabla f| \, dH^{n-1} \leq \varepsilon.$$

Proof. Let V be any open set containing S and let $\delta = 1/2(\text{dist } S, \mathbb{R}^n - V)$. Since $H^{n-7}(S) = 0$ and S is compact, there is a finite collection of open balls $\{B(x_i, r_i)\}_{i=1}^m$ such that $2r_i < \delta, B(x_i, r_i) \cap S \neq \emptyset, S \subset \bigcup_{i=1}^m B(x_i, r_i)$ and

$$\sum_{i=1}^m r_i^{n-7} < \frac{\varepsilon}{C},$$

C as in Lemma 5.5. We will assume that each ball $B(x_i, r_i)$ has been chosen so that $r_i < 1$ and that $2r_i$ satisfies Lemma 5.5. Let W denote the union of these balls and define f_i by

$$f_i(x) = \begin{cases} 1 & \text{if } \ |x - x_i| \leq r_i \\ 2 - \dfrac{|x - x_i|}{r_i} & \text{if } \ r_i \leq |x - x_i| \leq 2r_i \\ 0 & \text{if } \ 2r_i \leq |x - x_i| \,. \end{cases}$$

In view of Lemma 5.5, it follows that

$$\int_{B(x_i, r_i) \cap \partial E} |\nabla f_i| \, dH^{n-1} \leq C r_i^{n-2} < C r_i^{n-7}.$$

Now let $f := \max_{1 \leq i \leq m} f_i$. Then f is Lipschitz, $W \subset \{f = 1\}$, $\operatorname{spt} f \subset V$ and

$$\begin{aligned} \int_{\partial E} |\nabla f| \, dH^{n-1} &\leq \sum_{i=1}^m \int_{B(x_i, r_i) \cap \partial E} |\nabla f_i| \, dH^{n-1} \\ &< C \sum_{i=1}^m r_i^{n-7} < \varepsilon. \end{aligned}$$

□

Lemma 5.7. *Let T denote the $(n-1)$-rectifiable current determined by $(\partial E)^+$, the part of ∂E that lies above the equatorial disk $D := D_{B_\Omega}$ of B_Ω. Then ∂T is the $(n-2)$-sphere given by $\partial T = \partial D$.*

Proof. Clearly, the support of ∂T contains the $(n-2)$-sphere, but we must rule out the possibility of it containing points of S as well. For this purpose, choose $x \in S$ and let φ be any smooth differential form supported in some neighborhood of x that does not meet $(\partial E)^+ \cap \partial D$. It suffices to show that $T(d\varphi) = 0$. Let μ denote H^{n-1} restricted to $(\partial E)^+$. Appealing to Lemma 5.6, we can produce a sequence of Lipschitz functions $\{\omega_i\}$ such that

$$\begin{gathered}\omega_i \to 1 \ \mu \text{ a.e.} \\ |\nabla \omega_i| \to 0 \ \mu \text{ a.e.} \\ \omega_i \text{ vanishes in a neighborhood of } S \\ \int_{(\partial E)^+} |\nabla \omega_i| \ d\mu \to 0.\end{gathered}$$

Thus, we obtain

$$\begin{aligned}0 = T(d(\varphi\omega_i)) &= T(d\varphi \wedge \omega_i) + T(\varphi \wedge d\omega_i) \\ &= \int_{(\partial E)^+} d\varphi \wedge \omega_i + \int_{(\partial E)^+} \varphi \wedge d\omega_i.\end{aligned}$$

The first integral tends to

$$\int_{(\partial E)^+} d\varphi = T(d\varphi)$$

while the second tends to 0. Thus, $T(d\varphi) = 0$. □

Let E denote a minimizer of (5.1), where Ω is strictly convex with C^2 boundary. Since ∂E is locally an $(n-1)$-manifold of class C^1 except for a singular set S whose Hausdorff dimension does not exceed $n-8$, it follows that ∂E can be regarded as an oriented $n-1$ integral current whose boundary is 0; i.e. an oriented $n-1$ integral cycle.

Let T denote the $n-1$ integral current represented by $(\partial E) \cap H^+$. Since ∂E is of class $C^{1,1}$ in a neighborhood of each point of $(\partial E) \cap (\partial \Omega)$, it follows that the tangent cone to ∂E at such points is in fact a tangent plane. Consequently, ∂E is analytic near such points and therefore the singular set S of ∂E lies in the interior of $(\partial E) \cap H^+$. We know from Lemma 5.7 that the boundary of T is the $(n-2)$-sphere determined by ∂D_{B_Ω}, the equator of B_Ω. Let $p\colon \mathbb{R}^n \to T_{B_\Omega}$ denote the orthogonal projection and consider the current $R := p_\#(T)$. Note that $\partial R = p_\#(\partial T) = \partial D_{B_\Omega}$. Furthermore, D_{B_Ω} is the unique current in T_{B_Ω} whose boundary is ∂D_{B_Ω} and therefore, we conclude that $R = D_{B_\Omega}$. Let us consider the action of R operating on an $(n-1)$-form φ. For this we will let $\alpha(x)$ denote the Grassman $(n-1)$-vector of norm one that is in the tangent plane orthogonal to $\nu(E,x)$, the exterior normal to E at x. The vector $\alpha(x)$ is chosen in such a way that $\alpha(x) \wedge \nu(E,x)$ forms the Grassman unit n-vector that induces a positive orientation of $\mathbb{R}^n$. Also, we let $dp(\alpha(x))$ denote the value of the differential of p operating on $\alpha(x)$. Then, with the help of (5.10), we have

$$\begin{aligned}
R(\varphi) &= T(p^\#\varphi) \\
&= \int_{(\partial E)\cap H^+} p^\#\varphi \cdot \alpha \\
&= \int_{(\partial E)\cap H^+} \varphi[p(x)] \cdot dp(\alpha(x))\, dH^{n-1}(x) \\
&= \int_{D_{B_\Omega}} \varphi(y)[N^+(p,\partial E,y) - N^-(p,\partial E,y)]\, dy
\end{aligned}$$

where $N^+(p,\partial E,y)$ denotes the number of points of $p^{-1}(y) \cap \partial E$ at which Jp is positive and similarly, $N^-(p,\partial E,y)$ denotes the number of points of $p^{-1}(y) \cap \partial E$ at which Jp is negative. Since $R = D_{B_\Omega}$, we conclude that

$$N^+(p,\partial E,y) - N^-(p,\partial E,y) = 1 \tag{5.11}$$

for almost all $y \in D_{B_\Omega}$. □

Lemma 5.8. *Assume that Ω is bounded, strictly convex, has a C^2 boundary, and satisfies a great circle condition. Let H denote the convex hull for any minimizer E of the variational problem* (5.1). *Then there is a constant K such that $\mathcal{H}_{\partial H} = K$ at H^{n-1}-almost all points of $(\partial H) \cap \Omega$.*

Proof. First, we recall that $\partial E \cap \overline{\Omega}$ is C^1 at all of its points except for a singular set $S \subset \partial E \cap \Omega$ whose Hausdorff dimension does not exceed $n-8$. Furthermore, we know that $\partial E \cap \Omega$ is real analytic at all points away from S and that ∂H is $C^{1,1}$. Finally, we know that E contains B_Ω. Let $(\partial E)^+$ and $(\partial H)^+$ denote the parts of ∂E and ∂H respectively that lie above the equatorial plane P of B_Ω. Let $p\colon \mathbb{R}^n \to P$ denote the orthogonal projection. The mean curvature of ∂E is equal to a constant K at all points of $\partial E \cap (\Omega - S)$. Let X denote the vertical unit vector. We wish to apply (4.3) with $(\partial E)^+$ replacing M. Referring to the proof of Lemma 5.7, we see that this can be done in spite of the singular set $S \in (\partial E)^+$. Thus, applying (4.3), we obtain

$$\int_{(\partial H)^+} \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = \int_{(\partial E)^+} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \tag{5.12}$$

where ν_H and ν_E denote the unit exterior normals to H and E respectively. Let

$$\begin{aligned} A &= (\partial E)^+ \cap (\partial H)^+, \\ B &= ((\partial H)^+ - A) \cap \{x : \mathcal{H}_{\partial H}(x) < K\}, \\ C &= ((\partial H)^+ - A) \cap \{x : \mathcal{H}_{\partial H}(x) = K\}. \end{aligned}$$

Since $\mathcal{H}_{\partial H} \leq K \; H^{n-1}$-a.e. in $(\partial H)^+ \cap \Omega$, it suffices to prove that

$$H^{n-1}(B) = 0. \tag{5.13}$$

Observe that both B and C are subsets of ∂H^+. Note also that A, B, and C are mutually disjoint subsets of $(\partial H)^+$ with $H^{n-1}[(\partial H)^+ - (A \cup B \cup C)] = 0$. Thus, $p(A), p(B)$ and $p(C)$ are mutually disjoint and their union occupies almost all of D_{B_Ω}. Clearly, ν_E and ν_H as well as $\mathcal{H}_{\partial H}$ and $\mathcal{H}_{\partial E}$ agree H^{n-1}-almost everywhere on A. Therefore,

$$\int_A \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = \int_A \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \,. \tag{5.14}$$

Since $X \cdot \nu_H$ is the Jacobian of the mapping $p \colon \partial H^+ \to D_{B_\Omega}$, it follows from (5.10) that

$$\begin{aligned} &\int_B \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} < K H^{n-1}[p(B)], \\ &\int_C \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = K H^{n-1}[p(C)]. \end{aligned}$$

Now let

$$\begin{aligned} A^* &= ((\partial E)^+) \cap p^{-1}[p(A)], \\ B^* &= ((\partial E)^+) \cap p^{-1}[p(B)], \\ C^* &= ((\partial E)^+) \cap p^{-1}[p(C)]. \end{aligned}$$

Next, observe that both B^* and C^* are subsets of Ω. To see this, consider $x \in B^*$. If it were true that $x \in B^* \cap \partial\Omega$, then $x \in (\partial H)^+$ and thus $x \in A$.

This is impossible since $p(A)$ and $p(B)$ are disjoint. A similar argument holds for C^*. Referring to (5.10) and (5.11), we obtain

$$\begin{aligned}
&\int_{B^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&\quad = K \int_{B^* \cap \{x \colon X \cdot \nu_E(x) > 0\}} X \cdot \nu_E \, dH^{n-1} \\
&\qquad + K \int_{B^* \cap \{x \colon X \cdot \nu_E(x) < 0\}} X \cdot \nu_E \, dH^{n-1} \\
&\quad = K \int_{p(B^*)} N^+(p, \partial E, y) - N^-(p, \partial E, y) \, dH^{n-1}(y) \\
&\quad = K H^{n-1}[p(B^*)] \\
&\quad = K H^{n-1}[p(B)].
\end{aligned}$$

Similarly,

$$\int_{C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} = K H^{n-1}[p(C^*)] = K H^{n-1}[p(C)]$$

and

$$\int_{A^*} K X \cdot \nu_E \, dH^{n-1} = K H^{n-1}(p(A)).$$

Finally, because $A \subset (\partial H)^+$ and consequently $N^+(p, A, y) = 1$ and $N^-(p, A, y) = 0$ for H^{n-1}-almost all $y \in p(A)$, we obtain

$$\int_{A} K X \cdot \nu_E \, dH^{n-1} = K H^{n-1}(p(A)).$$

Now, using the facts that $A^* - A \subset \Omega$ and $\mathcal{H}_{\partial E} = K$ on $A^* - A - S$, we obtain

$$\begin{aligned}
&\int_{A^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&\quad = \int_{A^*} K X \cdot \nu_E \; dH^{n-1} + \int_{A^*} (\mathcal{H}_{\partial E} - K) X \cdot \nu_E \, dH^{n-1} \\
&\quad = \int_{A^*} K X \cdot \nu_E \, dH^{n-1} + \int_{A} (\mathcal{H}_{\partial E} - K) X \cdot \nu_E \, dH^{n-1} \\
&\quad = K H^{n-1}(p(A)) - K H^{n-1}(p(A)) + \int_{A} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&\quad = \int_{A} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1}.
\end{aligned}$$

Under the assumption $H^{n-1}(B) > 0$, we would obtain

$$\begin{aligned}
\int_{(\partial H)^+} & \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} \\
&< \int_A \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} + KH^{n-1}[p(B)] + KH^{n-1}[p(C)] \\
&= \int_A \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + KH^{n-1}[p(B^*)] + KH^{n-1}[p(C^*)] \\
&= \int_{A^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + KH^{n-1}[p(B^*)] + KH^{n-1}[p(C^*)] \\
&= \int_{A^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + \int_{B^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&\quad + \int_{C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&= \int_{A^* \cup B^* \cup C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\
&\leq \int_{(\partial E)^+} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1},
\end{aligned}$$

where we have used that A^*, B^* and C^* are mutually disjoint. This would contradict (5.12), thus establishing (5.13). □

A function $u \in C^1(W)$ is called a *weak subsolution* (*supersolution*) *of the equation of constant* K *mean curvature* if

$$Mu(\varphi) = \int_W \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} - K\varphi \, dx \leq 0 \quad (\geq 0)$$

whenever $\varphi \in C_0^1(W)$, $\varphi \geq 0$.

We note that if $u \in C^{1,1}$ and classically satisfies the equation of constant mean curvature equation almost everywhere, then u is a weak solution.

The following result will be stated in the context of $\mathbb{R}^{n-1}$ because of its applications in the subsequent development.

Theorem 5.9. *Suppose Ω is a bounded, strictly convex domain with C^2 boundary that satisfies a great circle condition. Then any minimizer E of the variational problem* (5.1) *is convex.*

With the results above, by means of an approximation procedure, it can be shown that E is convex with $C^{1,1}$ boundary assuming only that Ω satisfies a great circle condition. The proof of this can be found in [StZ].

6 The inner trace of Sobolev functions

In this and the next two sections, I will discuss results that are based on [SwZ].

If $\Omega \subset \mathbb{R}^n$ is an open set, $W^{k,p}(\Omega)$, $p \geq 1$, will denote the Sobolev space of functions $f \in L^p(\Omega)$ whose distributional derivatives of order up to and including k are also elements of $L^p(\Omega)$. The norm on $W^{k,p}(\Omega)$ is defined by

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\sigma| \leq k} \int_\Omega |D^\sigma f|^p \, dx \right)^{1/p}$$

and $W_0^{k,p}(\Omega)$ is defined as the closure in $W^{k,p}(\Omega)$ of the family of C^∞ functions in Ω with compact support. It is well known that the space of Bessel potentials

$$L^{k,p}(\mathbb{R}^n) := \{f : f = G_k * g, \, g \in L^p(\mathbb{R}^n)\}$$

with norm $\|f\|_{k,p} := \|g\|_p$ is isometric to $W^{k,p}(\mathbb{R}^n)$. For arbitrary $\alpha > 0$, the Bessel kernel G_α is that function whose Fourier transform is

$$\widehat{G}_\alpha(x) = (2\pi)^{-n/2}(1 + |x|^2)^{-\alpha/2}.$$

The Bessel capacity of an arbitrary set $E \subset \mathbb{R}^n$ is defined as

$$C_{k,p}(E) := \inf\{\|g\|_p : g \in L^p(\mathbb{R}^n), \, g \geq 0, \, G_k * g \geq 1 \text{ on } E\}$$

where the infimum is taken over all non-negative functions $g \in L^p(\mathbb{R}^n)$ such that $G_k * g \geq 1$ on E. When $k = 1$ and $1 < p < n$, this capacity is equivalent to the p-capacity, γ_p, whose definition for bounded sets $E \subset \mathbb{R}^n$ is given by

$$\gamma_p(E) = \inf \left\{ \int_{\mathbb{R}^n} (|f|^p + |Df|^p) \, dx \right\}$$

where the infimum is taken over all $f \in W^{1,p}(\mathbb{R}^n)$ for which E is contained in the interior of $\{f \geq 1\}$. When $p \geq n$ the p-capacity of any non-empty set is positive. The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and $B(x,r)$ is the open ball of radius r centered at x. It will be clear from context the dimension of the Euclidean space on which Lebesgue measure is defined. Hausdorff $(n-1)$-dimensional measure will be denoted by H^{n-1}. The integral average of a function f over a set E is denoted by

$$⨍_E f = \frac{1}{|E|} \int_E f(x) \, dx.$$

An integrable function f is said to possess a Lebesgue point at x_0 if there is a number $l = l(x_0)$ such that

$$\lim_{r\to 0} \fint_{B(x_0,r)} |f(y) - l| \, dy = 0.$$

Recall that $l = f$ almost everywhere. Also, f is said to be approximately continuous at x_0 if there is a measurable set E with metric density one at x_0 such that

$$\lim_{\substack{x\to x_0\\ x\in E}} |f(x) - f(x_0)| = 0.$$

Note that if f has a Lebesgue point at x_0 and $l(x_0) = f(x_0)$, then f is approximately continuous at x_0.

If $f \in W_0^{k,p}(\Omega)$, then the function f^* defined as

$$f^*(x) := \begin{cases} f(x) & \text{if } \; x \in \Omega \\ 0 & \text{if } \; x \notin \Omega \end{cases} \tag{6.1}$$

is an element of $W^{k,p}(\mathbb{R}^n)$. It is well known that a Sobolev function $f \in W^{k,p}(\mathbb{R}^n)$ possesses a Lebesgue point everywhere except for a $C_{k,p}$ null set, cf. [Z], Theorem 3.3.3. Furthermore, if $f \in W_0^{k,p}(\Omega)$, it is not difficult to prove that

$$\lim_{r\to 0} \fint_{B(x,r)} f^*(y)\, dy = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} f(y)\, dy = 0 \tag{6.2}$$

for $C_{k,p}$-q.e. $x \in \mathbb{R}^n \setminus \Omega$, in particular for $C_{k,p}$-q.e. $x \in \partial\Omega$. The converse of this is one of the main results in [AH] which states the following.

Theorem 6.1 ([AH], Theorem 9.1.3)**.** *Let k be a positive integer, let $1 < p < \infty$ and let $f \in W^{k,p}(\mathbb{R}^n)$. If $\Omega \subset \mathbb{R}^n$ is an arbitrary open set, then $f \in W_0^{k,p}(\Omega)$ if and only if*

$$\lim_{r\to 0} \fint_{B(x,r)} |D^\beta f(y)| \, dy = 0 \tag{6.3}$$

for $C_{k-|\beta|,p}$-q.e. $x \in \mathbb{R}^n \setminus \Omega$ and for all multiindices β, $0 \le |\beta| \le k-1$.

For $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, this result is due independently to Havin [H] and Bagby [B].

A natural question arises whether the assumption that $f \in W^{k,p}(\mathbb{R}^n)$ can be replaced by the weaker one, $f \in W^{k,p}(\Omega)$, in which case (6.3) would have to be replaced by

$$\lim_{r\to 0} r^{-n} \int_{B(x,r)\cap\Omega} |D^\beta f(y)|\, dy = 0.$$

A similar question is raised in [AH], Section 9.12.1, concerning a different result. The purpose of this note is to provide an affirmative answer to this question.

In the course of this development, we will utilize the space BV, the class of functions of bounded variation.

Definition 6.2. The space $BV(\Omega)$ consists of all real-valued integrable functions f defined on Ω with the property that the distributional partial derivatives of f are totally finite Radon measures. The total variation measure of the vector valued measure associated with the gradient of f is denoted by $\|Df\|$. When viewed as a linear functional, its value on a non-negative real-valued continuous function g supported in Ω is

$$\|Df\|(g) = \sup\left\{ \int_\Omega g \operatorname{div} v\, dx : v \in C_c^\infty(\Omega;\mathbb{R}^n), |v(x)| \le f(x), x \in \Omega \right\},$$

and its value on a set E is $\|Df\|(E)$. The space $BV_{\mathrm{loc}}(\Omega)$ consists of all functions f defined on Ω with the property that $f \in BV(\Omega')$ for every open set Ω' compactly contained in Ω. The *measure theoretic boundary* of a set $E \subset \mathbb{R}^n$ is defined as

$$\partial_m E = \left\{ x : 0 < \limsup_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} \right\} \bigcap \left\{ x : \liminf_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} < 1 \right\}.$$

If $H^{n-1}(\partial_m E \cap \Omega) < \infty$, then E is said to have *finite perimeter in* Ω.

Functions in $BV(\mathbb{R}^n)$ can be characterized in terms of their behavior as functions of one variable. For this, consider a real valued function g defined on the interval $[a,b]$. The *essential variation* of g on $[a,b]$ is defined as

$$\operatorname{ess} V_a^b(g) := \sup\left\{ \sum_{i=1}^{k} |g(t_i) - g(t_{i-1})| \right\}$$

where the supremum is taken over all finite partitions of $[a, b]$ induced by $a < t_0 < t_1 < t_2 < \cdots < t_k < b$ where g is approximately continuous at each point of $\{t_0, t_1, \ldots, t_k\}$.

Now let us consider $f \in BV(\mathbb{R}^n)$ as a function of a single variable x_n while keeping fixed the remaining $n-1$ variables. Thus, let $\widehat{x}_n := (x_1, x_2, \ldots, x_{n-1})$ and define $f_{\widehat{x}_n}(t) := f(\widehat{x}_n, t)$. In a similar manner, we can define the remaining functions $f_{\widehat{x}_1}, f_{\widehat{x}_2}, \ldots, f_{\widehat{x}_{n-1}}$. A function $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if for almost every $\widehat{x}_k \in \mathbb{R}^{n-1}$, $\operatorname{ess} V_{a_k}^{b_k} f_{\widehat{x}_k}(\cdot) < \infty$ and

$$\int_R \operatorname{ess} V_{a_k}^{b_k} f_{\widehat{x}_k}(\cdot)\, d\widehat{x}_k < \infty \tag{6.4}$$

for each rectangular cell $R \subset \mathbb{R}^{n-1}$, $k \in \{1, 2, \ldots, n\}$, $-\infty < a_k < b_k < \infty$.

Another characterization of $BV(\Omega)$ is due to Fleming and Rishel [FR], and its statement most suitable for our purposes can be found in [Z], Theorem 5.4.4.

Theorem 6.3. *If $\Omega \subset \mathbb{R}^n$ is open and $f \in BV(\Omega)$, then*

$$\|Df\|(\Omega) = \int_{\mathbb{R}^1} H^{n-1}(\partial_m A_t \cap \Omega)\, dt, \tag{6.5}$$

where $A_t := \{x : f(x) > t\}$. Conversely, if $f \in L^1(\Omega)$ and A_t has finite perimeter in Ω for almost all t with

$$\int_{\mathbb{R}^1} H^{n-1}(\partial_m A_t \cap \Omega)\, dt < \infty, \tag{6.6}$$

then $f \in BV(\Omega)$.

In addition we will need the following known results concerning BV and Sobolev functions.

Theorem 6.4 ([F2], Theorem 4.5.9(29))**.** *If $f \in BV(\mathbb{R}^n)$ is approximately continuous at H^{n-1}-almost all points of $\mathbb{R}^n$, then f is continuous on almost all lines parallel to the coordinate axes.*

Theorem 6.5 ([GZ], Theorem 7.45)**.** *A function f defined on $[a, b]$ is absolutely continuous if and only if f is of bounded variation, continuous, and carries sets of measure zero into sets of measure zero.*

Theorem 6.6 ([Z], Theorem 2.1.4)**.** *Suppose $f \in W^{1,p}(\Omega)$, $p \geq 1$. Let $\Omega' \subset\subset \Omega$. Then f has a representative $\widetilde{f}$ that is absolutely continuous on* *almost all line segments of Ω' that are parallel to the coordinate axes and the classical partial derivatives of $\widetilde{f}$ agree almost everywhere with the distributional derivatives of u. Conversely, if f has such a representative and the classical partial derivatives $D_1 f, \ldots, D_n f$ together with f are in $L^p(\Omega')$ then $f \in W^{1,p}(\Omega')$.*

7 The inner trace

We are now in a position to prove our theorem.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and let f be a function defined on Ω with the property that $f \in BV(\Omega')$ for every open bounded subset $\Omega' \subset \Omega$. If f^* is approximately continuous H^{n-1}-a.e. in $\mathbb{R}^n$, then $f^* \in BV_{loc}(\mathbb{R}^n)$.*

Proof. Let $A_t := \{f > t\}$ and $A_t^* := \{f^* > t\}$. For each $t \neq 0$ we claim that $H^{n-1}[\partial_m A_t^* \setminus \Omega] = 0$. For this purpose, let $x_0 \in \mathbb{R}^n \setminus \Omega$ be a point of approximate continuity of f^*. Then $f^*(x_0) = 0$ and

$$\lim_{\substack{x \to x_0 \\ x \in E}} f^*(x) = 0 \tag{7.1}$$

for some set $E \subset \mathbb{R}^n$ whose metric density is one at x_0. If $t > 0$ this implies that

$$\lim_{r \to 0} \frac{|A_t^* \cap B(x_0, r)|}{|B(x_0, r)|} = 0$$

and therefore that $x_0 \notin \partial_m A_t^*$. Similarly, if $t < 0$ let $B_t^* := \{f^* < t\}$. Then equation (7.1) implies that

$$\lim_{r \to 0} \frac{|B_t^* \cap B(x_0, r)|}{|B(x_0, r)|} = 0 \quad \text{and therefore} \quad \lim_{r \to 0} \frac{|A_t^* \cap B(x_0, r)|}{|B(x_0, r)|} = 1,$$

thus showing that $x_0 \notin \partial_m A_t^*$. Since H^{n-1}-a.e. point of $\mathbb{R}^n \setminus \Omega$ is a point of approximate continuity of f^*, this shows that $H^{n-1}[\partial_m A_t^* \setminus \Omega] = 0$ for all $t \neq 0$.

Having established our claim, it follows that for any bounded open set $U \subset \mathbb{R}^n$,

$$\begin{aligned}
\int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t^* \cap U)\, dt &= \int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t^* \cap \Omega \cap U)\, dt \\
&= \int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t \cap \Omega \cap U)\, dt \\
&= \|Df\|\,(\Omega \cap U) < \infty
\end{aligned}$$

where the third equality is implied by (6.5) and is finite by the assumption that $f \in BV(\Omega \cap U)$. That $f^* \in BV(U)$ now follows from the first equality and (6.6). Since U is arbitrary, we conclude that $f^* \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, as desired. □

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and assume that $f \in W^{1,p}(\Omega)$, $1 < p < \infty$, has the property that*

$$\lim_{r \to 0} r^{-n} \int_{B(x,r) \cap \Omega} |f(y)| \, dy = 0 \tag{7.2}$$

for γ_p-q.e. $x \in \partial\Omega$. Then $f \in W_0^{1,p}(\Omega)$.

Except for a factor of $1/2$, the left side of (7.2) could be interpreted as the *inner trace* of f on domains with sufficient regularity, for example, on domains of finite perimeter. Thus our theorem states that if the inner trace of f is zero γ_p-q.e. on $\partial\Omega$, then $f \in W_0^{1,p}(\Omega)$.

Proof. Define f^* as in (6.1). The proof consists of the following steps.

Step 1. f^* is approximately continuous H^{n-1}-a.e. in $\mathbb{R}^n$.

Recall that f has a Lebesgue point at γ_p-q.e. point in Ω. Furthermore, for any set E, $\gamma_p(E) = 0$ implies $H^{n-p+\varepsilon}(E) = 0$ for all $\varepsilon > 0$, cf. [Z], Theorem 2.6.16. In particular, $H^{n-1}(E) = 0$. Consequently, f^* has a Lebesgue point at H^{n-1}-almost all points in Ω. Furthermore, for γ_p-q.e. $x \in \partial\Omega$, we know that

$$\lim_{r \to 0} \fint_{B(x,r)} |f^*(y)| \, dy = \lim_{r \to 0} r^{-n} \int_{B(x,r) \cap \Omega} |f(y)| \, dy = 0,$$

so f^* has a Lebesgue point at H^{n-1}-a.e. point in $\partial\Omega$. Finally, f^* is identically zero on $\mathbb{R}^n \setminus \Omega$ and therefore we conclude that f^* is approximately continuous at H^{n-1}-a.e. on $\mathbb{R}^n$.

Step 2. We know from Theorem 7.1 that $f^* \in BV_{\mathrm{loc}}(\mathbb{R}^n)$.

Step 3. f^* is continuous on almost all line segments parallel to the coordinate axes.

This follows from Steps 1, 2 and Theorem 6.4.

Step 4. f^* is of bounded variation on each bounded interval of almost all lines parallel to the coordinate axes.

This follows from Step 2 and (6.4).

Step 5. f^* is absolutely continuous on almost all line segments parallel to the coordinate axes.

In view of Theorem 6.5 we must show that on almost all line segments parallel to the coordinate axes, f^* (as a function of one variable) carries sets of Lebesgue measure zero (linear measure zero) into sets of Lebesgue measure zero. For this, consider for example a line segment λ parallel to the n^{th} coordinate axis passing through the point $x = (\widehat{x}, x_n)$ with the property that $f^*(\widehat{x}, \cdot)$ is continuous and of bounded variation and that $f(\widehat{x}, \cdot)$ is absolutely continuous on each bounded interval contained in $\lambda \cap \Omega$. Recall from steps 3 and 4 and Theorem 6.6 that almost all $\widehat{x}$ in $\mathbb{R}^{n-1}$ have this property. Let $E \subset \lambda$ be a set of linear measure zero and let I be any bounded, open interval of $\lambda \cap \Omega$. For any closed interval $J \subset I$, it follows from Theorem 6.6 that $f^*(J \cap E)$ is of measure zero and therefore, by a limiting process, $f^*(I \cap E)$ is of measure zero. Hence, $E \cap \lambda \cap \Omega$ is carried into a set of measure zero. Finally, f^* is constantly zero on $E \cap \lambda \cap (\mathbb{R}^n \setminus \Omega)$, and so f^* carries sets of measure zero into measure zero.

Step 6. From Step 5 we see that the distributional partial derivatives of f^* are functions and Step 2 implies that $|Df^*| \in L^1_{\mathrm{loc}}(\mathbb{R}^1)$. Since the classical partial derivatives of f^* exist almost everywhere on $\mathbb{R}^n$, we have that $Df^* = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$ and that $Df^* = Df$ on Ω. Consequently, $|Df^*| \in L^p(\mathbb{R}^n)$. Theorem 6.6 implies that $f^* \in W^{1,p}(\mathbb{R}^n)$ and since

$$\lim_{r \to 0} \fint_{B(x,r)} |f^*(y)|\, dy = 0$$

for γ_p-q.e. $x \in \mathbb{R}^n \setminus \Omega$, it follows from Theorem 6.1 that $f^* \in W^{1,p}_0(\Omega)$. As $f^* = f$ on Ω, it follows that $f \in W^{1,p}_0(\Omega)$ as desired. □

8 Extensions to $W^{k,p}(\Omega)$

As in Theorem 7.2, we address the problem of replacing the requirement that $f \in W^{k,p}(\mathbb{R}^n)$ with $f \in W^{k,p}(\Omega)$. This will be an easy consequence of Theorems 6.1 and 7.1.

For this, we begin with the following observation. If $\Omega \subset \mathbb{R}^n$ is an arbitrary open set and $f \in W^{k,p}_0(\Omega)$, then $f^* \in W^{k,p}(\mathbb{R}^n)$ and

$$D^\alpha f^* = (D^\alpha f)^* \tag{8.1}$$

for each multiindex $0 \leq |\alpha| \leq k$.

We now are in a position to prove the following.

Theorem 8.1. *Let k be a positive integer, let $1 < p < \infty$ and let $f \in W^{k,p}(\Omega)$. If $\Omega \subset \mathbb{R}^n$ is an arbitrary open set, then $f \in W_0^{k,p}(\Omega)$ if and only if*

$$\lim_{r \to 0} r^{-n} \int_{B(x,r)\cap\Omega} |D^\beta f(y)|\, dy = 0 \tag{8.2}$$

for $C_{k-|\beta|,p}$-q.e. $x \in \mathbb{R}^n \setminus \Omega$ and for all multiindices β, $0 \le |\beta| \le k-1$.

Proof. The proof of sufficiency is immediate and thus we will consider only necessity. This proceeds by induction on k with the case $k = 1$ having been established by Theorem 7.2. Assume that $f \in W^{k,p}(\Omega)$ satisfies condition (8.2). Then $f \in W^{k-1,p}(\Omega)$, and since $C_{k-1-|\beta|,p} \le C_{k-|\beta|,p}$ for every multiindex β, $0 \le |\beta| \le k-2$, it follows that f satisfies condition (8.2) as an element of $W^{k-1,p}(\Omega)$. Thus by the induction hypothesis we conclude that $f \in W_0^{k-1,p}(\Omega)$ and hence that $f^* \in W^{k-1,p}(\mathbb{R}^n)$.

Let β be a multiindex with $|\beta| = k-1$, and define $g := D^\beta f$. Then $g \in W^{1,p}(\Omega)$ satisfies the hypotheses of Theorem 7.2, which implies that $g^* \in W^{1,p}(\mathbb{R}^n)$. Thus by (8.1), we have that $D^\beta f^* = (D^\beta f)^* \in W^{1,p}(\mathbb{R}^n)$ whenever $|\beta| = k-1$. It follows that $f^* \in W^{k,p}(\mathbb{R}^n)$. Now we may apply Theorem 6.1 to conclude that $f^* \in W_0^{k,p}(\Omega)$. This yields our desired conclusion since $f^* = f$ on Ω. □

References

[AH] D.R. Adams and L.I. Hedberg, *Function spaces and potential theory.* Springer-Verlag, Berlin 1996.

[B] T. Bagby, *Quasi topologies and rational approximation.* J. Funct. Anal. **10** (1972), 259–268.

[BDG] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem.* Invent. Math. **7** (1969), 255–267.

[BK] H. Brezis and D. Kinderlehrer, *The smoothness of solutions to nonlinear variational inequalities.* Indiana Univ. Math. J. **23** (1974), 831–844.

[F1] H. Federer, *Curvature measures.* Trans. Amer. Math. Soc. **93** (1959), 418–491.

[F2] H. Federer, *Geometric measure theory.* Springer-Verlag, New York 1969.

[FR] W.H. Fleming and R. Rishel, *An integral formula for total gradient variation.* Arch. Math. **11** (1960), 218–222.

[GZ] R.F. Gariepy and W.P. Ziemer, *Modern real analysis.* PWS Publishing Co., 1994.

[GT] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order.* Springer-Verlag, Second ed., New York 1983.

[G1] E. Giusti, *Boundary behavior of nonparametric minimal surfaces.* Indiana Univ. Math. J. **22** (1977), 435–444.

[G2] E. Giusti, *Minimal surfaces and functions of bounded variation.* Birkhäuser, 1985.

[GMT1] E. Gonzalez, U. Massari and I. Tamanini, *Minimal boundaries enclosing a given volume.* Manuscripta Math. **34** (1981), 381–395.

[GMT2] E. Gonzalez, U. Massari and I. Tamanini, *On the regularity of sets minimizing perimeter with a volume constraint.* Indiana Univ. Math. J. **32** (1983), 25–37.

[GH] J. M. Greenberg and J. R. Harper, *Algebraic topology.* Benjamin-Cummings Press, 1981.

[Gr] M. Grüter, *Boundary regularity for solutions of a partitioning problem.* Arch. Rational Mech. Anal. **97** (1987), 261–270.

[H] R. Hardt, *Uniqueness of nonparametric area minimizing currents.* Indiana Univ. Math. J. **26** (1977), 65–71.

[Ha] V. P. Havin, *Approximation in the mean by analytic functions.* Soviet Math. Dokl. **9** (1968), 245–248.

[ISZ] T. Ilmanen, P. Sternberg and W. Ziemer, *Equilibrium solutions to generalized motion by mean curvature.* Accepted by J. Geom. Anal.

[KS] R. V. Kohn and G. Strang, *The constrained least gradient problem.* Nonclassical Continuum Mechanics. R. Knops and A. Lacy (eds.), Cambridge Univ. Press, 1987, 226–243.

[LS] P. Laurence and E. W. Stredulinsky, *On quasiconvex equimeasurable rearrangement, a counterexample and an example.* J. Reine Angew. Math. **447** (1994), 63–81.

[MM] U. Massari and M. Miranda, *Minimal surfaces of codimension one.* Mathematics Studies 91, North Holland, Amsterdam 1984.

[M] M. Miranda, *Sul minimo dell'integrale del gradiente di una funzione.* Ann. Scuola Norm. Sup. Pisa **19** (1965), 627–656.

[Mo] M. P. Moschen, *Principio di massimo forte per le frontiere di misura minima.* Ann. Univ. Ferrara Sez. VII, Sc. Mat. **23** (1977), 165–168.

[P1] H. Parks, *Explicit determination of area minimizing hypersurfaces.* Duke Math. J. **44** (1977), 519–534.

[P2] H. Parks, *Explicit determination of area minimizing hypersurfaces, II.* Mem. Amer. Math. Soc. **342** (1986).

[PZ] H. Parks and W. Ziemer, *Jacobi fields and functions of least gradient.* Ann. Scuola Norm. Sup. Pisa **XI** (1984), 505–527.

[S1] L. Simon, *Lectures on geometric measure theory.* Proc. Centre Math. Analysis, ANU, Canberra, 3(1983).

[S2] L. Simon, *A strict maximum principle for area minimizing hypersurfaces.* J. Differential Geom. **26** (1987), 327–335.

[SWZ1] P. Sternberg, G. Williams and W. Ziemer, $C^{1,1}$ *regularity of constrained area-minimizing hypersurfaces* J. Differential Equations **94** (1991), 83–94.

[SWZ2] P. Sternberg, G. Williams and W. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient.* J. Reine Angew. Math. **430** (1992), 35–60.

[SWZ3] P. Sternberg, G. Williams and W. Ziemer, *The constrained least gradient problem in* $\mathbb{R}^n$. Trans. Amer. Math. Soc. **339** (1993), 403–433.

[SZ] P. Sternberg and W. Ziemer, *The Dirchlet problem for functions of least gradient.* Invited paper to celebrate James Serrin's 65$^{\text{th}}$ birthday. IMA Volumes in Math. and Appl. **47** (1993), 197–214.

[StZ] E. Stredulinsky and W. Ziemer, *Area minimizing sets subject to a volume constraint in a convex set.* Accepted by J. Geom. Anal.

[SwZ] D. Swanson and W. Ziemer, *Sobolev functions whose inner trace at the boundary is zero.* Accepted by Ark. Mat.

[T] I. Tamanini, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector.* J. Reine Angew. Math. **334** (1982), 27–39.

[W] G. Williams, *Regularity for solutions of the minimal surface equation with continuous boundary values.* Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 411–429.

[Z] W. P. Ziemer, *Weakly differentiable functions.* Springer-Verlag, GTM Series, 120, Berlin 1989.