Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

QUATERNIONIC CONTACT STRUCTURES

OLIVIER BIQUARD

This article is a survey on the notion of *quaternionic contact structures*, which I defined in [2]. Roughly speaking, quaternionic contact structures are quaternionic analogues of integrable CR structures.

Definition and first examples

Let X be a manifold and V a distribution in X, so at each point $x \in X$ we have a subspace V_x of T_xX. One can define a nilpotent Lie algebra structure on $V_x \oplus (T_xX/V_x)$ by

$$[a,b] = \begin{cases} \pi_{T_xX/V_x}[a,b] & \text{if } a,b \in V_x, \\ 0 & \text{otherwise,} \end{cases}$$

where on the RHS we have the bracket of vector fields.

The Heisenberg algebra is defined as the vector space $\mathbf{C}^m \oplus \mathbf{R}$ with a Lie bracket $[\mathbf{C}^m, \mathbf{C}^m] \subset \mathbf{R}$ given by

$$\left[\sum_1^m x_i e_i, \sum_1^m y_i e_i \right] = \operatorname{Im} \sum_1^m \bar{x}_i y_i.$$

The same formula gives also the Lie bracket of the quaternionic Heisenberg algebra $\mathbf{H}^m \oplus \operatorname{Im}\mathbf{H}$.

A contact structure on X^{2m+1} is a codimension 1 distribution V such that at each point x the nilpotent Lie algebra $V_x \oplus T_xX/V_x$ is isomorphic to the Heisenberg algebra. Similarly, a quaternionic contact structure on X^{4m+3} is concerned with a codimension 3 distribution V such that at each point x the nilpotent Lie algebra $V_x \oplus T_xX/V_x$ is isomorphic to the quaternionic Heisenberg algebra.

There is an equivalent, more concrete, description of such distributions: there exists a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbf{R}^3$, such that $V = \ker \eta$ and the three 2-forms

$(d\eta_i|_V)$ are the fundamental 2-forms of a quaternionic structure on V: this means that there exists a metric γ on V such that

$$d\eta_i|_V = \gamma(I_i\cdot,\cdot), \tag{$\dagger$}$$

and the I_i are complex structures on V satisfying the commutation relations of the quaternions $I_1I_2I_3 = -1$. The 1-form η is given only up to the action of SO_3 on $\mathbf{R}^3$ and to a conformal factor, thus we get a CSp_mSp_1-structure on V.

Definition 1. *A* quaternionic contact structure *on X^{4m+3} is the data of a codimension 3 distribution V, equipped with a CSp_mSp_1-structure, such that the CSp_mSp_1-structure and the contact form with values in $\mathbf{R}^3$ satisfy the compatibility relation* ($\dagger$).

Let us point an important difference between contact structures and quaternionic contact structures: a quaternionic contact structure always define a (conformal) metric on the distribution, but a contact structure defines only a symplectic structure, and one needs to choose some compatible complex structure on the distribution (that is, a CR structure) in order to get a metric. That is why I consider quaternionic contact structures as a quaternionic analogue of CR structures.

The sphere $S^{4m-1} \subset \mathbf{R}^{4m}$ has a canonical quaternionic contact structure, defined as follows: the flat manifold $\mathbf{R}^{4m}$ is hyperkähler with three complex structures I_1, I_2, I_3 satisfying $I_1I_2I_3 = -1$; then on S^{4m-1} the contact form η with values in $\mathbf{R}^3$ is

$$\eta_i = I_i dr,$$

where r is the radius in $\mathbf{R}^{4m}$; the associated metric γ is the restriction to $V = \ker \eta$ of the standard metric.

More generally, any 3-Sasakian manifold has a canonical quaternionic contact structure; as we shall see later, this is a very special case, since 3-Sasakian manifolds are rigid [3], but quaternionic contact structures come in infinite dimensional families.

Conformal infinities of Einstein metrics

Submanifolds of complex manifolds are integrable CR manifolds. The example of $S^{4m-1} \subset \mathbf{R}^{4m}$ could suggest the same for quaternionic contact structures, but this is actually not true. A better interpretation of this example is to see S^{4m-1} as the boundary at infinity of the quaternionic hyperbolic space $\mathbf{H}H^m$. If we pick up a point $*$ in $\mathbf{H}H^m$, we may identify $\mathbf{H}H^m - \{*\} = \mathbf{R}_+^* \times S^{4m-1}$, and the hyperbolic metric can be written as

$$g = dr^2 + \sinh^2(2r)\tilde{\gamma} + \sinh^2(4r)\eta^2,$$

where $\tilde{\gamma}$ is the extension of γ to the whole TS^{4m-1} by 0 on the fibers of the map

$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4m-1} \\ & & \downarrow \\ & & \mathbf{H}P^{m-1} \end{array}$$

At infinity, we recover γ as

$$\gamma = \lim_{r\to\infty} e^{-2r} g|_{\{r\}\times S_r};$$

note that this limit is infinite, except on $V = \ker\eta$; so from this point of view, we should consider γ as a Carnot-Carathéodory metric, that is a metric which is infinite outside some distribution whose brackets generate the whole tangent space. Also, the change of base point $*$ induces a conformal change on the limit, so that only the conformal class $[\gamma]$ of γ is defined: we call it the *conformal infinity* of g.

The first motivation for studying quaternionic contact structures is the following result on Einstein deformations of $\mathbf{H}H^m$.

Theorem 2. *If a quaternionic contact structure (V,γ) on S^{4m-1} is close enough to the standard one, then it is the conformal infinity of a complete Einstein metric g.*

More precisely, g has conformal infinity $[\gamma]$ means that, near infinity, one has

$$g \sim dr^2 + e^{2r}\gamma + e^{4r}\eta^2.$$

Also, this theorem is a generalization of a theorem of Graham-Lee [7] on Einstein deformations of real hyperbolic space; for other rank one symmetric spaces, see [2] and the survey [1].

Twistor construction for quaternionic contact structures

Recall that the twistor space of $\mathbf{H}P^m$ is $\mathbf{C}P^{2m+1}$, with projection given in homogeneous coordinates by

$$[z_1 : \cdots : z_{2m+2}] \longrightarrow [z_1 + jz_2 : \cdots : z_{2m+1} + jz_{2m+2}].$$

One may realize $\mathbf{H}H^m$ inside $\mathbf{H}P^m$ as

$$\mathbf{H}H^m = \{[q_1 : \cdots : q_{m+1}], |q_1|^2 + \cdots + |q_m|^2 < |q_{m+1}|^2\}$$

and its twistor space $T(\mathbf{H}H^m)$ is a domain in $\mathbf{C}P^{2m+1}$:

$$T(\mathbf{H}H^m) = \{[z_1 : \cdots : z_{2m+2}], |z_1|^2 + \cdots + |z_{2m}|^2 < |z_{2m+1}|^2 + |z_{2m+2}|^2\}.$$

Remark that the twistorial fibration restricts on the boundary to give

$$\begin{array}{ccc} \partial T(\mathbf{H}H^m) & \subset \mathbf{C}P^{2m+1} \\ \downarrow & \\ S^{4m-1} & \subset \mathbf{H}P^m \end{array}$$

So we see that the sphere S^{4m-1} has some kind of twistor space which is a real hypersurface in $\mathbf{C}P^{2m+1}$. This generalizes to quaternionic contact structures in the following way.

Theorem 3. *If X^{4m-1} has a quaternionic contact structure, with $4m-1>7$, then there is a twistor space T^{4m+1} with a projection*

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & T \\ & & \big\downarrow{\scriptstyle\pi} \\ & & X \end{array}$$

such that

(i) T has an integrable CR structure, and the fibers of π are holomorphic;

(ii) T has a holomorphic contact structure, orthogonal to the fibers;

(iii) T has a real structure compatible with the other structures.

One should precise what a holomorphic contact structure is for an integrable CR manifold M^{4m+1}: consider the bundle $T'M = T_{\mathbf{C}}M/T^{0,1}M$, which can be identified with $T^{1,0}M \oplus \mathbf{C}R$ for some choice of a Reeb vector field R. When M is the boundary of a complex manifold, $T'M$ is the restriction of the holomorphic tangent bundle to the boundary. In general, it has a canonical holomorphic structure defined by

$$X \in T^{0,1}M, \quad \sigma \in T'M, \quad \bar\partial_X \sigma = [X,\sigma];$$

this is well defined because of the integrability condition $[T^{0,1}, T^{0,1}] \subset T^{0,1}$. Now a holomorphic contact structure is a codimension 1 holomorphic distribution of $T'M$, given locally by a complex 1-form η^c such that $d\eta^c$ is (complex) symplectic on the distribution.

This theorem generalizes a twistorial construction of LeBrun [11] for conformal 3-dimensional metrics. It is probable that, as in dimension 3, the converse of the theorem holds, that is a fibration by $\mathbf{C}P^1$'s satisfying the three conditions of the theorem is a twistor space of a quaternionic contact structure, but I have not completely checked this statement.

Let us now give an idea of the proof of theorem 3. In the construction of the twistor space of a conformal 3-dimensional metric, or in Salamon's construction [13] of the twistor space of a quaternionic-Kähler manifold, one uses the Levi-Civita connection to define a horizontal subspace for the fibration, and then the complex structure. In the case of a quaternionic contact structure, there is a priori no canonical connection; fortunately, the following theorem provides a connection, which is analogous to the Tanaka-Webster connection [15] in CR geometry.

Theorem 4. *If X^{4m-1} (for $4m-1>7$) has a quaternionic contact structure V, with a choice of metric γ on V in the conformal class, then there exists a unique connection ∇ on X and a unique supplementary subspace W of V in TX, such that*

(i) ∇ preserves the decomposition $V \oplus W$ and the metric;

(ii) for $A, B \in V$, one has $T^\nabla_{A,B} = -[A,B]_W$;

(iii) ∇ preserves the $Sp_{m-1}Sp_1$-structure on V;

(iv) for $R \in W$, the endomorphism $\cdot \to (T^\nabla_{R,\cdot})_V$ of V lies in the orthogonal of $\mathfrak{sp}_1 \oplus \mathfrak{sp}_m$;

(v) the connection on W is induced by the natural identification of W with the subspace $\mathfrak{sp}_1$ of the endomorphisms of V.

This theorem shows us that quaternionic contact structures, in dimension greater than 7, have some kind of integrability hidden in the definition, which enables to construct a natural connection preserving the $Sp_{m-1}Sp_1$-structure. This is why they are the quaternionic analogue of *integrable* CR structures. But in dimension 7 (see also below), some integrability condition probably remains to understand.

Given the connection of theorem 4, one can construct the CR structure on the twistor space in the usual way, but then integrability is a difficult task, because the connection has nonzero torsion, so one has to prove some complicated algebraic identities on torsion, in order to get theorem 3.

Now, we outline some steps for the proof of theorem 4. In particular, we want to explain what does not work in dimension 7. The main issue is to understand the derivation ∇_A on V for $A \in V$; this part of the connection and the supplementary subspace W are characterized by properties (i), (ii) and (iii): given some W, the properties (i) and (ii) define a unique connection on V along V; property (iii) can be expressed saying that the fundamental 4-form $\Omega = \sum_1^3 (d\eta_i|_V)^2$ is parallel; the method consists in decomposing $\nabla\Omega$ in irreducible components under the action of $Sp_{m-1}Sp_1$ (see the book [14]): some components vanish because Ω, from its definition, is somehow closed, and the remaining obstructions correspond exactly to fixing a choice of W.

The proof does not work for dimension 7 because V is then 4-dimensional and condition (iii) becomes empty; instead of a condition on the connection, one rather needs some kind of selfduality condition on the curvature, but I have not done that.

One interesting point is that the supplementary subspace W can be described explicitly: choose 1-forms (η_1, η_2, η_3), then W is generated by vector fields R_1, R_2, R_3 such that

$$\eta_i(R_j) = \delta_{ij}, \quad (i_{R_i} d\eta_i)|_V = 0. \tag{‡}$$

The space generated by such R_1, R_2, R_3 a priori depends on the action of SO_3 on the 1-forms; actually, it is fixed, and one has the stronger identity

$$(i_{R_i} d\eta_j + i_{R_j} d\eta_i)|_V = 0;$$

this property together with (‡) is invariant under SO_3.

More generally, our quaternionic contact structures, after a choice of conformal factor, fit in the theory of Carnot-Carathéodory metrics with strong bracket generating hypothesis (see [8]), and this could lead to a slightly different proof of theorem 4: the supplementary subspace defined by (‡) enables to define a metric on the whole tangent space; this metric becomes canonical after averaging over SO_3, and the orthogonal of V furnishes the canonical supplementary subspace W; it remains to verify that the connection satisfying (i) and (ii) also satisfies the integrability (iii). This approach could be interesting in order to understand the case of dimension 7.

Construction of quaternionic-Kähler metrics

Theorem 5. *If X^{4m-1} (for $4m-1>7$) has a real analytic quaternionic contact structure, then it is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of X.*

Here are some comments on this theorem:

1. This theorem is the natural generalization to higher dimension of LeBrun's theorem [10] on filling of 3-dimensional real analytic conformal manifolds by selfdual Einstein metrics.
2. The theorem remains true in dimension 7, under the additional assumption of the existence of the twistor space of X. More generally, quaternionic contact structures in dimension 7 could be related to "symplectic quaternionic structures" in dimension 8, that is Sp_2Sp_1-structures such that the fundamental 4-form is closed.
3. LeBrun [9] has constructed an infinite dimensional family of complete quaternionic-Kähler deformations of $\mathbf{H}H^m$; as we shall see below, these metrics have conformal infinities which are quaternionic contact structures, so the uniqueness statement implies that they coincide with the metric constructed by the theorem. Also, they provide an infinite dimensional family of examples of quaternionic contact structures.
4. Under the assumption of theorem 2, the quaternionic-Kähler metric usually does not coincide with the Einstein metric; instead, the quaternionic-Kähler metric gives a high order approximation of the Einstein metric at infinity. Obviously, an interesting problem is to understand which quaternionic contact structures can be filled by complete quaternionic-Kähler metrics (this problem is also unsolved for conformal metrics in dimension 3).

The basic construction in the proof of this theorem is the following. One has the twistor fibration

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & T \\ & & \big\downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

with the connection ∇; at least locally, one may complexify X as $X^{\mathbf{C}}$ and extend the fibration T as a holomorphic fibration U on $X^{\mathbf{C}}$,

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & U \\ & & \big\downarrow{\scriptstyle \pi} \\ & & X^{\mathbf{C}} \end{array}$$

The complex dimension of U if $4m$; the connection ∇ extends to a connection on U and the CR structure on T induces a complex endomorphism J with $J^2 = -1$ on a $(4m-1)$-dimensional distribution. Now one considers the distribution F which is a subspace of the horizontal distribution Hor^∇ of ∇:

$$F = T_J^{0,1} \cap Hor^\nabla.$$

One can see that F is actually a $(2m-1)$-dimensional holomorphic integrable distribution, so there is a $(2m+1)$-dimensional space of leaves N, which will be the twistor space of the manifold M that we want to construct.

In dimension 3, remark that U is the bundle of null directions in $TX^{\mathbf{C}}$ and the leaves of the foliation are null geodesics, so one recovers LeBrun's construction.

The space U has two projections

$$\begin{array}{ccc} U & \xrightarrow{q} & N \\ {\scriptstyle p}\big\downarrow & & \\ X^{\mathbf{C}} & & \end{array}$$

for $x \in X^{\mathbf{C}}$, $C_x = q(p^{-1}(x))$ is a holomorphic line in N, with normal bundle $\mathcal{O}(1) \otimes \mathbf{C}^{2m}$, so the space $M^{\mathbf{C}}$ of deformations of these lines is $4m$-dimensional, when $X^{\mathbf{C}}$ is only a $(4m-1)$-dimensional submanifold; also, N has a holomorphic contact structure, and C_x is transverse to the contact distribution except for $x \in X^{\mathbf{C}}$; this kind of situation is analyzed in a general context in the following proposition, from which the theorem follows.

Proposition 6. *Suppose that N is a $(2m+1)$-dimensional complex manifold, with*
(i) a holomorphic contact structure;
(ii) a family $M^{\mathbf{C}}$ of holomorphic lines $(C_m)_{m\in M}$ with normal bundle $\mathcal{O}(1)\otimes\mathbf{C}^{2m}$, such that C_m is transverse to the contact distribution except on a hypersurface $S^{\mathbf{C}}$;
(iii) a real structure, compatible with the other structures;
then N is the twistor space of a quaternionic-Kähler metric on $M-S$, with conformal infinity a quaternionic contact structure on S; the twistor space of this quaternionic contact structure is $N|_S$.

Explicit examples

From theorem 5 and the quotient construction for quaternionic-Kähler metrics [4, 6], one may deduce that there is a quotient construction for groups acting on quaternionic contact structures. Actually, some known quotients of the quaternionic hyperbolic space by Hitchin and Galicki have a counterpart on their conformal infinities, as I shall now explain.

The isometry group of $\mathbf{H}H^2$ is $Sp_{2,1}$; there is an action of $\mathbf{R}$ on $\mathbf{H}H^2$ by

$$\begin{pmatrix} e^{ix} & & \\ & \cosh(\ell x) & \sinh(\ell x) \\ & \sinh(\ell x) & \cosh(\ell x) \end{pmatrix} \in Sp_1 \times SO_{1,1} \subset Sp_{2,1}$$

and the quotient is Pedersen's selfdual Einstein metric [12] on the 4-ball, with conformal infinity the Berger sphere. This means that the quotient of S^7 by this action is the 3-sphere with the Berger metric. Note that the action on S^7 preserves the quaternionic contact structure, not the metric.

This example can be generalized to quotients of $\mathbf{H}H^m$ by subgroups of $Sp_{m,1}$, see [5]; at infinity, this gives quotients of S^{4m-1} by subgroups of $Sp_{m,1}$, some of which could probably be made explicit.

References

1. O. Biquard, *Einstein deformations of hyperbolic metrics*, Essays on Einstein Manifolds (C. LeBrun and M. Wang, eds.), Suveys in Differential Geometry, vol. VI, International Press, 1999, pp. 235–246.
2. ______, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **165** (2000).
3. C. P. Boyer and K. Galicki, *3-Sasakian manifolds*, Essays on Einstein Manifolds (C. LeBrun and M. Wang, eds.), Suveys in Differential Geometry, vol. VI, International Press, 1999, pp. 123–184.
4. K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Commun. Math. Phys. **108** (1987), 117–138.
5. ______, *Multi-centre metrics with negative cosmological constant*, Class. Quantum Grav. **8** (1991), 1529–1543.
6. K. Galicki and B. H. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1–21.
7. C. R. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.
8. Ursula Hamenstädt, *Some regularity theorems for Carnot-Carathéodory metrics*, J. Differential Geom. **32** (1990), 819–850.
9. C. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63** (1991), 723–743.
10. C. R. LeBrun, *$\mathcal{H}$-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380** (1982), 171–185.
11. ______, *Twistor CR manifolds and three-dimensional conformal geometry*, Trans. Amer. Math. Soc. **284** (1984), 601–616.
12. H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274** (1986), 35–59.
13. S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.
14. ______, *Riemannian geometry and holonomy groups*, Longman Scientific & Technical, Harlow, 1989.
15. S. M. Webster, *Pseudo-hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1979), 25–41.

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René Descartes, F-67084 Strasbourg Cedex

E-mail address: olivier.biquard@math.u-strasbg.fr