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QUATERNION KÄHLER FLAT MANIFOLDS

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This note is a revised version of the talk given by the author at the meeting *Quaternionic structures in Mathematics and Physics* at Rome in September, 1999. The results presented here are part of [4], a joint work with R. Miatello.

1. Introduction

A Riemannian manifold is *quaternion Kähler* if its holonomy group is contained in $Sp(n)Sp(1)$. It is known that quaternion Kähler manifolds are Einstein, so the scalar curvature s splits these manifolds according to whether $s > 0$, $s = 0$ or $s < 0$. Ricci flat quaternion Kähler manifolds include hyperkähler manifolds, that is, those with full holonomy group contained in $Sp(n)$. Such a manifold can be characterized by the existence of a pair of integrable anticommuting complex structures, compatible with respect to the Riemannian metric, and parallel with respect to the Levi-Civita connection.

It is the main purpose of this lecture to indicate a rather general method to construct quaternion-Kähler compact flat manifolds. This construction will give many families of quaternion Kähler manifolds of dimensions $n \geq 8$, which admit no Kähler structure (see Section 3). This will follow from the explicit calculation of the Betti numbers of the manifolds involved.

The simplest model of hyperkähler manifolds (and in particular, of quaternion Kähler manifolds) is provided by $\mathbb{R}^{4n}$ with the standard flat metric and a pair J, K of orthogonal anticommuting complex structures. This hyperkähler structure descends to the $4n$-torus $T_\Lambda := \Lambda\backslash\mathbb{R}^{4n}$, for any lattice Λ in $\mathbb{R}^{4n}$. The main idea in the construction consists of finding finite groups F acting freely on the torus, endowed with the standard hyperkähler structure, in such a way that $F\backslash T^{4n}$ becomes quaternion Kähler but its cohomology changes in such a way that the resulting manifold will not admit any Kähler structure.

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2. Construction of quaternion Kähler flat manifolds.

One way of constructing free actions of finite groups on tori is via Bieberbach groups. A Bieberbach group Γ is a crystallographic group (i.e. a discrete cocompact subgroup of $\mathrm{I}(\mathbb{R}^n)$) which is torsion-free. The quotient $M_\Gamma := \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ. If $v \in \mathbb{R}^n$, let L_v denote translation by v. By Bieberbach's first theorem, if Γ is a crystallographic group then $\Lambda = \{v : L_v \in \Gamma\}$ is a lattice in $\mathbb{R}^n$. The translation lattice $L_\Lambda = \{L_v : v \in \Lambda\}$ is a normal and maximal abelian subgroup of Γ and the quotient $F := L_\Lambda\backslash\Gamma$ is a finite group acting freely on $\Lambda\backslash\mathbb{R}^n$; it represents the linear holonomy group of the flat Riemannian manifold M_Γ and is called the holonomy group of Γ. We will usually write Λ in place of L_Λ.

Any element $\gamma \in \mathrm{I}(\mathbb{R}^n)$ decomposes uniquely $\gamma = BL_b$, with $B \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$ and the lattice Λ is B-stable for each $BL_b \in \Gamma$. The restriction to Γ of the canonical projection from $\mathrm{I}(\mathbb{R}^n)$ to $\mathrm{O}(n)$, mapping BL_b to B, has kernel Λ and the image is a finite subgroup of $\mathrm{O}(n)$, called the point group of Γ. We shall often identify the holonomy group F with the point group of Γ. The action of F on Λ defines an integral representation of F, usually called the holonomy representation.

If $M_\Gamma = \Gamma\backslash\mathbb{R}^{4n}$ is a compact flat manifold such that the holonomy action of $F = \Lambda\backslash\Gamma$ centralizes (resp. normalizes) the algebra generated by J, K, then M_Γ inherits a hyperkähler (resp. quaternion Kähler) structure. To produce Bieberbach groups having the previous property we introduced in [1] a "doubling" procedure for Bieberbach groups which allows to produce many flat hyperkähler (even Clifford Kähler) manifolds. In particular, we showed that any finite group is the holonomy group of a hyperkähler flat manifold. The main goal will be to give a variant of this construction which produces quaternion Kähler manifolds which are generically not Kähler.

Let Γ be a Bieberbach group with holonomy group F and translation lattice $\Lambda \subset \mathbb{R}^n$. Let $\phi : F \to \mathbb{R}^n$ be a 1-cocycle modulo Λ, that is, ϕ satisfies $\phi(B_1B_2) = B_2^{-1}\phi(B_1) + \phi(B_2)$, modulo Λ, for each $B_1, B_2 \in F$. Then ϕ defines a cohomology class in $H^1(F; \mathbb{R}^n/\Lambda) \simeq H^2(F; \Lambda)$ and one may associate to ϕ a crystallographic group with holonomy group F and translation lattice Λ. Furthermore, this group is torsion-free if and only if the class of ϕ is a special class (see [2]).

Definition 2.1. Let Γ be a Bieberbach group with holonomy group F and translation lattice $\Lambda \subset \mathbb{R}^n$. Let $\phi : F \to \mathbb{R}^n$ be any 1-cocycle modulo Λ. We let $\mathrm{d}_\phi\Gamma$ be the subgroup of $\mathrm{I}(\mathbb{R}^{2n})$ generated by elements of the form $\left[\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right] L_{(\phi(B),b)}$ and $L_{(\lambda,\mu)}$, for $\gamma = BL_b \in \Gamma$ and $(\lambda, \mu) \in \Lambda \oplus \Lambda$.

Proposition 2.2. *(compare with* [1]*, Theorem 3.1) Let Γ, ϕ and $d_\phi\Gamma$ be as in Definition 2.1 Then*

(i) $d_\phi\Gamma$ is a Bieberbach group with holonomy group F, translation lattice $\Lambda \oplus \Lambda$ and $d_\phi\Gamma\backslash\mathbb{R}^{2n}$ is a Kähler compact flat manifold.

(ii) If $\Gamma\backslash\mathbb{R}^n$ has a locally invariant Kähler structure, then $d_\phi\Gamma\backslash\mathbb{R}^{2n}$ is hyperkähler. In particular, if $\phi' : F \to \mathbb{R}^{2n}$ is any 1-cocycle modulo $\Lambda \oplus \Lambda$, then $d_{\phi'} d_\phi\Gamma\backslash\mathbb{R}^{4n}$ is hyperkähler. Any finite group is the holonomy group of a hyperkähler compact flat manifold.

We shall work mostly with the choice $\phi = 0$ and we shall then write $\mathrm{d}_0\Gamma$. Other natural choice is to let ϕ be the 1-cocycle associated to Γ, as in [1]; we denote $\mathrm{d}_\phi\Gamma$ by $\mathrm{d}\Gamma$ in this case.

It is clear that the procedure in (ii) of Proposition 2.2 can be iterated. If we assume that $\phi = 0$, for simplicity, and we set $\mathrm{d}_0^m\Gamma = \mathrm{d}_0\mathrm{d}_0^{m-1}\Gamma$, we get that $\mathrm{d}_0^m\Gamma$ is a Bieberbach subgroup of $\mathrm{I}(\mathbb{R}^{2^m n})$ with holonomy group F, diagonal holonomy representation and translation lattice Λ^{2^m}. Furthermore the holonomy representation commutes with m anticommuting complex structures on $\mathbb{R}^{2^m n}$, hence $\mathrm{d}_0^m\Gamma\backslash\mathbb{R}^{2^m n}$ has a Clifford structure of order m (compare [1], 3.1).

We wish to enlarge $\mathrm{d}_\phi\Gamma$ into a Bieberbach group $\mathrm{d}_{q,\phi}\Gamma$ in such a way that some element in the holonomy group of $\mathrm{d}_{q,\phi}\Gamma$ anticommutes with the complex structure J_{2n} in $\mathbb{R}^{2n}$. Once this is done, then by repeating the procedure twice, we shall get a Bieberbach group such that any element in the holonomy group will either commute or anticommute with each one of a pair of anticommuting complex structures, hence the quotient manifold will be a quaternion Kähler flat manifold which in general, will not be Kähler.

In order for this second construction to work we will restrict to Bieberbach groups with holonomy group $\mathbb{Z}_2^k$. We will make use of the following result from [3], Proposition 2.1 (see also [5], Proposition 1.1).

Proposition 2.3. *Assume that $\Gamma = \langle\gamma_1, \dots, \gamma_k, \Lambda\rangle$ is a subgroup of $Aff(\mathbb{R}^n)$, with $\gamma_i = B_i L_{b_i}$, $b_i \in \mathbb{R}^n$, $B_i \in Gl(n, \mathbb{R})$ such that $\langle B_1, \dots, B_r\rangle$ is isomorphic to $\mathbb{Z}_2^k$ and Λ is a lattice in $\mathbb{R}^n$ stable by the B_i's. Then Γ is torsion-free with translation lattice Λ if and only if the following two conditions hold:*

(i) *For each pair i, j, $1 \le i, j \le k$, $(B_i - Id)b_j - (B_j - Id)b_i \in \Lambda$.*
(ii) *For each $I = (i_1, \dots, i_s)$ with $1 \le i_1 \le i_2 \le \cdots \le i_s \le k$, let $B_{i_1} L_{b_{i_1}} \dots B_{i_s} L_{b_{i_s}} = B_I L_{b(I)} \in \Gamma$, with $B_I := B_{i_1} \dots B_{i_s}$ and $b(I) = B_{i_s} \dots B_{i_2} b_{i_1} + B_{i_s} \dots B_{i_3} b_{i_2} + \cdots + B_{i_s} b_{i_{s-1}} + b_{i_s}$. Then*
$$(B_I + Id)\, b(I) \in \Lambda \setminus (B_I + Id)\, \Lambda.$$

Finally, if Γ satisfies conditions (i) and (ii), then Γ is isomorphic to a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^k$.

In what follows we state the definitions and main results used to construct quaternion Kähler compact flat manifolds.

Definition 2.4. Let Γ be a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^k$, with translation lattice Λ and such that $b \in \frac{1}{2}\Lambda$ for any $\gamma = BL_b \in \Gamma$. Let $\phi : F \to \mathbb{R}^n$ be a 1-cocycle modulo Λ. Set $E_n = \left[\begin{smallmatrix} Id & \\ & -Id \end{smallmatrix}\right] \in \mathrm{I}(\mathbb{R}^{2n})$. Set $\mathrm{d}_{q,\phi}(\Gamma, v) = \langle \mathrm{d}_\phi\Gamma, E_n L_{(v,0)}\rangle$, where $v \in \mathbb{R}^n$.

As we shall see, under rather general conditions, $\mathrm{d}_{q,\phi}(\Gamma, v)$ contains $\mathrm{d}_\phi\Gamma$ as a normal subgroup of index 2, hence if $v \in \mathbb{R}^n$ can be chosen so that $\mathrm{d}_{q,\phi}(\Gamma, v)$ is torsion free, $M_{\mathrm{d}_{q,\phi}(\Gamma,v)}$ will be a compact flat manifold with holonomy group $F \times \mathbb{Z}_2$ having as a double cover the Kähler manifold $M_{\mathrm{d}_\phi\Gamma}$ (see 2.1). Furthermore F commutes with J, but E_n only anticommutes with J. If we use this construction twice we will get a Bieberbach group $\mathrm{d}_q^2(\Gamma, v, u) := \mathrm{d}_{q,\phi'}(\mathrm{d}_{q,\phi}(\Gamma, v), u) \subset \mathrm{I}(\mathbb{R}^{4n})$ such that the holonomy group normalizes two anticommuting complex structures, J_1, J_2, on $\mathbb{R}^{4n}$, hence $\mathrm{d}_q^2(\Gamma, v, u)\backslash\mathbb{R}^{4n}$ will be a quaternion Kähler manifold. Thus, our main goal will be to give conditions on $v \in \mathbb{R}^n$ that ensure that $\mathrm{d}_{q,\phi}(\Gamma, v)$ is torsion free. We also note that if n is even, $M_{\mathrm{d}_{q,\phi}(\Gamma,v)}$ will always be orientable. We will show that this can be done for a family $\mathcal{F}$ of Bieberbach groups with holonomy group $\mathbb{Z}_2^k$ (for a description of $\mathcal{F}$, which is technical, see [4]).

Theorem 2.5. *Let Γ, ϕ be as in 2.4. Then*

(i) *If $v \in \mathbb{R}^n$ is such that $2v \in \Lambda$ and satisfies*

$$(B - Id)v \in \Lambda \text{ for each } \gamma = BL_b \in \Gamma,$$

then $d_{q,\phi}\Gamma$ is a crystallographic group with translation lattice $\Lambda\oplus\Lambda$ and holonomy group $\mathbb{Z}_2^{k+1}$. Furthermore, $d_{q,\phi}\Gamma$ is torsion-free if and only if $v \notin \Lambda$ and for each $\gamma = BL_b \in \Gamma$ we have:

$$(B + Id)(\phi(B) + v) \in \Lambda \setminus (B + Id)\Lambda, \text{ or } (B - Id)b \notin (B - Id)\Lambda.$$

(ii) *If every element in the holonomy group F commutes or anticommutes with a translation invariant complex structure and v satisfies the conditions in (i), then $d_{q,\phi}(\Gamma, v)\backslash\mathbb{R}^{2n}$ is quaternion Kähler.*

(iii) *If v satisfies the conditions in (i) we have that $\beta_1(d_{q,\phi}(\Gamma, v)\backslash\mathbb{R}^{2n}) = \beta_1(\Gamma\backslash\mathbb{R}^n)$ and $\beta_2(d_{q,\phi}(\Gamma, v)\backslash\mathbb{R}^{2n}) = 2\beta_2(d_{q,\phi}(\Gamma, v)\backslash\mathbb{R}^{2n})$. Hence, if $\beta_1(\Gamma\backslash\mathbb{R}^n)$ is odd, or if $\beta_2(\Gamma\backslash\mathbb{R}^n) = 0$ and if F satisfies the condition in (ii), then $d_{q,\phi}(\Gamma, v)\backslash\mathbb{R}^{2n}$ is quaternion Kähler and not Kähler.*

(iv) *Assume $\phi = 0$ and $\Gamma \in \mathcal{F}$. Then the vector $v = \frac{1}{2}\sum_{i=1}^n e_i$ satisfies the conditions in (i), hence $d_{q,0}(\Gamma, v)$ is a Bieberbach group. Furthermore, $d_{q,0}(\Gamma, v) \in \mathcal{F}$.*

Corollary 2.6. *In the notation of Theorem 2.5, assume $v \in \mathbb{R}^n$ is such that $d_{q,\phi}(\Gamma, v)$ is a Bieberbach group. Let ϕ' be a cocycle on F modulo $\Lambda \oplus \Lambda$. If $u \in \mathbb{R}^{2n}$ can be chosen so that $d^2_{q,\phi,\phi'}(\Gamma, v, u) := d_{q,\phi'}(d_{q,\phi}(\Gamma, v), u)$ is torsion-free, then the quotient of $\mathbb{R}^{4n}$ by $d^2_{q,\phi,\phi'}(\Gamma, v, u)$ is a quaternion Kähler manifold. In particular, if Γ is a Bieberbach group in $\mathcal{F}$ and we take $\phi = 0$, $v = \sum_{i=1}^n e_i$ and $u = \sum_{i=2n+1}^{3n} e_i$, then $d_{q,0}(\Gamma, v) \in \mathcal{F}$ and $d^2_{q,0,0}(\Gamma, v, u)\backslash\mathbb{R}^{4n}$ is a quaternion Kähler manifold.*

As it will be seen in the examples of the next section the vector v satisfying the conditions in the theorem is by no means unique, in general.

3. Quaternion Kähler flat manifolds of low dimensions

We will now illustrate the construction and results in the previous section by looking at particular Bieberbach groups in low dimensions. For more, and different examples we refer to [4]. In the examples below we will use $\phi = 0$ and we will write $\mathrm{d}^2_{q,0}(\Gamma, v, u)$ in place of $\mathrm{d}_{q,0}(\mathrm{d}_{q,0}(\Gamma, v), u)$. Furthermore it will be convenient, for any C in $\mathrm{O}(n)$, to denote by $C' \in \mathrm{O}(2n)$ the matrix $C' = \left[\begin{smallmatrix} C & \\ & C \end{smallmatrix}\right]$. Also, $C'' \in \mathrm{O}(4n)$ will have a similar meaning and Λ_n will denote the canonical lattice in $\mathbb{R}^n$.

Examples We let Γ be the Klein bottle Bieberbach group, for $n = 2$. By applying $\mathrm{d}_{q,0}$ twice to Γ, we shall obtain several 8-dimensional compact flat manifolds with holonomy group $\mathbb{Z}_2^3$ which are quaternion Kähler and not Kähler. This will follow from the explicit computation of the real cohomology.

We take $\Gamma = \langle BL_b, \Lambda_2 \rangle$, where $B = \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$, $b = \dfrac{e_1}{2}$. Then $\Gamma \backslash \mathbb{R}^2$ is a Klein bottle. If $v = \frac{1}{2}(m_1 e_1 + m_2 e_2)$, $m_1, m_2 \in \mathbb{Z}$, then

$$\mathrm{d}_{q,0}(\Gamma, v) = \langle B' L_{b'}, E_2 L_{(v,0)}, \Lambda_4 \rangle,$$

with $B' = \left[\begin{smallmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{smallmatrix}\right]$, $E_2 = \left[\begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{smallmatrix}\right]$ and $b' = \dfrac{e_3}{2}$.

We wish to find all $m_1, m_2 \in \mathbb{Z}$ such that the conditions in (i) of Theorem 2.5 are satisfied, so that $\mathrm{d}_{q,0}(\Gamma, v)$ is torsion-free.

The first condition in (i) of 2.5 clearly holds for any choice of v since $(B - Id)v = -m_2 e_2 \in \Lambda_2$. Furthermore, $v \in \frac{1}{2}\Lambda \setminus \Lambda$ if and only if at least one of the m_i's is odd. We also need that $(B + Id)v = m_1 e_1 \notin (B + Id)\Lambda_2 = \mathbb{Z}2e_1$, hence m_1 must be odd. Thus, the possible solutions, modulo Λ_2 are $v_1 = \frac{e_1}{2}$ and $v_2 = \frac{e_1+e_2}{2}$. By computing the first integral homology groups in both cases, one can show that these solutions lead to flat manifolds non homeomorphic to each other.

We now form $\mathrm{d}^2_{q,0}(\Gamma, v_i, u)$ with $i = 1, 2$ and $u = \frac{1}{2}\sum_{j=1}^4 m_j e_j$, with $m_j \in \mathbb{Z}$ to be determined. Again we need that at least one of the m_j's be odd. We now consider the second condition in (i)of 2.5 for each choice of v.

We have that

$$\mathrm{d}^2_{q,0}\left(\Gamma, \frac{e_1}{2}, u\right) = \langle B'' L_{\frac{e_7}{2}},\ E_2' L_{\frac{e_5}{2}},\ E_4\ L_{(u,0)},\ \Lambda_8 \rangle$$

$$\mathrm{d}^2_{q,0}\left(\Gamma, \frac{e_1 + e_2}{2}, u\right) = \langle B'' L_{\frac{e_7}{2}},\ E_2' L_{\frac{e_5+e_6}{2}},\ E_4\ L_{(u,0)},\ \Lambda_8 \rangle$$

where

$$B'' = \begin{bmatrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& 1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& -1 \end{bmatrix} \quad E_2' = \begin{bmatrix} Id_2 &&& \\ & -Id_2 && \\ && Id_2 & \\ &&& -Id_2 \end{bmatrix} \quad E_4 = \begin{bmatrix} Id_4 & \\ & -Id_4 \end{bmatrix}.$$

The first condition in (i) of 2.5 is clearly satisfied in both cases, for any choice of $u \in \frac{1}{2}\Lambda$, since the matrices B', E_2 are diagonal. For the second condition we also need:

$$(B' + Id)u = m_1e_1 + m_3e_3 \notin (B' + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_3),$$
$$(E_2 + Id)u = m_1e_1 + m_2e_2 \notin (E_2 + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_2),$$
$$(B'E_2 + Id)u = m_1e_1 + m_4e_4 \notin (B'E_2 + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_4).$$

These conditions are satisfied if and only if, either m_1 is odd, or if each one of m_2, m_3 and m_4 are odd. This yields the following solutions modulo Λ_4: either $u = u_Q = \frac{e_1+e_Q}{2}$, where $e_Q = \sum_{j\in Q} e_j$ and Q runs through all subsets of $\{e_2, e_3, e_4\}$, or $u = u' := \frac{e_2+e_3+e_4}{2}$. We get 9 distinct solutions, the same set for both choices $v = v_1, v = v_2$. It will be convenient to order the subsets Q as follows:

$$\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}\{2, 4\}\{3, 4\}, \{2, 3, 4\}$$

and then to set $u_j = u_Q$, for $j = 1, \dots, 8$ according to this ordering, letting $u_9 = u'$.

In this way we obtain 18 Bieberbach groups $\Gamma_{i,j} := \mathrm{d}^2_{q,0}(\Gamma, v_i, u_j)$ with $1 \leq i \leq 2$, $1 \leq j \leq 9$, so that the quotients $\Gamma_{i,j}\backslash\mathbb{R}^8$ are quaternion Kähler manifolds. We note that none of these manifolds is Kähler, since for all i, j, $\beta_1(\Gamma_{i,j}\backslash\mathbb{R}^8) = \beta_1(\Gamma\backslash\mathbb{R}^2) = 1$ and $\beta_2(\Gamma_{i,j}\backslash\mathbb{R}^8) = 2\beta_2(\Gamma\backslash\mathbb{R}^2) = 0$, by (iii) in 2.5. We also note that some of the groups may possibly be isomorphic to each other, however we see in [4] that many of them are pairwise non isomorphic, by computing $\Gamma_{i,j}/[\Gamma_{i,j}, \Gamma_{i,j}]$ in each case.

We shall first determine all Betti numbers, by giving generators of $\Lambda^h{\mathbb{R}^8}^{\mathrm{F}}$, for $1 \leq h \leq 8$.

It is clear that the space of F-invariants in $\mathbb{R}^8$ is spanned by e_1 and furthermore $\Lambda^2{\mathbb{R}^8}^{\mathrm{F}} = 0$. If $h = 3$, it is easy to see that a basis for the F-invariants is given by $e_3 \wedge e_5 \wedge e_7, e_2 \wedge e_3 \wedge e_4, e_3 \wedge e_6 \wedge e_8, e_2 \wedge e_5 \wedge e_6, e_2 \wedge e_7 \wedge e_8, e_4 \wedge e_6 \wedge e_7, e_4 \wedge e_5 \wedge e_8$, hence $\beta_3 = \beta_5 = 7$.

By Poincaré duality we have that $\chi(\Gamma_{i,j}\backslash\mathbb{R}^8) = 2 - 2\beta_1 + 2\beta_2 - 2\beta_3 + \beta_4 = 0$, hence (since $\beta_1 = 1, \beta_2 = 0, \beta_3 = 7$) we get $\beta_4 = 2\beta_3 = 14$. We may check this value by finding a basis for the F-invariants in $\Lambda^4\mathbb{R}^8$. This is given by vectors of the form $e_i \wedge e_j \wedge e_k \wedge e_l$, with $\{i, j, k, l\}$ running through the sets

$$\{1, 3, 5, 7\},\ \{2, 4, 6, 8\},\ \{1, 2, 5, 6\},\ \{3, 4, 7, 8\},\ \{2, 3, 5, 8\},\ \{1, 2, 3, 4\},\ \{5, 6, 7, 8\},$$
$$\{1, 3, 6, 8\},\ \{1, 2, 7, 8\},\ \{2, 4, 5, 7\},\ \{1, 4, 6, 7\},\ \{2, 3, 6, 7\},\ \{2, 4, 5, 7\},\ \{1, 4, 5, 8\}$$

Summing up, we get that the Poincaré polynomial of each one of the flat manifolds $\Gamma_{i,j}\backslash\mathbb{R}^8$ is $p(t) = 1 + t + 7t^3 + 14t^4 + 7t^5 + t^7 + t^8$.

We thus have 2-fold coverings $M_{\mathrm{d}^2_0\Gamma} \to M_{\Gamma_{ij}}$, where $M_{\mathrm{d}^2_0\Gamma}$ is hyperkähler, by Proposition 3.2, and $M_{\Gamma_{i,j}}$ does not admit any Kähler structure, since $\beta_1(M_{\Gamma_{i,j}}) = 1$, for all i, j.

To conclude this example, one can show (see [4]) that many of the manifolds $M_{\Gamma_{ij}}$ are non homeomorphic to each other, by computing the first integral homology groups, $H_1(M_{\Gamma_{ij}}, \mathbb{Z}) \simeq \Gamma_{ij}/[\Gamma_{ij}, \Gamma_{ij}]$.

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