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HYPERHOLOMORPHIC FUNCTIONS IN $\mathbb{R}^4$

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ABSTRACT. Let $\mathbb{H}$ be the algebra of quaternions generated by e_1, e_2 and e_{12} satisfying $e_1 e_2 = e_{12}$ and $e_i e_j + e_j e_i = -2\delta_{ij}$ for $i = 1, 2, 12$. Any element x in $\mathbb{H} \oplus \mathbb{H}$ may be decomposed as $x = Px + Qxe_3$ for quaternions Px and Qx. The generalized Cauchy-Riemann operator in $\mathbb{R}^4$ is defined by $D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3$. Leutwiler noticed that the power function $(x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3)^m$ is the solution of the generalized Cauchy-Riemann system $x_3 Df + 2f_3 = 0$ which has connections to the hyperbolic metric. We study solutions of the equation $x_3 Df + 2Q'(f) = 0$ (the prime $'$ is the main involution) called hyperholomorhic functions. If $f = f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3$ for some real functions f_0, f_1, f_2, f_3 then f is the solution of the generalized Cauchy-Riemann system stated earlier.

1. INTRODUCTION

Let $\mathbb{H}$ be the associative algebra of quaternions generated by e_1, e_2 and e_{12} satisfying the usual relations

$$e_1^2 = e_2^2 = e_{12}^2 = -1$$
$$e_1 e_2 = e_{12} = -e_{12}.$$

Set $e_1 e_2 = e_{12}$. The conjugation $\overline{q}$ of the quaternion $q = t + xe_1 + ye_2 + ze_{12}$ is defined by

$$\bar{q} = t - xe_1 - ye_2 - ze_{12}.$$

The involution $' : \mathbb{H} \to \mathbb{H}$ is the isomorphism defined by

$$q' = t - xe_1 - ye_2 + ze_{12}.$$

The second involution $^* : \mathbb{H} \to \mathbb{H}$, called **reversion**, is the anti-isomorphism defined by

$$q^* = t + xe_1 + ye_2 - ze_{12}.$$

We consider the set $\mathbb{H} \oplus \mathbb{H}$ with the usual addition and the multiplication defined by

$$(u_1, u_2)(v_1, v_2) = (u_1 v_1 - u_2 v_2', u_1 v_2 + u_2 v_1'). \tag{1}$$

It is known that

$$\mathbb{H}\oplus\mathbb{H} \cong C\ell_{0,3},$$

where $C\ell_{0,3}$ is the Clifford algebra generated by the elements e_1, e_2 and e_3 satisfying the relation $e_ie_j + e_je_i = -2\delta_{ij}$ for the usual Kronecker delta δ_{ij}. The isomorphism $\varphi : \mathbb{H}\oplus\mathbb{H} \to Cl_{0,3}$ is given by $\varphi(q_1, q_2) = q_1 + q_2e_3$. We identify the space $\mathbb{H}\oplus\mathbb{H}$ with $Cl_{0,3}$.

The elements $x = t + xe_1 + ye_2 + we_3$ for $t, x, y, w \in \mathbb{R}$ are called *paravectors* in $\mathbb{H}\oplus\mathbb{H}$. The space $\mathbb{R}^4$ is identified with the set of paravectors. We also denote $e_0 = 1$.

The involution $()'$ is extended to an isomorphism in $\mathbb{H}\oplus\mathbb{H}$ by

$$(q_1 + q_2e_3)' = q_1' - q_2'e_3 \quad (q_i \in \mathbb{H}) . \tag{2}$$

Note that

$$e_3q = q'e_3 \tag{3}$$

for any $q \in \mathbb{H}$.

The involution $*$ is extended to $\mathbb{H}\oplus\mathbb{H}$ as follows

$$e_3^* = e_3,\ (ab)^* = b^*a^*$$
$$(a+b)^* = a^* + b^*$$

and the conjugation by $\overline{a} = (a^*)' = (a')^*$. Note that $\overline{ab} = \overline{b}\,\overline{a}$.

Paravectors can be characterized as follows.

Lemma 1. *An element $x \in \mathbb{H}\oplus\mathbb{H}$ is a paravector if and only if*

$$\sum_{i=0}^{3} e_ixe_i = -2x'. \tag{4}$$

Proof. It is easy to see that the equation (4) holds for all e_i with $i = 0, ..., 3$. Using the linearity we infer that it holds for all paravectors. Conversely, calculate first

$$\sum_{i=0}^{3} e_ie_je_ke_i = 2e_je_k \quad \text{for } 0 < j < k \leq 3$$
$$\sum_{i=0}^{3} e_ie_1e_2e_3e_i = -2e_1e_2e_3.$$

Hence comparing the components of the left and right side of the equality (4) we note that the equality (4) implies that x has to be a paravector. □

The projection operators $P : \mathbb{H}\oplus\mathbb{H}\to\mathbb{H}$ and $Q : \mathbb{H}\oplus\mathbb{H}\to\mathbb{H}$ are defined by equations $P(q_1 + q_2 e_3) = q_1$ and $Q(q_1 + q_2 e_3) = q_2$ for $q_i \in \mathbb{H}$. Applying (2) we note that $P(a') = (Pa)'$ and $Q(a') = -(Qa)'$. By virtue of (1) we obtain

$$P(ab) = PaPb - Qa(Qb)', \tag{5}$$

$$Q(ab) = PaQb + Qa(Pb)'. \tag{6}$$

The notation $f \in \mathcal{C}^k(\Omega)$ for a function $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ means that the real coordinate functions of f are k-times continuously differentiable on $\Omega \subset \mathbb{R}^4$.

2. Hyperholomorphic functions

The Cauchy Riemann operator is defined by

$$Df = \sum_{i=0}^{3} e_i \frac{\partial f}{\partial x_i}$$

for a mapping $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$, $\Omega \subset \mathbb{R}^4$, whose components are partially differentiable. An operator $\overline{D}$ is defined by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - \sum_{j=1}^{3} e_j \frac{\partial f}{\partial x_j}.$$

Note that $D\overline{D} = \overline{D}D = \triangle$, where $\triangle$ is the Laplace operator in $\mathbb{R}^4$. If $Df = 0$ the function f is called (left) *monogenic*. For the reference on properties of monogenic functions in the general case see [1] and for quaternions [21].

Using (5) we obtain

$$P(Df) = \sum_{i=0}^{2} e_i \frac{\partial Pf}{\partial x_i} - \frac{\partial Q'f}{\partial x_3}. \tag{7}$$

Similarly, the property (6) implies that

$$Q(Df) = \sum_{i=0}^{2} e_i \frac{\partial Qf}{\partial x_i} + \frac{\partial P'f}{\partial x_3}. \tag{8}$$

The **modified Cauchy-Riemann operator** M is defined by

$$(Mf)(x) = (Df)(x) + \frac{2}{x_3} Q'f$$

and the operator $\overline{M}$ by

$$\left(\overline{M}f\right)(x) = \left(\overline{D}f\right)(x) - \frac{2}{x_3} Q'f.$$

Note that

$$(Mf)(x) + \left(\overline{M}f\right)(x) = (Df)(x) + \left(\overline{D}f\right)(x) = 2\frac{\partial f}{\partial x_0}$$

and

$$\overline{D}f = \left(D\left(f'\right)\right)', \quad \overline{M}f = \left(M\left(f'\right)\right)'. \tag{9}$$

Let $\Omega \subset \mathbb{R}^4$ be open. If $f \in \mathcal{C}^2(\Omega)$ and $Mf(x) = 0$ for any $x \in \Omega \backslash \{x \mid x_3 = 0\}$, the function f is called **hyperholomorphic** in Ω. If f is hyperholomorphic in Ω and $f = \sum_{i=0}^{3} f_i e_i$ for some real functions f_i the function f is called an ***H*-solution**.

The H-solutions in $\mathbb{R}^3$ were introduced by H. Leutwiler ([17]).They are notably studied in $\mathbb{R}^n$ by H. Leutwiler ([18], [19], [20]), H. Hempfling ([13], [14], [15]), J. Cnops ([4]), P. Cerejeiras ([3]) and S.-L. Eriksson-Bique ([6], [11], [8], [9]). In $\mathbb{R}^3$ the hyperholomorphic functions are researched by W. Hengartner and H. Leutwiler ([16]) and in $\mathbb{R}^n$ by H. Leutwiler and S.-L. Eriksson-Bique ([11]).

Applying (7) and (8) for $x_3 Mf = 0$ we obtain the following system.

Proposition 2. *Let Ω be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathbb{H} \oplus \mathbb{H}$ be a mapping with continuous partial derivatives. The equation $Mf = 0$ is equivalent with the system of equations*

$$\begin{aligned} &x_3 \left(D_2(Pf) - \tfrac{\partial (Qf)'}{\partial x_3}\right) + 2Q'f = 0, \\ &D_2(Qf) + \tfrac{\partial P'(f)}{\partial x_3} = 0, \end{aligned}$$

where $D_2 = \sum_{i=0}^{2} e_i \frac{\partial f}{\partial x_i}$.

Using the preceding result we infer the relation between monogenic and hyperholomorphic functions.

Proposition 3. *Let Ω be an open subset of $\mathbb{R}^4$. Then if $f : \Omega \to \mathbb{H} \oplus \mathbb{H}$ is monogenic and $Qf = 0$, then f is hyperholomorphic.*

Proposition 4. *Let $f : \Omega \to \mathbb{H} \oplus \mathbb{H}$ be twice continuously differentiable on an open subset Ω of $\mathbb{R}^4$. Then*

$$\begin{aligned} \overline{M}Mf = M\overline{M}f &= \triangle(Pf) - \frac{2}{x_3}\frac{\partial Pf}{\partial x_n} \\ &+ \left(\triangle(Qf) - \frac{2}{x_3}\frac{\partial Qf}{\partial x_3} + \frac{2}{x_3^2}Qf\right)e_3, \end{aligned}$$

where $\triangle$ is the Laplacian in $\mathbb{R}^4$.

Proof. The property (8) implies that

$$QMf = QDf + \frac{2}{x_3}Q\left(Q'f\right) = D_2 Qf + \frac{\partial P'f}{\partial x_3}.$$

Hence we have

$$\overline{M}Mf = \overline{D}Df + 2\overline{D}\left(\frac{Q'f}{x_3}\right) - \frac{2}{x_3}\left((D_2Qf)' + \frac{\partial Pf}{\partial x_3}\right).$$

Since by (9) and (3)

$$\begin{aligned}\overline{D}\left(\frac{Q'f}{x_3}\right) &= \frac{\overline{D}Q'f}{x_3} + \frac{e_3Q'f}{x_3^2} = \frac{(DQf)'}{x_3} + \\ &= \frac{(D_2Qf)'}{x_3} - \frac{1}{x_3}\frac{\partial Qf}{\partial x_3}e_3 + \frac{Qf}{x_3^2}e_3,\end{aligned}$$

we obtain

$$\overline{M}Mf = \triangle Pf - \frac{2}{x_3}\frac{\partial Pf}{\partial x_3} + \left(\triangle Qf - \frac{2}{x_3}\frac{\partial Qf}{\partial x_3} + 2\frac{Qf}{x_3^2}\right)e_3.$$

From the definitions we note that $Mf + \overline{M}f = 2\frac{\partial f}{\partial x_0}$. Hence we obtain

$$\begin{aligned}M\overline{M}f + M^2f &= 2M\left(\frac{\partial f}{\partial x_0}\right) = 2\frac{\partial Mf}{\partial x_0} \\ &= \left(M + \overline{M}\right)(Mf) = \overline{M}Mf + M^2f,\end{aligned}$$

which implies $M\ \overline{M}f = \overline{M}Mf$, completing the proof. □

Corollary 5. *If $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ is hyperholomorphic then*

$$x_3\triangle Pf - 2\frac{\partial Pf}{\partial x_3} = 0 \tag{10}$$

and

$$-2Qf = x_3^2\triangle(Qf) - 2x_3\frac{\partial Qf}{\partial x_3}. \tag{11}$$

Conversely, if the equations (10) and (11) hold, then $\overline{M}f$ is hyperholomorphic.

The equation (10) is the Laplace-Beltrami equation associated with the hyperbolic metric

$$ds^2 = x_3^{-2}\left(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^3\right).$$

The second equation (11) presents the eigenfunctions of the Laplace-Beltrami operator corresponding to the eigenvalue -2.

Using standard methods we obtain the following observation.

Lemma 6. *Let $u : \Omega \to \mathbb{R}$ be twice continuously differentiable on an open subset Ω of $\mathbb{R}^4$. If u satisfies the equation*

$$-2u(x) = x_3^2\triangle u(x) - 2x_3\frac{\partial u}{\partial x_3}(x), \quad x = (x_0, x_1, x_2, x_3) \tag{12}$$

then the mapping $g : \Omega \to \mathbb{R}$ *defined by*

$$g(x) = \begin{cases} \frac{u(x)}{x_3}, & \text{if } x_3 \neq 0, \\ \frac{\partial u}{\partial x_3}(x), & \text{if } x_3 = 0, \end{cases}$$

is harmonic on $\Omega \setminus \{x_3 = 0\}$*. Moreover, if* $u \in \mathcal{C}^3(\Omega)$*, then* g *is harmonic on* Ω*.*

Proof. If $x_3 \neq 0$, then

$$\triangle g = \frac{\triangle u}{x_3} - \frac{2\partial u}{x_3^2 \partial x_3} + \frac{2u}{x_3^3}.$$

Using (12) we note that $\triangle g = 0$. Since by (12) $\frac{\partial g}{\partial x_3} = \frac{1}{2} \triangle u$, we see that $g \in \mathcal{C}^2(\Omega)$ and therefore $\triangle g = 0$ on Ω provided that $u \in \mathcal{C}^3(\Omega)$. $\square$

Proposition 7. *If* $f : \Omega \to \mathbb{H} \oplus \mathbb{H}$ *is hyperholomorphic and* $f \in \mathcal{C}^3(\Omega)$*, then* $\triangle f$ *is monogenic and therefore also harmonic.*

Proof. Assume that $f : \Omega \to \mathbb{H} \oplus \mathbb{H}$ is hyperholomorphic. Using preceding Lemma and $\triangle = D\overline{D}$ we obtain

$$0 = D\overline{D}Mf = D\overline{D}Df + 2\triangle\left(\frac{Q'f}{x_3}\right) = D(\triangle f).$$

Hence $\triangle f$ is monogenic and furthermore harmonic. $\square$

Example 8. *Let* $f = u + iv$ *be holomorphic in an open set* $\Omega \subset \mathbf{C}$*. We define the mapping* $\pi : \mathbb{R}^4 \to \mathbb{C}$ *by*

$$\pi(x_0, x_1, x_2, x_3) = x_0 + x_1 i$$

and the mapping $\rho : \mathbb{R}^4 \to \mathbb{C}$ *by*

$$\rho(x_0, x_1, x_2, x_3) = x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then the function $f \circ \pi$ *is hyperholomorphic in the set* $\{x \mid \pi(x) \in \Omega\}$*. Moreover the function* $\widetilde{f}$ *defined by*

$$\widetilde{f}(x) = u \circ \rho(x) + \frac{x_1 e_1 + x_2 e_2 + x_3 e_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} v \circ \rho(x)$$

is hyperholomorphic in the set $\{x \mid \rho(x) \in \Omega\}$ *(see* [17, p.157]*).*

Proposition 9. *The space of hyperholomorphic functions in an open subset* Ω *of* $\mathbb{R}^4$ *forms a right quaternionic vector space.*

Proof. Let $q \in \mathbb{H}$. Using (6) we notice that

$$M(fq) = (Df)q + 2\frac{Q'(fq)}{x_3} = (Mf)q,$$

implying the assertion. $\square$

The following result is easy to see.

Lemma 10. *Let Ω be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then $\frac{\partial f}{\partial x_l}$ is hyperholomorphic for $l = 0, 1, 2$.*

Product of hyperholomorphic functions is not necessarily hyperholomorphic.

Theorem 11. *Let Ω be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then the product $f(x)\,x$ is hyperholomorphic if and only if f is an H-solution.*

Proof. Assume that $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ and $f(x)\,x$ are hyperholomorphic. Then by (6) we have

$$
\begin{aligned}
0 = M(fx) &= (Df)\,x + \sum_{i=0}^{3} e_i f e_i + 2\frac{Q'(fx)}{x_3} \\
&= (Df)\,x + 2\frac{Q'f(Px)}{x_3} + 2\frac{P'(f)\,Q'x}{x_3} + \sum_{i=0}^{3} e_i f e_i.
\end{aligned}
$$

Since $Px = x - x_3 e_3$ and $Qx = x_3$ we obtain

$$
0 = (Mf)\,x + \sum_{i=0}^{3} e_i f e_i - 2Q' f e_3 + 2P'(f) = \sum_{i=0}^{3} e_i f e_i + 2f'.
$$

By virtue of Lemma 1 we find out that f is para vector valued. The converse statement is proved similarly. $\square$

Corollary 12. *The function x^m is an H-solution.*

Theorem 13. *Let Ω be an open subset of $\mathbb{R}^4$ and $F : \Omega \to \mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then the function $f(x) = F(x)\,x^{-1}$ is hyperholomorphic in $\Omega\backslash\{0\}$ if and only if it is paravector valued.*

Proof. Assume that $F : \Omega \to \mathbb{H}\oplus\mathbb{H}$ and $f(x) = F(x)\,x^{-1}$ are hyperholomorphic. Since $F(x) = \left(F(x)\,x^{-1}\right)x$ we obtain from the preceding theorem that $\left(F(x)\,x^{-1}\right)$ is para vector valued. On the other hand, assume that f is vector valued and F is hyperholomorphic. Then by (6) and Lemma 1, we conclude

$$
0 = MF = (Mf)\,x + \sum_{j=0}^{n} e_j f e_j + 2f' = (Mf)\,x
$$

and therefore f is also hyperholomorphic. $\square$

Corollary 14. *The function x^{-m} is an H-solution.*

Theorem 15. *Let Ω be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ twice continuously differentiable. Then f is hyperholomorphic if and only if for any $a \in \Omega$ and any ball*

$B(a,r)$ with $B(a,r) \subset \Omega$ there exists a continuous differentiable mapping H from $B(a,r)$ into $\mathbb{H}$ satisfying the equations

$$f = \overline{D}H \tag{13}$$

and

$$x_n \triangle H - 2\frac{\partial H}{\partial x_n} = 0 \tag{14}$$

on $B(a,r)$.

Proof. Assume that a mapping $H : B(x,r) \to \mathbb{H}$ satisfies (14) and $f = \overline{D}H$. Since $D\overline{D} = \triangle$ we have $x_3 Df = x_3 \triangle H$. The equality $-2\frac{\partial H}{\partial x_n} = Q'f$ follows from

$$Qfe_3 = Q\overline{D}H = -e_3\frac{\partial H}{\partial x_n}.$$

Hence $Mf = 0$ and therefore f is hyperholomorphic.

Conversely assume that $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ is hyperholomorphic. Set $Pf = f_0 + f_1e_1 + f_2e_2 + f_{12}e_{12}$. Let $B(a,r)$ be a ball in $\mathbb{R}^4$ centered at $a = (a_0, ..., a_n)$ satisfying $B(a,r) \subset \Omega$. Let $s_i : (B(a,r) \cap \{x \mid x_3 = a_3\})^{\sim} \to \mathbb{R}$ be a twice continuously differentiable solution of the Poisson equation (which exists for example by [2, p.171])

$$\triangle s_i(\tilde{x}) = f_i(\tilde{x}, a_3).$$

for $i = 0, 1, 2, 12$ and a ball $B((a_0, a_1, a_2), r)$. Set $s = \sum_{i\subset\{0,1,2,12\}} s_i e_i$ and define a mapping $H : B(a,r) \to \mathbb{H}$ by

$$H(x) = -\int_{a_3}^{x_3} Q'f(\tilde{x}, t)\,dt + D_2 s.$$

Then we have by Proposition 2

$$\begin{aligned}\overline{D}H(x) &= e_3 2Q'f(x) - \int_{a_n}^{x_n} \overline{D_2}Q'f(\tilde{x}, t)\,dt + \overline{D_2}D_2 s\\ &= Q_2 f(x)\,e_n + \int_{a_n}^{x_n} \frac{\partial Pf}{\partial x_n}(\tilde{x}, t)\,dt + (Pf)(\tilde{x}, a_n)\\ &= f(x).\end{aligned}$$

Since the image space of H is $\mathbb{H}$ we note that

$$Qfe_3 = -e_3\frac{\partial H}{\partial x_3}.$$

Using the assumption f is hyperholomorphic we obtain

$$0 = x_3 Df + 2Q'f = x_3 D\overline{D}H - 2\frac{\partial H}{\partial x_3}.$$

Hence the mapping H satisfies (14), completing the proof. □

Hyperholomorphic functions may be obtained from H-solutions as follows.

Theorem 16. *A mapping f is hyperholomorphic on a ball $B(a,r) \subset \mathbb{R}^4$ if and only if there exist H-solutions g_i such that*

$$f = g_0 + g_1 e_1 + g_2 e_2 + g_{12} e_{12}.$$

Proof. Assume that f is hyperholomorphic on a ball $B(a,r) \subset \mathbb{R}^4$. Applying Theorem 15 we find a mapping H from $B(a,r)$ into $\mathbb{H}$ satisfying the equations (14) and $f = \overline{D}H$. Denote $H = h_0 + h_1 e_1 + h_2 e_2 + h_{12} e_{12}$ for real functions h_i. Then the mapping $g_i = \overline{D} h_i$ is vector valued and therefore an H-solution. Clearly we have $f = g_0 + g_1 e_1 + g_2 e_2 + g_{12} e_{12}$. □

Corollary 17. *If f is hyperholomorphic on Ω, then it is real analytic on $\Omega \setminus \{x_3 = 0\}$. Moreover, if $f \in \mathcal{C}^3(\Omega)$is hyperholomorphic on Ω, then it is real analytic on Ω.*

Proof. The H-solutions are real-analytic by [8, Theorem 4]. Hence the preceding theorem implies the statement. □

The fundamental homogeneous polynomial H-solutions are defined as follows.

Definition 18. *The homogeneous polynomials L_m^α are defined for any multi-index $\alpha \in \mathbb{N}_0^2$ and a non-negative integer m by*

$$L_m^\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} x^{m+|\alpha|}}{\partial x^\alpha}.$$

The homogeneous polynomial H-solution T_m^α of degree m is defined by

$$\frac{\partial T_m^\alpha}{\partial x^\alpha} = \alpha! x_3^2 e_3$$

and $T_m^\alpha(x) = 0$ for any x with $x_3 = 0$.

Theorem 19. *The set $\{L_m^\alpha \mid |\alpha| \le m\} \cup \{T_m^\alpha \mid |\alpha| = m - n + 1\}$ is a basis of the right $\mathbb{H}$-module of homogeneous hyperholomorphic polynomials of degree m.*

Proof. Using [8, Theorem 4] we obtain that the set in question is a basis of the right $\mathbb{H}$-module generated by the homogeneous polynomial H-solutions of degree m. If f is a homogenous hyperholomorphic polynomial of degree m, then by Theorem 16 there exists homogeneous polynomial H-solutions p_i of degree m satisfying $f = \sum_{i \in \{0,1,2,12\}} p_i e_i$. Hence it is also a basis of the right $\mathbb{H}$-module of homogeneous hyperholomorphic polynomials of degree m. □

Theorem 20. *Let $f \in \mathcal{C}^3(\Omega)$ be hyperholomorphic in a neighborhood of a point $x = (x_0, x_1, x_2, 0)$. Then there exist constants $b_k(\alpha), c(\alpha) \in \mathbb{H}$ such that*

$$f = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=0}^{k} L_k^\alpha b_k(\alpha) + \sum_{|\alpha|=k-2} T_k^\alpha c_k(\alpha) \right)$$

Proof. Assume that $f \in \mathcal{C}^3(\Omega)$ is hyperholomorphic in a neighborhood of a point $x = (x_0, x_1, x_2, 0)$. If $T(y) = y + a$ for $a \in \mathbb{R}^4$ with the last coordinate $a_3 = 0$ then $f \circ T$ is also hyperholomorphic. Hence we may assume that $x = 0$. Since f is hyperholomorphic, f is real analytic and therefore admits the presentation

$$f(y) = \sum_{\beta \in \mathbb{N}_0^{n+1}} a(\beta)\, y^{\beta}.$$

in some neighborhood $B_r(0)$. Applying M we obtain

$$0 = Mf(y) = \sum_{k=0}^{\infty} M\left(\sum_{|\beta|=k} a(\beta)\, y^{\beta}\right).$$

Since $M\left(\sum_{|\beta|=k} a(\beta)\, y^{\beta}\right)$ is a homogeneous polynomial of degree k we infer

$$M\left(\sum_{|\beta|=k} a(\beta)\, y^{\beta}\right) = 0.$$

This implies that $\sum_{|\beta|=k} a(\beta)\, y^{\beta}$ is hyperholomorphic. Applying Theorem 19 we obtain the result. $\square$

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