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DIFFERENTIAL GEOMETRY OF CIRCLES IN A COMPLEX PROJECTIVE SPACE

SADAHIRO MAEDA AND TOSHIAKI ADACHI

ABSTRACT. It is well-known that all geodesics in a Riemannian symmetric space of rank one are congruent each other under the action of isometry group. In this paper we are interested in *circles* in a quaternionic projective space $\mathbb{Q}P^n$. Recently we have known that each circle in $\mathbb{Q}P^n$ is congruent to a circle in $\mathbb{C}P^n$ which is a totally geodesic submanifold of $\mathbb{Q}P^n$. This fact leads us to the study about circles in $\mathbb{C}P^n$.

1. INTRODUCTION

A smooth curve $\gamma : \mathbb{R} \to M$ parametrized by its arclength s in a complete Riemannian manifold M is called a *circle* of curvature $\kappa(\geqq 0)$, if there exists a field of unit vectors Y_s along the curve satisfying the following equations: $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s$ and $\nabla_{\dot\gamma}Y_s = -\kappa\dot\gamma$, where κ is a non-negative constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. A circle of null curvature is nothing but a geodesic. For given a point $x \in M$, orthonormal pair of vectors $u, v \in T_xM$ and for given each positive constant κ, we have a unique circle $\gamma = \gamma(s)$ of curvature κ satisfying the initial condition that $\gamma(0) = x$, $\dot\gamma(0) = u$ and $(\nabla_{\dot\gamma}\dot\gamma)(0) = \kappa v$. It is known that in a complete Riemannian manifold every circle can be defined for $-\infty < s < \infty$ (cf. [N]).

In general, a circle in a Riemannian manifold is not closed. Here, a curve $\gamma = \gamma(s)$ is said to be *closed* if there exists a positive s_0 with $\gamma(s + s_0) = \gamma(s)$ for every s. For a circle γ, the definition of closedness of γ can be rewritten as follows: A circle γ is said to be closed if there exists a positive s_0 with

$$\gamma(s_0) = \gamma(0),\ \dot\gamma(s_0) = \dot\gamma(0) \quad \text{and} \quad (\nabla_{\dot\gamma}\dot\gamma)(s_0) = (\nabla_{\dot\gamma}\dot\gamma)(0).$$

Of course, any circles of positive curvature in Euclidean n-space $\mathbb{R}^n$ are closed. And also any circles in Euclidean n-sphere $S^n(c)$ are closed. But in a real hyperbolic space n-space $H^n(c)$, there exist many open circles. In fact, a circle of curvature κ is closed if and only if $\kappa > \sqrt{|c|}$ (see [C]).

In this paper we make mention of length of circles. For a closed curve γ, we call the minimum positive constant s_0 with the condition $\gamma(s + s_0) = \gamma(s)$ for every s its *length*, and denote by Length(γ). For an open circle, a circle which is not closed, we put its length as Length$(\gamma) = \infty$. In order to get rid of the influence of the action of the full isometry group, we shall consider the moduli space of circles under the action of isometries. We say that two circles γ_1 and γ_2 are congruent each other if there exist an isometry φ and a constant s_1 with $\gamma_2(s) = \varphi \circ \gamma_1(s + s_1)$ for each s. The moduli space Cir(M) of circles is the quotient space of the set of all circles in M under this congruence relation. The *length spectrum* of circles in M is the map $\mathfrak{L}: \ \mathrm{Cir}(M) \to \mathbb{R} \cup \{\infty\}$ defined by $\mathfrak{L}([\gamma]) = \mathrm{Length}(\gamma)$. Sometimes we also call the image $\mathrm{LSpec}(M) = \mathfrak{L}(\mathrm{Cir}(M)) \cap \mathbb{R}$ in the real line the lenth spectrum of circles on M.

In a real space form $M^n(c) (= S^n(c),\ \mathbb{R}^n$ or $H^n(c))$ of constant sectional curvature c, circles are well-understood. In these spaces, two circles are congruent each other if and only if they have the same curvature. If the curvature of a circle is κ, then its length is $\frac{2\pi}{\sqrt{\kappa^2+c}}$ in $S^n(c)$, $\frac{2\pi}{\kappa}$ in $\mathbb{R}^n$ and $\frac{2\pi}{\sqrt{\kappa^2+c}}$ when $\kappa > \sqrt{|c|}$ in $H^n(c)$. Therefore length spectrum of these spaces are $\mathrm{LSpec}(S^n(c)) = \left(0, \frac{2\pi}{\sqrt{c}}\right]$, $\mathrm{LSpec}(\mathbb{R}^n) = \mathrm{LSpec}(H^n(c)) = (0, \infty)$. So we treat an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c and a quaternionic projective space $\mathbb{Q}P^n(c)$ of constant quaternionic sectional curvature c as model spaces. We are particularly interested in the following problem:

Problem In a complex projective space $\mathbb{C}P^n(c)$ (resp. a quaternionic projective space $\mathbb{Q}P^n(c)$), for each positive ℓ does there exist a unique closed circle γ whose length is ℓ up to an isometry of $\mathbb{C}P^n(c)$ (resp. $\mathbb{Q}P^n(c)$)?

In order to give an answer to this problem, we shall study the length spectrum of circles in $\mathbb{C}P^n(c)$ in detail (see section 4).

2. Congruence theorem for circles

In order to state the congruence theorem for circles in a complex projective space, we introduce an important invariant for circles in a Kähler manifold. Let (M, J) be a Kähler manifold with complex structure J. For a circle $\gamma = \gamma(s)$ in M satisfying the equations $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s$ and $\nabla_{\dot\gamma} Y_s = -\kappa\dot\gamma$, we call $\tau = \langle \dot\gamma, JY_s\rangle$ its *complex torsion*. The complex torsion τ is constant along γ. Indeed,

$$\begin{aligned}\nabla_{\dot\gamma}\langle\dot\gamma, JY_s\rangle &= \langle\nabla_{\dot\gamma}\dot\gamma, JY_s\rangle + \langle\dot\gamma, J\nabla_{\dot\gamma}Y_s\rangle \\ &= \kappa\cdot\langle Y_s, JY_s\rangle - \kappa\cdot\langle\dot\gamma, J\dot\gamma\rangle = 0.\end{aligned}$$

Clearly it satisfies $|\tau| \leqq 1$. We denote by $M_n(c)$ an n-dimensional complete simply connected complex space form of constant holomorphic sectional curvature c. It is well-known that any isometry φ of a non-flat complex space form $M_n(c) (= \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c))$ is holomorphic or anti-holomorphic. The congruence theorem for circles in $M_n(c),\ c \neq 0$ is stated as follows (see Theorem 5.1 in [MO]):

Proposition 2.1. Two circles in a non-flat complex space form $M_n(c)$ are congruent if and only if they have the same curvatures and the same absolute values of complex torsions.

For a circle γ in a quaternionic Kähler manifold $(M, \{I, J, K\})$ with quaternionic Kähler structure $\{I, J, K\}$, the corresponding invariant *structure torsion* τ is defined by

$$\tau = \sqrt{\langle\dot{\gamma}, IY_s\rangle^2 + \langle\dot{\gamma}, JY_s\rangle^2 + \langle\dot{\gamma}, KY_s\rangle^2}.$$

On a quaternionic projective space and on a quaternionic hyperbolic space, this invariant can be interpreted in terms of the sectional curvature $\mathrm{Riem}(\dot{\gamma}, Y)$ of the plane spanned by $\dot{\gamma}$ and Y: $\mathrm{Riem}(\dot{\gamma}, Y) = \frac{c}{4}(1 + 3\tau^2)$, where c is the quaternionic sectional curvature of the base manifold.

Proposition 2.2. Two circles in a quaternionic projective space or in a quaternionic hyperbolic space are congruent if and only if they have the same curvatures and the same structure torsions.

Since a quaternionic projective (resp. hyperbolic) space contains a complex projective (resp. hyperbolic) space as a totally geodesic submanifold, we are enough to study circles in a complex space form (c.f. [A1]). For a circle γ in a Cayley plane and in a Cayley hyperbolic plane we can define its invariant by $\mathrm{Riem}(\dot{\gamma}, Y)$ and obtain congruence theorem of the same type (see [MT]). In the following, we only study on a complex projective space $\mathbb{C}P^n(c)$. But all the results similarly hold for a quaternionic projective space $\mathbb{Q}P^n(c)$ of constant quaternionic sectional curvature c and for a Cayley plane of maximal sectional curvature c.

3. When is a circle closed in $\mathbb{C}P^n(c)$?

We first suppose that a complex projective space $\mathbb{C}P^n$ is furnished with the standard metric of constant holomorphic sectional curvature 4. First of all we are devoted to the study about circles of curvature $\frac{1}{\sqrt{2}}$.

Our main tool is the following parallel isometric imbedding h of $S^1 \times S^{n-1}/\phi$ into $\mathbb{C}P^n(4)$. Here the identification ϕ is defined by

$$\phi((e^{i\theta}, a_1, \ldots, a_n)) = (-e^{i\theta}, -a_1, \ldots . -a_n),$$

where $\Sigma a_j^2 = 1$. The isometric imbedding $h: \ S^1 \times S^{n-1}/\phi \to \mathbb{C}P^n(4)$ is defined by

$$h(e^{i\theta}; a_1, \ldots, a_n) = \pi\left(\begin{pmatrix} \frac{1}{3}(e^{-2i\theta/3} + 2a_1 e^{i\theta/3}) \\ \frac{\sqrt{2}}{3}(e^{-2i\theta/3} - a_1 e^{i\theta/3}) \\ \frac{2}{\sqrt{6}} i a_2 e^{i\theta/3} \\ \vdots \\ \frac{2}{\sqrt{6}} i a_n e^{i\theta/3} \end{pmatrix}\right),$$

where $\pi: \ S^{2n+1}(1) \to \mathbb{C}P^n(4)$ is the Hopf fibration.

We recall that the map h is injective and that for each geodesic γ on $M = S^1 \times S^{n-1}/\phi$ the curve $h \circ \gamma$ is a circle of $\frac{1}{\sqrt{2}}$ in $\mathbb{C}P^n(4)$ (for details, see [N]). Hence, investigating all geodesics on M, we obtain the following theorem which gives us information about all circles of curvature $\frac{1}{\sqrt{2}}$ in $\mathbb{C}P^n$.

Theorem 2.3. For any unit vector $X = \alpha u + v \in T_x(S^1 \times S^{n-1}/\phi) \backsimeq T_{x_1}S^1 \oplus T_{x_2}S^{n-1}$ at a point x, we denote by γ_X the geodesic along X on $S^1 \times S^{n-1}/\phi$. Then the circle $h \circ \gamma_X$ on $\mathbb{C}P^n(4)$ satisfies the following properties:

1. The curvature of $h \circ \gamma_X$ is $\frac{1}{\sqrt{2}}$.
2. The complex torsion of $h \circ \gamma_X$ is $4\alpha^3 - 3\alpha$ for $-1 \leqq \alpha \leqq 1$.
3. The circle $h \circ \gamma_X$ is closed if and only if either $\alpha = 0$ or $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$ is rational.
4. When $\alpha = 0$, the length of the closed circle is $\frac{2\sqrt{6}}{3}\pi$.
5. When $\alpha \neq 0$ and $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$ is rational, we denote by $\frac{p}{q}$ the irreducible fraction defined by $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$. Then the length ℓ of a closed circle $h \circ \gamma_X$ is as follows;
6. When pq is even, ℓ is the least common multiple of $\frac{2\sqrt{2}}{3|\alpha|}\pi$ and $\frac{2\sqrt{2}}{\sqrt{3(1-\alpha^2)}}\pi$. In particular, when $\alpha = \pm 1$, then $\ell = \frac{2\sqrt{2}}{3}\pi$.
7. When pq is odd, ℓ is the least common multiple of $\frac{\sqrt{2}}{3|\alpha|}\pi$ and $\frac{\sqrt{2}}{\sqrt{3(1-\alpha^2)}}\pi$.

Next, we prepare the following in order to consider circles of arbitrary positive curvature. Let N be the outward unit normal on $S^{2n+1}(1)(\subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1})$. We here mix the complex structures of $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n(4)$. We shall study circles in $\mathbb{C}P^n(4)$ by making use of the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$. For the sake of simplicity we identify a vector field X on $\mathbb{C}P^n(4)$ with its horizontal lift X^* on $S^{2n+1}(1)$. Then the relation between the Riemannian connection ∇ of $\mathbb{C}P^n(4)$ and the Riemannian connection $\widetilde{\nabla}$ of $S^{2n+1}(1)$ is as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle X, JY \rangle JN$$

for any vector fields X and Y on $\mathbb{C}P^n(4)$, where $\langle\ ,\ \rangle$ is the natural metric of $\mathbb{C}^{n+1}$. By using this relation we can see that for each circle γ of positive curvature any horizontal lift $\tilde{\gamma}$ of γ in $S^{2n+1}(1)$ is a helix in $S^{2n+1}(1)$.

Proposition 2.4. Let γ denote a circle with curvature $\kappa(> 0)$ and complex torsion τ in $\mathbb{C}P^n(4)$ satisfying that $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s$ and $\nabla_{\dot\gamma} Y_s = -\kappa\dot\gamma$. Then a horizontal lift $\tilde{\gamma}$ of γ in $S^{2n+1}(1)$ is a helix of order $2, 3$ or 5 corresponding to $\tau = 0, \tau = \pm 1$ or $\tau \neq 0, \pm 1$, respectively. Moreover, it satisfies the following differential equations:

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa Y_s, \\ \widetilde{\nabla}_{\dot\gamma}Y_s = -\kappa\dot\gamma + \tau JN, \\ \widetilde{\nabla}_{\dot\gamma}(JN) = -\tau Y_s + \sqrt{1-\tau^2}Z_s, \\ \widetilde{\nabla}_{\dot\gamma}Z_s = -\sqrt{1-\tau^2}JN + \kappa W_s, \\ \widetilde{\nabla}_{\dot\gamma}W_s = -\kappa Z_s, \end{cases}$$

where $Z_s = \frac{1}{\sqrt{1-\tau^2}}(J\dot\gamma + \tau Y_s), W_s = \frac{1}{\sqrt{1-\tau^2}}(JY_s - \tau\dot\gamma)$.

Note that a curve $\gamma = \gamma(s)$ in $\mathbb{C}P^n(4)$ is closed if and only if there exists a positive constant s_* such that a horizontal lift $\tilde\gamma = \tilde\gamma(s)$ of γ in $S^{2n+1}(1)$ satisfies $\tilde\gamma(s+s_*) = e^{i\theta_s}\tilde\gamma(s)$ with some $\theta_s \in [0, 2\pi)$ for every s. Then by studying a horizontal lift $\tilde\gamma$ of a circle γ in $\mathbb{C}P^n(4)$ we establish the following.

Theorem 2.5. Let γ be a circle of curvature $\kappa(>0)$ and of complex torsion τ in a complex projective space $\mathbb{C}P^n(4)$. Then the following hold:

1. When $\tau = 0$, a circle γ is a simple closed curve with length $\frac{2\pi}{\sqrt{\kappa^2+1}}$.
2. When $\tau = \pm 1$, a circle γ is a simple closed curve with length $\frac{2\pi}{\sqrt{\kappa^2+4}}$.
3. When $\tau \neq 0, \pm 1$, we denote by a, b and d $(a < b < d)$ the nonzero solutions for
$$\lambda^3 - (\kappa^2+1)\lambda + \kappa\tau = 0.$$
Then we find the following:
4. If one of (hence all of) the three ratios $\frac{a}{b}$, $\frac{b}{d}$ and $\frac{d}{a}$ is rational, then γ is a simple closed curve. Its length is the least common multiple of $\frac{2\pi}{b-a}$ and $\frac{2\pi}{d-a}$.
5. If each of the three ratios $\frac{a}{b}$, $\frac{b}{d}$ and $\frac{d}{a}$ is irrational, then γ is a simple open curve.

Let γ be a circle of curvature κ in a Riemannian manifold (M, g). When we change the metric g homothetically to $m^2 \cdot g$ for some positive constant m, the curve $\sigma(s) = \gamma(\frac{s}{m})$ is a circle of curvature $\frac{\kappa}{m}$ in $(M, m^2 \cdot g)$. Under the operation $g \to m^2 \cdot g$, the length of a closed curve changes to m-times of the original length. Hence, by virtue of Theorem 3.3 we can conclude the following which is the main result in this section.

Theorem 2.6. Let γ be a circle with curvature $\kappa(>0)$ and with complex torsion τ in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. Then the following hold:

1. When $\tau = 0$, a circle γ is a simple closed curve with length $\frac{4\pi}{\sqrt{4\kappa^2+c}}$.
2. When $\tau = \pm 1$, a circle γ is a simple closed curve with length $\frac{2\pi}{\sqrt{\kappa^2+c}}$.
3. When $\tau \neq 0, \pm 1$, we denote by a, b and d $(a < b < d)$ the nonzero solutions for
$$c\lambda^3 - (4\kappa^2+c)\lambda + 2\sqrt{c}\kappa\tau = 0.$$
Then we find the following:

4. If one of (hence all of) the three ratios $\frac{a}{b}$, $\frac{b}{d}$ and $\frac{d}{a}$ is rational, γ is a simple closed curve. Its length is the least common multiple of $\frac{4\pi}{\sqrt{c}(b-a)}$ and $\frac{4\pi}{\sqrt{c}(d-a)}$.
5. If each of the three ratios $\frac{a}{b}$, $\frac{b}{d}$ and $\frac{d}{a}$ is irrational, γ is a simple open curve.

Remarks. A circle $\gamma = \gamma(s)$ with complex torsion τ is a plane curve in $\mathbb{C}P^n(c)$ (that is, γ is locally contained on some real 2-dimensional totally geodesic submanifold of $\mathbb{C}P^n(c)$) if and only if $\tau = 0$ or $\tau = \pm 1$.

1. When $\tau = 0$, the circle γ lies on $\mathbb{R}P^2(\frac{c}{4})$ which is a totally real totally geodesic submanifold of $\mathbb{C}P^n(c)$.
2. When $\tau = 1$ or -1, the circle γ lies on $\mathbb{C}P^1(c)$ which is a holomorphic totally geodesic submanifold of $\mathbb{C}P^n(c)$.

Circles of complex torsion ± 1 are called *holomorphic circles*, and circles of null complex torsion are called *totally real circles.*

3. Length spectrum of circles in $\mathbb{C}P^n(c)$

In this section, we study the length spectrum of circles in $\mathbb{C}P^n(c)$. For a spectrum $\lambda \in \mathrm{LSpec}(M)$ the cardinality $m_M(\lambda)$ of the set $\mathfrak{L}^{-1}(\lambda)$ is called the *multiplicity* of the length spectrum $\mathfrak{L}$ at λ. When $m_M(\lambda) = 1$, we say that λ is *simple.* For example, every length spectrum of circles in a real space form is simple. When the multiplicity of $\mathfrak{L}$ is greater than one at some point λ, this means that we can find circles which are not congruent each other but have the same length λ.

Rewriting Theorem 3.1, we find the following which is our main tool in this section.

Proposition 3.1. In $\mathbb{C}P^n(c)$ a circle γ of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau = 3\alpha - 4\alpha^3$ $(0 < |\alpha| < \frac{1}{2})$ is closed if and only if $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$ is rational. In this case if we denote $\sqrt{\frac{1-\alpha^2}{3\alpha^2}} = \frac{p}{q}$ by relatively prime positive integers p and q, then its length is

$$\mathrm{Length}(\gamma) = \begin{cases} \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is even,} \\ \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is odd.} \end{cases}$$

We denote by $[\gamma_{\kappa,\tau}]$ the congruency class of circles of curvature κ and complex torsion $\tau(\geqq 0)$ in $\mathbb{C}P^n(c)$. The moduli space of circles have a natural stratification by their curvatures. We denote by $\mathrm{Cir}_\kappa(M)$ the moduli space of circles of curvature κ in M and by $\mathfrak{L}_\kappa$ the restriction of $\mathfrak{L}$ on this space.

For a positive constant κ we define a canonical transformation

$$\Phi_\kappa : \mathrm{Cir}_\kappa(\mathbb{C}P^n(c)\setminus\{[\gamma_{\kappa,1}]\} \to \mathrm{Cir}_{\sqrt{2c}/4}(\mathbb{C}P^n(c))\setminus\{[\gamma_{\sqrt{2c}/4,1}]\}$$

by

$$\Phi_\kappa([\gamma_{\kappa,\tau}]) = [\gamma_{\sqrt{2c}/4,3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2}}].$$

The following lemma guarantees that the structure of the length spectrum $\mathfrak{L}_\kappa$ of circles of curvature κ essentially does not depend on κ.

Lemma 3.2. The canonical transformation Φ_κ satisfies

$$\mathfrak{L}([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3c}{2(4\kappa^2+c)}} \cdot \mathfrak{L}(\Phi_\kappa([\gamma_{\kappa,\tau}]))$$

for every τ $(0 \leqq \tau < 1)$.

We denote by $\mathrm{LSpec}_\kappa(M) = \mathfrak{L}\,(\mathrm{Cir}_\kappa(M)) \cap \mathbb{R}$ the length spectrum of circles of curvature κ in M. This lemma yields that

$$\begin{aligned}\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c)) = &\left\{\frac{2\pi}{\sqrt{\kappa^2+c}}, \frac{4\pi}{\sqrt{4\kappa^2+c}}\right\}\\ &\bigcup \left\{ \frac{4\pi\sqrt{3p^2+q^2}}{3(4\kappa^2+c)} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is even and } p > \alpha_\kappa q > 0 \end{array} \right\},\\ &\bigcup \left\{ \frac{2\pi\sqrt{3p^2+q^2}}{3(4\kappa^2+c)} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is odd and } p > \alpha_\kappa q > 0 \end{array} \right\},\end{aligned}$$

where α_κ $(\geqq 1)$ denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\alpha_\kappa^2 - 1}{(3\alpha_\kappa^2+1)^{3/2}}.$$

Note that the constant α_κ satisfies

1. $\alpha_{\sqrt{2c}/4} = 1$,
2. monotone decreasing when $0 < \kappa \leqq \frac{\sqrt{2c}}{4}$, and monotone increasing when $\kappa \geqq \frac{\sqrt{2c}}{4}$,
3. $\lim_{\kappa\to 0} \alpha_\kappa = \lim_{\kappa\to\infty} \alpha_\kappa = \infty$.

Lemma 4.2 also guarantees that

$$\mathrm{LSpec}(\mathbb{C}P^n(c)) = \left(0, \frac{4\pi}{\sqrt{c}}\right) \cup \bigcup \left\{ I_{p,q} \;\middle|\; \begin{array}{c} p > q, p \text{ and } q \text{ are relatively} \\ \text{prime positive integers} \end{array} \right\},$$

where

$$I_{p,q} = \begin{cases} \left(\frac{4\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{4\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is even,} \\ \left(\frac{2\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{2\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is odd.} \end{cases}$$

We denote by $\mathrm{Cir}^\tau(M)$ the moduli space of circles with complex torsion τ in a Kähler manifold M by $\mathfrak{L}^\tau$ the restriction of $\mathfrak{L}$ onto this space. From these expressions on length spectrum of circles we establish the following main result.

Theorem 3.3. For a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ of constant holomorphic sectional curvature c, the length spectrum of circles has the following properties.

1. Both the sets
$$\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c)) = \mathfrak{L}(\mathrm{Cir}_\kappa(\mathbb{C}P^n(c))) \cap \mathbb{R}$$
and
$$\mathrm{LSpec}^\tau(\mathbb{C}P^n(c)) = \mathfrak{L}(\mathrm{Cir}^\tau(\mathbb{C}P^n(c))) \cap \mathbb{R}$$
are unbounded discrete subsets of $\mathbb{R}$ for each $\kappa(>0)$ and $0<\tau<1$.
2. The length spectrum $\mathrm{LSpec}(\mathbb{C}P^n(c))$ of circles coincides with the real positive line $(0,\infty)$.
3. For $\kappa>0$ the bottom of $\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c))$ is $\frac{2\pi}{\sqrt{\kappa^2+c}}$, which is the length of the holomorphic circle of curvature κ. The second lowest spectrum of $\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c))$ is $\frac{4\pi}{\sqrt{4\kappa^2+c}}$, which is the length of the totally real circle of curvature κ. They are simple for $\mathfrak{L}_\kappa$.
4. The multiplicity of $\mathfrak{L}$ is finite at each point $\lambda \in \mathbb{R}$.
5. $\lambda(\in \mathbb{R})$ is simple for $\mathfrak{L}$ if and only if $\lambda \in \left(\frac{2}{\sqrt{c}}\pi, \frac{4}{3}\sqrt{\frac{5}{c}}\pi\right]$.
6. The multiplicity of $\mathfrak{L}_\kappa$ $(\kappa>0)$ is not uniformly bounded;
$$\limsup_{\lambda\to\infty} \sharp(\mathfrak{L}_\kappa^{-1}(\lambda)) = \infty.$$
The growth order of the multiplicity with respect to λ is not so rapid. It satisfies $\lim_{\lambda\to\infty}\lambda^{-\delta}\sharp(\mathfrak{L}_\kappa^{-1}(\lambda)) = 0$ for arbitrary positive δ.

The statements (2) and (5) in our theorem give the complete answer to the problem in the introduction.

Remark It follows from Proposition 4.1 that a circle of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion τ in $\mathbb{C}P^n(c)$ is closed if and only if
$$\tau = \tau(p,q) = q(9p^2-q^2)(3p^2+q^2)^{-3/2},$$
for some relatively prime positive integers p and q with $p>q$. We find that the length spectrum $\mathfrak{L}_{\frac{\sqrt{2c}}{4}}$ is not simple at the following points for examples.

1. Let γ_1 be a circle of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau=\tau(27,7)=\frac{5698}{559\sqrt{559}}$ and γ_2 be a circle of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau=\tau(25,19)=\frac{12502}{559\sqrt{559}}$. These two closed circles have the same curvature and the same length $\frac{4\sqrt{1118}}{3\sqrt{c}}\pi$. But they are not congruent.
2. Let γ_i be a circle of the same curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau_i=\tau(p_i,q_i)$, $i=1,2,3$. Here we set $(p_1,q_1)=(129,71)$, $(p_2,q_2)=(131,59)$ and $(p_3,q_3)=(135,17)$. Note that $3p_i^2+q_i^2=54964$ for $i=1,2,3$. Then these three circles have the same curvature and the same length. But these three circles are not congruent each other.

Finally we investigate the asymptotic behaviour of the number of congruency classes of closed circles of curvature κ. Let $n_M(\lambda;\kappa)$ denotes the number of congruency classes of closed circles of curvature κ in M with length not greater than λ.

Theorem 3.4. For a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ of constant holomorphic sectional curvature c, we have for $\kappa > 0$

$$\lim_{\lambda\to\infty} \frac{n_{\mathbb{C}P^n(c)}(\lambda;\kappa)}{\lambda^2} = \frac{3\sqrt{3}(4\kappa^2+c)}{8\pi^4}\tan^{-1}\left(\frac{1}{\sqrt{3}\alpha_\kappa}\right),$$

where α_κ $(\geqq 1)$ denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\alpha_\kappa^2-1}{(3\alpha_\kappa^2+1)^{3/2}}.$$

In particular,

$$\lim_{\lambda\to\infty} \frac{n_{\mathbb{C}P^n(c)}(\lambda;\sqrt{2c}/4)}{\lambda^2} = \frac{3\sqrt{3}c}{32\pi^3}.$$

Sketch of the proof. For a positive integer d, we put $n_\alpha(\lambda)$ and $k_\alpha(\lambda;d)$ the cardinalities of the sets

$$\left\{(p,q)\in\mathbb{Z}\times\mathbb{Z}\,\middle|\,\begin{array}{c}p\text{ and }q\text{ are relatively prime integers with}\\ 3p^2+q^2\le\lambda^2\text{ and }p>\alpha q>0\end{array}\right\}$$

and

$$K_\alpha(\lambda;d) = \left\{(p,q)\in d\mathbb{Z}\times d\mathbb{Z}\,\middle|\,3p^2+q^2\le\lambda^2,\ p>\alpha q>0\right\},$$

respectively. Here $d\mathbb{Z}$ denotes the set $\{dj \mid j\in\mathbb{Z}\}$. Since the correspondence $(p,q)\mapsto(dp,dq)$ of $K_\alpha(\lambda/d;1)$ to $K_\alpha(\lambda;d)$ is bijective, we find the following relation between $n_\alpha(\lambda)$ and $k_\alpha(\lambda;1)$ by using the Möbius function μ;

$$n_\alpha(\lambda) = \sum_{d\ge1}\mu(d)k_\alpha(\lambda;d) = \sum_{1\le d\le[\lambda/2]}\mu(d)k_\alpha(\lambda/d;1),$$

where $[\delta]$ denotes the integer part of a real number δ. Put

$$C = \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}\alpha}\right),$$

which is the area of the set $\{(x,y)\in\mathbb{R}^2 \mid 3x^2+y^2\leqq\lambda^2, x\geqq\alpha y\geqq0\}$. One can easily find positive constants C_1, C_2 with $|k_\alpha(\lambda;1)-C\lambda^2| < C_1\lambda+C_2$. Thus we obtain

$$\lim_{\lambda\to\infty}\frac{n_\alpha(\lambda)}{\lambda^2} = C\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} = \frac{C}{\zeta(2)} = \frac{6C}{\pi^2},$$

where ζ denotes the Riemann zeta function.

We now put $n_\alpha^{\mathrm{o}}(\lambda)$ and $n_\alpha^{\mathrm{e}}(\lambda)$ the cardinalities of the sets

$$\left\{(p,q)\in\mathbb{Z}\times\mathbb{Z}\,\middle|\,\begin{array}{c}p\text{ and }q\text{ are relatively prime integers which satisfy}\\ pq\text{ is odd, }3p^2+q^2\leqq\lambda^2\text{ and }p>\alpha q>0\end{array}\right\}$$

and

$$\left\{ (p,q) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, \begin{array}{c} p \text{ and } q \text{ are relatively prime integers which satisfy} \\ pq \text{ is even, } \alpha p^2 + \beta q^2 \le \lambda^2 \text{ and } p > \alpha q > 0 \end{array} \right\},$$

respectively. By similar argument we obtain

$$\lim_{\lambda\to\infty} \frac{n_\alpha^{\mathrm{o}}(\lambda)}{\lambda^2} = \frac{C}{4} \sum_{\substack{1 \le d < \infty, \\ d \text{ is odd}}} \frac{\mu(d)}{d^2} = \frac{2C}{\pi^2},$$

and

$$\lim_{\lambda\to\infty} \frac{n_\alpha^{\mathrm{e}}(\lambda)}{\lambda^2} = \lim_{\lambda\to\infty} \left(\frac{n_{\alpha,\beta}(\lambda)}{\lambda^2} - \frac{n_{\alpha,\beta}^{\mathrm{o}}(\lambda)}{\lambda^2} \right) = \frac{4C}{\pi^2}.$$

Since we have

$$n_{\mathbb{C}P^n(c)}(\lambda;\kappa) = 2 + n_{\alpha_\kappa}^{\mathrm{o}} \left(\frac{\sqrt{4\kappa^2+c}}{2\sqrt{3}\pi}\lambda \right) + n_{\alpha_\kappa}^{\mathrm{e}} \left(\frac{\sqrt{4\kappa^2+c}}{4\sqrt{3}\pi}\lambda \right)$$

for $\lambda > \frac{4\pi}{\sqrt{4\kappa^2+c}}$, we obtain the conclusion. □

Remark. The constant $c(\kappa) = \lim_{\lambda\to\infty} \lambda^{-2} n_{\mathbb{C}P^n(c)}(\lambda;\kappa)$ satisfies

$$\lim_{\kappa\to 0} c(\kappa) = 0 \quad \text{and} \quad \lim_{\kappa\to\infty} c(\kappa) = \frac{9c}{16\pi^4}.$$

We finally pose some problems on length spectrum of circles.

Problems.

1. Are there non-simple spectrum for $\mathfrak{L}^\tau$ $(0 < \tau < 1)$?
2. Whether is the multiplicity of $\mathfrak{L}^\tau$ $(0 < \tau < 1)$ uniformly bounded or not?
3. Give an explicit formula of the first spectrum for $\mathfrak{L}^\tau$ $(0 < \tau < 1)$.
4. Study the asymptotic behaviour of the number of congruency classes of closed circles of complex torsion $\tau (\ne 0, 1)$ with respect to length.
5. Find nice properties for the multiplicity $m_{\mathbb{C}P^n(c)}(\lambda)$ of the full length spectrum $\mathfrak{L}$. For a complex hyperbolic space, it is monotone increasing left continuous function with polynomial growth and its jumping step is not uniformly bounded (see [A2]).
6. Study the behaviour of $c(\kappa)$. What is the maximum value of this function $c(\kappa)$?
7. Study the geometric meaning of the constant $\lim_{\kappa\to\infty} c(\kappa)$.

References

A1. T. Adachi, *Circles on a quaternionic space form*, J. Math. Soc. Japan**48**, 205–227 (1996).

A2. T. Adachi, *Distribution of length spectrum of circles on a complex hyperbolic space*, Nagoya Math. J.**153** , 119–140 (1999).

AM1. T. Adachi and S. Maeda, *Length spectrum of circles in a complex projective space*, Osaka J. Math.**35**, 553–565 (1998).

AMU. T. Adachi, S. Maeda and S. Udagawa, *Circles in a complex projective space*, Osaka J. Math.**32**, 709–719 (1995).

C. A. Comtet, *On the Landau levels on the hyperbolic plane*, Ann. of Physics **173**, 185–209 (1987).

MA. S. Maeda and T. Adachi, *Holomorphic helices in a complex space form*, Proc. Amer. Math. Soc.**125**, 1197–1202 (1997).

MO. S. Maeda and Y. Ohnita, *Helical geodesic immersions into complex space forms*, Geometriae Dedicata **30**, 93–114 (1989).

MT. K. Mashimo and K. Tojo, *Circles in Riemannian symmetric spaces*, Kodai Math. J.**20**, 1–14 (1999).

N. H. Naitoh, *Isotropic immersions with parallel second fundamental form in $P^m(c)$*, Osaka J. Math.**18**, 427–464 (1981).

Sadahiro Maeda: Department of Mathematics, Shimane University, Matsue 690-8504, Japan

E-mail address: `smaeda@math.shimane-u.ac.jp`

Toshiaki Adachi: Department of Mathematics, Nagoya Institute of technology, Nagoya 466-8555, Japan

E-mail address: `adachi@math.kyy.nitech.ac.jp`