Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

EXAMPLES OF HYPER-KÄHLER CONNECTIONS WITH TORSION

YAT SUN POON

This article is a revised version of the author's lecture given at the Second Meeting on Quaternionic Structures in Mathematics and Physics at Roma, September 8, 1999. It will appear in the proceeding of this conference. The work presented here are jointly done with Gueo Grantcharov. This article is composed while the author visits the ESI.

1. It has been known for a fairy long time that when the Wess-Zumino term is present in the $N = 4$ supersymmetric one-dimensional sigma model, the internal space has a torsionful linear connection with holonomy in $\mathrm{Sp}(n)$ [3]. Such geometry also arises when one considers T-duality of toric hyper-Kähler manifolds [5]. In this conference, G. Papadopoulos explains the role of HKT-geometry in M-theory, or more specifically in IIA Superstring theory [10].

In this lecture, we discuss HKT-geometry entirely from a mathematical point of view, and present several methods to produce series of examples that may interest mathematicians.

2. The background object of HKT-geometry is a hyper-Hermitian manifold. Three complex structures I_1, I_2 and I_3 on a smooth manifold M form a hypercomplex structure if

$$I_1^2 = I_2^2 = I_3^2 = -1, \quad \text{and} \quad I_1 I_2 = I_3 = -I_2 I_1. \tag{0.1}$$

A triple of such complex structures is equivalent to the existence of a 2-sphere worth of integrable complex structures: $\mathcal{I} = \{a_1 I_1 + a_2 I_2 + a_3 I_3 : a_1^2 + a_2^2 + a_3^2 = 1\}$. When g is a Riemannian metric on the manifold M such that it is Hermitian with respect to every complex structure in the hypercomplex structure, $(M, \mathcal{I}, g)$ is called a hyper-Hermitian manifold.

On a hyper-Hermitian manifold, there are two natural torsion-free connections, namely the Levi-Civita connection and the Obata connection. However, in general the Levi-Civita connection does not preserve the hypercomplex structure and the

Obata connection does not preserve the metric. We are interested in the following type of connections.

Definition 1. A linear connection ∇ on a hyper-Hermitian manifold $(M, \mathcal{I}, g)$ is hyper-Hermitian if $\nabla g = 0$, and $\nabla I_1 = \nabla I_2 = \nabla I_3 = 0$.

Although a general hyper-Hermitian connection has torsion, physical requirement limits our discussion to a special type of hyper-Hermitian connections. We follow physicists' conventions of definitions. Recall that when T^∇ is the torsion tensor for a connection ∇, we can construct the $(3,0)$-tensor $c(X,Y,Z) = g(X, T^\nabla(Y,Z))$.

Definition 2. A linear connection ∇ on a hyper-Hermitian manifold $(M, \mathcal{I}, g)$ is hyper-Kähler with torsion (HKT) if it is hyper-Hermitian and its torsion $(3,0)$-tensor is totally skew-symmetric. It is a strong HKT-connection if its torsion 3-form is closed.

Example 1. Let q be the quaternion coordinate for the one-dimensional quaternion module $\mathbf{H}$. Through left multiplications by the unit quaterions i, j and k, one obtains a hypercomplex structure $\mathcal{I}$ on $\mathbf{H}$. The Euclidean metric $g = dqd\overline{q}$ is hyper-Kähler. The Levi-Civita connection is a HKT-connection.

A less obvious and more relevant example for us is to consider the following metric on $\mathbf{H}\backslash\{0\}$.

$$\hat{g} = \frac{dqd\overline{q}}{|q\overline{q}|}. \tag{0.2}$$

Considering the diffeomorphis $\mathbf{H}\backslash\{0\} = \mathbf{R}^+ \times S^3$, we choose a spherical coordinate (r, θ, ϕ, ψ). Let g_S be the metric on the round unit-sphere. Then

$$\hat{g} = \frac{dr^2}{r^2} + g_S. \tag{0.3}$$

Now, $(\mathbf{H}\backslash\{0\}, \mathcal{I}, \hat{g})$ is a HKT-structure. The torsion form c is the volume form of the sphere S^3. It is also a closed 3-form.

If one chooses to study the Hermitian geometry for one of the complex structures J in the hypercomplex structure, one should note that Gauduchon found a collection of canonical Hermitian connections on any Hermitian manifold. The collection forms an affine subspace of the space of linear connections [4]. This collection of Hermitian connections include Chern connection and Lichnerowicz's first canonical connection. Within this family, there exists exactly one connection whose torsion $(3,0)$-tensor is a 3-form. To describe it, we recall the following definitions and convention. For any n-form ω, $d^c\omega := (-1)^n JdJ\omega$ where $(J\omega)(X_1, \ldots, X_n) := (-1)^n\omega(JX_1, \ldots, JX_n)$. Then $\partial = \frac{1}{2}(d + id^c)$ and $\overline{\partial} = \frac{1}{2}(d - id^c)$. By [4], the Hermitian connection with totally skew-symmetric torsion $(3,0)$-tensor c is uniquely determined by the following identity.

$$c(X,Y,Z) = -\frac{1}{2}d^cF(X,Y,Z), \tag{0.4}$$

where $F(X,Y) = g(JX,Y)$ is the Kähler form for the complex structure J.

Now the HKT-connection serves as such a unique connection for each complex structure in the hypercomplex structure. Therefore, if we use F_a and d_a to represent the Kähler form and complex exterior differential for the complex structure I_a, $a = 1, 2, 3$, we have the following observation.

Proposition 1. A hyper-Hermitian manifold $(M, \mathcal{I}, g)$ admits a HKT-connection if and only if $d_1 F_1 = d_2 F_2 = d_3 F_3$. If it exists, it is unique.

In view of the uniqueness, we say that $(M, \mathcal{I}, g)$ is a HKT-structure if it admits a HKT-connection.

Example 2. A non-trivial class of HKT-structures can be found on semi-simple Lie groups and homogeneous spaces [13] [9]. For instance, the Killing-Cartan form $-B$ on the Lie group $\mathrm{SU}(2n+1)$ defines a bi-invariant metric $g = -B$. This group has a left-invariant hypercomplex structure $\mathcal{I}$ so that with the bi-invariant metric g, it forms a HKT-structure. The HKT-connection is the left-invariant connection defined by having all left-invariant vector fields to be parallel. The torsion of this connection is the Lie bracket, and the torsion tensor $c(X,Y,Z) = -B(X,[Y,Z])$ is totally skew-symmetric. Similar constructions can be applied to $\mathrm{U}(1) \times \mathrm{SU}(2n)$ and other homogeneous spaces.

3. To further our analysis of HKT-geometry, we note a holomorphic characterization of HKT-structures.

Proposition 2. Let $(M, \mathcal{I}, g)$ be a hyper-Hermitian manifold and F_a be the Kähler form for (I_a, g). Then $(M, \mathcal{I}, g)$ is a HKT-structure if and only if $\partial_1(F_2 + iF_3) = 0$; or equivalently $\overline{\partial}_1(F_2 - iF_3) = 0$.

Applying this proposition to any complex structure in the given hypercomplex structure, one obtains a section of twisted 2-form on the twistor space of the hypercomplex structures. However, this 2-form is only J_2-holomorphic in the sense of Eells-Salamon [2]. Since the almost complex structure J_2 is never integrable [2], we shall concentrate on the holomorphic characterization given above and ignore the twistor characterization.

Due to the absence of type $(3,0)$-form with respect to any complex structure on any real four-dimensional manifold, it is now apparent that any four-dimensional hyper-Hermitian manifold is a HKT-structure.

The holomorphic characterization also yields new examples of HKT-structures.

Example 3. Let $\{X_1, ..., X_{2n}, Y_1, ..., Y_{2n}, Z\}$ be a basis for $\mathbf{R}^{4n+1}$. Define commutators by $[X_i, Y_i] = 4Z$, and all others are zero. These commutators define on $\mathbf{R}^{4n+1}$ the structure of the Heisenberg Lie algebra $\mathfrak{h}$. Let $\mathbf{R}^3$ be the 3-dimensional Abelian algebra. The direct sum $\mathfrak{n} = \mathfrak{h} \oplus \mathbf{R}^3$ is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis $\{E_1, E_2, E_3\}$ for $\mathbf{R}^3$. Consider the endomorphisms I_1,

I_2 and I_3 of $\mathfrak{n}$ defined by left multiplications of the quaternions i, j and k on the module of quaternions $\mathbf{H}$, and the identifications

$$
\begin{aligned}
x_0X_{2a-1}+x_1X_{2a-1}+x_2Y_{2a-1}+x_3Y_{2a} &\to x_0+x_1i+x_2j+x_3k;\\
x_0Z+x_1E_1+x_2E_2+x_3E_3 &\to x_0+x_1i+x_2j+x_3k.
\end{aligned}
\tag{0.5}
$$

Through left translations, these endomorphisms define almost complex structures on the product of the Heisenberg group and the Abelian group $N=H\times\mathbf{R}^3$. It is clear from the definition that these almost complex structures satisfy the algebra (0.1). Moreover, for $a=1,2,3$ and $X,Y\in\mathfrak{n}$,

$$
[I_aX,I_aY]=[X,Y]
\tag{0.6}
$$

so I_a are Abelian complex structures on $\mathfrak{n}$ in the sense of [1]. In particular, they are integrable. It implies that $\{I_a : a=1,2,3\}$ is a left-invariant hypercomplex structure on the Lie group N. It is known [12] that the complex structures I_a on $\mathfrak{n}$ satisfy $d(\Lambda^{1,0}_{I_a}\mathfrak{n}^*)\subset\Lambda^{1,1}_{I_a}\mathfrak{n}^*$ where $\mathfrak{n}^*$ is the space of left-invariant 1-forms on N and $\Lambda^{i,j}_{I_a}\mathfrak{n}^*$ is the (i,j)-component of $\mathfrak{n}^*\otimes\mathbf{C}$ with respect to I_a. But then we have $d(\Lambda^{2,0}_{I_a}\mathfrak{n}^*)\subset\Lambda^{2,1}_{I_a}\mathfrak{n}^*$ and any left-invariant (2,0)-form is ∂_1-closed. Now consider the invariant metric on N for which the basis $\{X_i,Y_i,Z,E_a\}$ is orthonormal. Since it is compatible with the complex structures I_a, in view of the holomorphic characterization of HKT-geometry, we obtain a left-invariant HKT-structure on N.

Example 4. Based on the above computation, we could also see that there is a left-invariant HKT-structure on the product of the $4n+1$-dimensional Heisenberg group and the compact simple Lie group SU(2), an interesting mixture of the last example and Example 2.

Recall that the underlying manifold of the Heisenberg group H_{4n+1} is the manifold $\mathbf{R}^{4n+1}$. Consider it as the product space $\mathbf{R}^{2n}\times\mathbf{R}^{2n}\times\mathbf{R}$, the group law for the Heisenberg group is

$$
(\vec{x},\vec{y},z)*(\vec{x}',\vec{y}',z')=(\vec{x}+\vec{x}',\vec{y}+\vec{y}',z+z'+2\sum_{i,j=1}^{2n}(x_iy_j'-y_ix_j')).
\tag{0.7}
$$

The 1-forms $\alpha_j=dx_j,\beta_j=dy_j,\gamma=dz+2\sum(y_jdx_i-x_jdy_i)$ are left-invariant. Let $\{X_j,Y_j,Z\}$ be the dual left-invariant vector fields.

On SU(2), choose left-invariant vector fields A_1, A_2 and A_3 such that $[A_1,A_2]=2A_3$, etc., then the dual left-invariant 1-forms σ_1, σ_2 and σ_3 satisfy the identities

$$
d\sigma_1=2\sigma_2\wedge\sigma_3, d\sigma_2=2\sigma_3\wedge\sigma_1, d\sigma_3=2\sigma_1\wedge\sigma_2.
\tag{0.8}
$$

Now, using $\{A_1,A_2,A_3\}$ instead of $\{E_1,E_2,E_3\}$, we define endomorphisms I_1, I_2 and I_3 on $\mathfrak{h}\oplus\mathfrak{su}(2)$ as in (0.5). Through left translation, we define three almost complex structures on the product group $H\times\mathrm{SU}(2)$ satisfying the identities (0.1). To prove that these almost complex structures are integrable, one first notes that when $\mathfrak{c}$ is the center of the Heisenberg algebra, then the vector space $\mathfrak{h}\oplus\mathfrak{su}(2)$

has a direct sum decomposition $\mathfrak{t}_{4n} \oplus \mathfrak{c} \oplus \mathfrak{su}(2)$ where $\mathfrak{t}_{4n}$ is the linear span of all the X_j and Y_j. On $\mathfrak{t}_{4n}$, the almost complex structures satisfy the identity (0.6). Therefore, the Nijenhuis tensor vanishes on $\mathfrak{t}_{4n}$. On $\mathfrak{c} \oplus \mathfrak{su}(2)$, the almost complex structures are the standard ones for $\mathbf{H}\backslash\{0\}$. Therefore, the Nijenhuis tensor vanishes on this summand. Since $\mathfrak{c} \oplus \mathfrak{su}(2)$ commutes with $\mathfrak{t}_{4n}$, and both $\mathfrak{t}_{4n}$ and $\mathfrak{c} \oplus \mathfrak{su}(2)$ are invariant of the endomorphisms I_1, I_2 and I_3, the Nijenhuis tensor vanishes completely. Therefore, the left-invariant almost complex structures I_1, I_2 and I_3 define a left-invariant hypercomplex structure on the product group $H \times \mathrm{SU}(2)$. However the hypercomplex structure is no longer Abelian.

We define a left-invariant metric g on the product group by requiring the left-invariant vector fields $\{X_1, ..., X_{2n}, Y_1, ..., Y_{2n}, Z, E_1, E_2, E_3\}$ to be an orthonormal frame. Equivalently,

$$g = \sum_{a=1}^{n}(\alpha_{2a-1}^2 + \alpha_{2a}^2 + \beta_{2a-1}^2 + \beta_{2a}^2) + \gamma^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$$

This metric, along with the left-invariant hypercomplex structure, is a HKT-structure on the product group $H \times \mathrm{SU}(2)$. Indeed, when F_1, F_2 and F_3 are the three Kähler forms for the complex structures I_1, I_2 and I_3, $dF_a = 2i(d\gamma \wedge \sigma_a - \gamma \wedge \sigma_b \wedge \sigma_c)$, where (abc) is any even permutation of (123). Since $d\gamma = -4\sum_{a=1}^{n}(\alpha_{2a-1}\wedge\beta_{2a-1}+\alpha_{2a}\wedge\beta_{2a})$,

$$I_1 dF_1 = I_2 dF_2 = I_3 dF_3 = -2i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3). \tag{0.9}$$

Therefore, we have a HKT-structure. Since the torsion 3-form $c = i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3)$

$$dc = i d\gamma \wedge d\gamma. \tag{0.10}$$

This is not a closed 3-form, the corresponding HKT-structure is weak.

4. The holomorphic characterization shows that the form $F_2 + iF_3$ has a locally defined $(1,0)$-form β as its potential. Although the $(0,1)$-form $I_2\beta$ is not a priori ∂_1-closed, we consider the case when it is. From this observation, we extract the following definition.

Definition 3. Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler forms F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if

$$F_1 = \frac{1}{2}(dd_1 + d_2d_3)\mu, \quad F_2 = \frac{1}{2}(dd_2 + d_3d_1)\mu, \quad F_3 = \frac{1}{2}(dd_3 + d_1d_2)\mu. \tag{0.11}$$

Referring to the holomorphic characterization of HKT-geometry, we reformulate the definition of HKT-potential in the following way.

Proposition 3. Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler form F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if

$$F_2 + iF_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu. \tag{0.12}$$

Example 5. On the complex vector space $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\} \cong \mathbf{H}^n\backslash\{0\}$, let (z_α, w_α), $1 \leq \alpha \leq n$, be its coordinates. Define a hypercomplex structure $\mathcal{I}$ by right multiplication of the pure quaternions i, j and k. Let g be the flat metric. It is a hyper-Kähler metric with hyper-Kähler potential $\mu = \frac{1}{2}(|z|^2 + |w|^2)$. Consider a new metric

$$\hat{g} = \frac{1}{\mu}g - \frac{1}{4\mu^2}(d\mu \otimes d\mu + I_1 d\mu \otimes I_1 d\mu + I_2 d\mu \otimes I_2 d\mu + I_3 d\mu \otimes I_3 d\mu).$$

Then the hyper-Hermitian structure $(\mathcal{I}, \hat{g})$ is a HKT-structure. Moreover, the function $\ln(\mu)$ is its potential.

Now, for any real number r, with $0 < r < 1$, and $\theta_1, \dots, \theta_n$ modulo 2π, we consider the integer group $< \gamma >$ generated by the following action on $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$.

$$\gamma(z_\alpha, w_\alpha) = (re^{i\theta_\alpha} z_\alpha, re^{-i\theta_\alpha} w_\alpha). \tag{0.13}$$

Since γ is a hyper-holomorphic isometry, the HKT-structure on $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ descends to a HKT-structure on the quotient space with respect to the group $< \gamma >$. As this quotient space is diffeomorphic to $S^1 \times S^{4n-1}$ [11], and the quotient hypercomplex structure is not homogeneous, we obtain a family of inhomogeneous HKT-structures on the manifold $S^1 \times S^{4n-1}$.

It should be noted that this method of generating HKT-geometry through a transformation from HKT-potentials to HKT-potentials can easily generate large classes of inhomogeneous HKT-structures on homogeneous manifolds especially when we start from well known hyper-Kähler metrics with hyper-Kähler potentials.

Remark To produce more examples, one may develop a reduction theory along the line of hyper-Kähler reduction [7]. One can also prove that Joyce's twist construction of hypercomplex manifolds [8] carries HKT-manifolds to HKT-manifolds. We do not present details of these theories here. Details of our work can be found in [6].

References

1. M. L. Barberis, I. G. Dotti Miatello, & R. J. Miatello. *On certain locally homogeneous Clifford manifolds*, Ann. Global. Anal. Geom. **13** (1995) 289–301.
2. J. Eells & S. M. Salamon. *Twistorial construction of harmonic maps of surfaces into four manifolds*, Ann. Scuola Norm. Sup. Pisa Cl.Sci. **12** (1985), 589–640.
3. S. J. Gates, Jr., C. M. Hull & M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nucl. Phys. **B248** (1984), 157–186.
4. P. Gauduchon. *Hermitian connections and Dirac operators*, Bollettino U.M.I., **11B** (1997) 257–288.
5. G.W. Gibbons, G. Papadopoulos & K.S. Stelle. *HKT and OKT geometries on soliton black hole moduli spaces*, Nucl. Phys. B **508** (1997) 623-658.
6. G. Grantcharov & Y. S. Poon. *Geometry of Hyper-Kähler Connections with Torsion*, preprint, math.DG/9908015.
7. N.J. Hitchin, A. Karlhede, U. Lindström & M. Roček. *Hyper-Kähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987) 535–589.

8. D. Joyce. *Compact hypercomplex and quaternionic manifolds*, J. Differential Geom. **35** (1992) 743–761.
9. A. Opfermann & G. Papadopoulos. *Homogeneous HKT and QKT manifolds*, preprint, math-ph/9807026.
10. G. Papadopoulos. *Brane solitons and quaternionic geometries*, Lecture at the Second Meeting on Quaternionic Structures in Mathematics and Physics, Roma, September, 1999.
11. H. Pedersen & Y. S. Poon. *Inhomogeneous hypercomplex structures on homogeneous manifolds*, J. reine angew. Math. (to appear).
12. S. M. Salamon. *Complex structures on nilpotent Lie algebras*, preprint, math.DG/ 9808025.
13. Ph. Spindel, A. Sevrin, W. Troost & A. Van Proeyen. *Extended supersymmetric σ-models on group manifolds* , Nucl. Phys. **B308** (1988) 662–698.

Department of Mathematics, University of California at Riverside, Riverside, CA 92521, U.S.A.

E-mail address: ypoon@math.ucr.edu