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A NEW WEIGHT SYSTEM ON CHORD DIAGRAMS VIA HYPERKÄHLER GEOMETRY

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ABSTRACT. A weight system on graph homology was constructed by Rozansky and Witten using a compact hyperkähler manifold. A variation of this construction utilizing holomorphic vector bundles over the manifold gives a weight system on chord diagrams. We investigate these weights from the hyperkähler geometry point of view.

1. INTRODUCTION

New invariants of hyperkähler manifolds were introduced by Rozansky and Witten in [10]. They occur as the weights in a Feynman diagram expansion of the partition function

$$Z^{\mathrm{RW}}(M) = \sum b_\Gamma(X) I_\Gamma^{\mathrm{RW}}(M)$$

of a three-dimensional physical theory. In this expansion the terms $I_\Gamma^{\mathrm{RW}}(M)$ depend on the three-manifold M but not on the compact hyperkähler manifold X, whereas the weights $b_\Gamma(X)$ depend on X but not on M. Both terms are indexed by the trivalent graph Γ, though $b_\Gamma(X)$ actually only depends on the *graph homology* class which Γ represents. There are many similarities with Chern-Simons theory, for which a Feynman diagram expansion of the partition function

$$Z^{\mathrm{CS}}(M) = \sum c_\Gamma(\mathfrak{g}) I_\Gamma^{\mathrm{CS}}(M)$$

gives us the more 'familiar' weights $c_\Gamma(\mathfrak{g})$ on graph homology constructed from a Lie algebra $\mathfrak{g}$ (in this case, the Lie algebra of the gauge group). We wish to further exploit the analogies.

For example, in Chern-Simons theory we can introduce Wilson lines, ie. a link embedded in the three-manifold. This leads to correlation functions which are invariants of the link, depending on representations V_a of the Lie algebra $\mathfrak{g}$ which are attached to the components of the link (the Wilson lines). Perturbatively we get

$$Z^{\mathrm{CS}}(M;\mathcal{L}) = \sum c_D(\mathfrak{g}; V_a) Z_D^{\mathrm{Kont}}(M;\mathcal{L})$$

where we sum over all *chord diagrams* D (unitrivalent graphs whose univalent vertices lie on a collection of oriented circles), the weights $c_D(\mathfrak{g}; V_a)$ depend on the Lie algebra $\mathfrak{g}$ and its representations V_a, but not on the three-manifold M or the link $\mathcal{L}$, and

$$Z^{\text{Kont}}(M;\mathcal{L}) = \sum Z_D^{\text{Kont}}(M;\mathcal{L})D$$

is the Kontsevich integral of the link $\mathcal{L}$ in M. We would like to imitate this construction in Rozansky-Witten theory, but although Rozansky and Witten give a construction using spinor bundles, it is not clear in general how to associate observables to Wilson lines using arbitrary holomorphic vector bundles over X. However, in this article we show that 'perturbatively' this is possible. In other words, we construct explicitly a weight system $b_D(X; E_a)$ on chord diagrams from a collection of holomorphic vector bundles E_a over a compact hyperkähler manifold X. This leads to potentially new invariants of links

$$Z^{\text{RW}}(M;\mathcal{L}) = \sum b_D(X; E_a) Z_D^{\text{Kont}}(M;\mathcal{L}).$$

Rather than investigate these invariants of links, our main purpose in this paper is to use these ideas to obtain new results in hyperkähler geometry. For example, the weights $b_\Gamma(X)$ are invariant under deformations of the hyperkähler metric, and for particular choices of Γ give characteristic numbers. We can use the formalism of graph homology to relate certain invariants, in particular arriving at a formula for the norm of the curvature of X in terms of characteristic numbers and the volume of X. This is our most fruitful application of this theory to hyperkähler geometry, though the result should extend to the invariants of holomorphic vector bundles over X.

Some of the ideas presented in this paper have already been described in more detail in Hitchin and Sawon [7], and the entire work is a continuation of the research first presented in [11]. A complete account may be found in the author's PhD thesis [12].

The author wishes to thank his PhD supervisor and collaborator N. Hitchin. Conversations with D. Bar-Natan, J. Ellegard Andersen, S. Garoufalidis, L. Göttsche, M. Kapranov, G. Thompson, and S. Willerton have been very helpful. Support from Trinity College (Cambridge) and the local organizers to attend this meeting in Rome is gratefully acknowledged.

2. Hyperkähler geometry and definitions

Let X be a compact hyperkähler manifold of real-dimension $4k$. This means there is a metric on X whose Levi-Civita connection has holonomy contained in $\mathrm{Sp}(k)$. Such a manifold admits the following structures:

- complex structures I, J, and K acting like the quaternions on the tangent bundle T,
- a *hyperkähler* metric g Kählerian wrt I, J, and K,
- corresponding Kähler forms ω_1, ω_2, and ω_3.

Although there is a whole sphere of complex structures compatible with the hyperkähler metric, there is no natural way to choose one of them. However, since we wish to use the techniques of complex geometry we shall choose to regard X as a complex manifold with respect to I. Then we can construct a holomorphic symplectic form

$$\omega = \omega_2 + i\omega_3 \in \mathrm{H}^0(X, \Lambda^2 T^*)$$

on X, whose dual is

$$\tilde{\omega} \in \mathrm{H}^0(X, \Lambda^2 T).$$

Note that in local complex coordinates $\tilde{\omega}$ has matrix ω^{ij} which is minus the inverse of the matrix ω_{ij} of ω. The Riemann curvature tensor of the Levi-Civita connection of g is

$$K \in \Omega^{1,1}(\mathrm{End}T)$$

which has components $K^i{}_{jk\bar{l}}$ with respect to local complex coordinates. Using ω to identify T and T^*, we get

$$\Phi_{ijk\bar{l}} = \sum_m \omega_{im} K^m{}_{jk\bar{l}}.$$

This tensor is symmetric in j and k as the Levi-Civita connection is torsion-free and complex structure preserving. It is also symmetric in i and j due to the $\mathrm{Sp}(2k, \mathbb{C})$ reduction of the frame bundle which accompanies the hyperkähler structure. Therefore

$$\Phi \in \Omega^{0,1}(\mathrm{Sym}^3 T^*).$$

Let E be a holomorphic vector bundle over X of complex-rank r, and choose a Hermitian structure h on E. The unique connection ∇ on E which is compatible with both the Hermitian and holomorphic structures is called the *Hermitian connection*. The curvature

$$R \in \Omega^{1,1}(\mathrm{End}E)$$

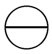
of this connection is of pure Hodge type and has components $R^I{}_{Jk\bar{l}}$ with respect to local complex coordinates on X and a local basis of sections of E.

Let Γ be an oriented trivalent graph with $2k$ vertices. The orientation means an equivalence class of orientations of the edges and an ordering of the vertices; if two such differ by a permutation π of the vertices and a reversal of the orientation on n edges then they are equivalent if $\mathrm{sign}\pi = (-1)^n$. Due to an argument of Kapranov [8] this notion of orientation is equivalent to the usual one given by an equivalence class of cyclic orderings of the outgoing edges at each vertex, with two such equivalent if they differ at an even number of vertices. Hence any trivalent graph drawn in the plane has a canonical orientation given by taking the anticlockwise cyclic ordering at each vertex. Note that Γ need not be connected, but we do not allow connected components which simply consist of closed circles.

Place a copy of Φ at each vertex of Γ and attach the holomorphic indices i, j, and k to the outgoing edges in any way. Place a copy of $\tilde{\omega}$ on each edge of Γ and attach the holomorphic indices i and j to the ends of the edges in a way compatible with the orientations of the edges. The ends of each edge will then have two indices attached to them, one coming from Φ and one coming from $\tilde{\omega}$. Now multiply all these copies of Φ and $\tilde{\omega}$, with the Φs multiplied in a way compatible with the ordering of the vertices, and then contract the indices at the ends of each edge. Finally, project to the exterior product to get an element

$$\Gamma(\Phi) \in \Omega^{0,2k}(X).$$

For example, suppose that Γ is the two-vertex graph



which we denote by the Greek letter Θ and call *theta*. The canonical orientation of this graph corresponds to ordering the vertices 1 and 2 with the three edges all oriented from 1 to 2 (or any equivalent arrangement). Therefore in local complex coordinates $\Theta(\Phi) \in \Omega^{0,2}(X)$ looks like

$$\Theta(\Phi)_{\bar{l}_1\bar{l}_2} = \Phi_{i_1j_1k_1\bar{l}_1}\Phi_{i_2j_2k_2\bar{l}_2}\omega^{i_1i_2}\omega^{j_1j_2}\omega^{k_1k_2}.$$

Note that X must be four real-dimensional in this example, ie. either a K3 surface S or a torus.

Returning to the general case, we multiply $\Gamma(\Phi)$ by ω^k which is a trivializing section of $\Lambda^{2k}T^*$. This gives us an element of $\Omega^{2k,2k}(X)$ which we can integrate to get a number.

Definition *The Rozansky-Witten invariant of X corresponding to the oriented trivalent graph Γ is*

$$b_\Gamma(X) = \frac{1}{(8\pi^2)^k k!}\int_X \Gamma(\Phi)\omega^k. \tag{1}$$

Now let D be a *chord diagram*, which consists of an oriented unitrivalent graph whose univalent (or *external*) vertices lie on a collection of oriented circles which we call the *skeleton* of the diagram. The orientation is given by an equivalence class of cyclic orderings of the outgoing edges at each trivalent (or *internal*) vertex, with two such equivalent if they differ at an even number of vertices. Including the skeleton, we can regard the entire diagram as being a trivalent graph with some extra information. Since the skeleton consists of *oriented* circles, it induces a canonical cyclic ordering of the outgoing edges at the external vertices, so this trivalent graph is also oriented. The corresponding ordering of the vertices and orientations of the edges can be chosen in a way compatible with the orientation of the skeleton since we are working in an equivalence class. Note that D may be disconnected; we even allow circles in the skeleton with no external vertex on them. We assume that D has $2k$ vertices (internal and external).

Let $E_1, \ldots, E_m$ be a collection of holomorphic vector bundles over X, one for each circle in the skeleton of D. Choose Hermitian structures on these bundles and denote the curvatures of the corresponding Hermitian connections by

$$R_a \in \Omega^{1,1}(\mathrm{End} E_a).$$

As before, we place a copy of Φ at each internal vertex of D and a copy of $\tilde{\omega}$ at each edge, and attach indices as before. Each circle in the skeleton will have a vector bundle E_a associated with it, and we place a copy of the curvature R_a of that vector bundle at each external vertex lying on that circle. Recall that in local complex coordinates, and with respect to a local basis of sections of E_a, this curvature has components $(R_a)^{I_a}{}_{J_a k\bar{l}}$. Then k should be attached to the outgoing edge, I_a to the *incoming* part of the skeleton, and J_a to the *outgoing* part of the skeleton (recall that the skeleton consists of *oriented* circles). Now multiply all these copies of Φ, $\tilde{\omega}$, and $R_1, \ldots, R_m$, with the Φs and R_as multiplied in a way compatible with the ordering of the vertices, and then contract the indices as before. For the curvatures attached to the external vertices, we contract indices like

$$\cdots (R_a)^{I_a}{}_{J_a k\bar{l}}(R_a)^{J_a}{}_{K_a m\bar{n}} \cdots$$

in an order which is compatible with the orientations of the circles making up the skeleton. If one of the circles has no external vertices lying on it, then we simply include a factor given by minus the rank of the vector bundle attached to that circle. Finally, projecting to the exterior product we get an element

$$D(\Phi; R_a) \in \Omega^{0,2k}(X).$$

As before, multiplying by ω^k gives us an element of $\Omega^{2k,2k}(X)$ which we can integrate.

Definition *The weight on the chord diagram D given by the vector bundles E_a over X is*

$$b_D(X; E_a) = \frac{1}{(8\pi^2)^k k!}\int_X D(\Phi; R_a)\omega^k. \tag{2}$$

For example, let S be a K3 surface, E a vector bundle over S, and D the chord diagram which is like the trivalent graph Θ, but with the outer circle being the skeleton. We usually break the skeleton at some (arbitrary) point and draw it as a directed line, and hence D looks like

In local complex coordinates $D(\Phi; R) \in \Omega^{0,2}(S)$ looks like

$$D(\Phi; R)_{\bar{l}_1\bar{l}_2} = R^I{}_{Jk_1\bar{l}_1}R^J{}_{Ik_2\bar{l}_2}\omega^{k_1k_2}$$

and

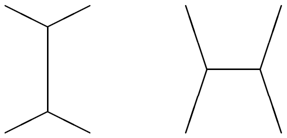
$$b_D(S; E) = \frac{1}{8\pi^2}\int_S D(\Phi; R)\omega.$$

This construction may be varied by replacing the curvatures $R_1, \ldots, R_m$, and K by the Dolbeault cohomology classes which they represent; these are the Atiyah classes of the bundles $E_1, \ldots, E_m$, and T, and are independent of the choices of Hermitian structures and hyperkähler metric (respectively) on these bundles. We can then calculate using cohomology instead of differential forms. In fact, we can define the Rozansky-Witten invariants $b_\Gamma(X)$ and the weights $b_D(X; E_a)$ for any holomorphic symplectic manifold X (ie. not necessarily Kähler). This approach is due to Kapranov [8].

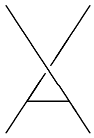
3. Properties of $b_\Gamma(X)$ and $b_D(X; E_a)$

Let us first mention some of the basic properties of $b_\Gamma(X)$.

1. Recall that we chose to regard X as a complex manifold with respect to I. In fact, $b_\Gamma(X)$ is independent of this choice of compatible complex structure. Furthermore, it is a real number.
2. If we deform the hyperkähler metric $b_\Gamma(X)$ remains invariant. In other words, $b_\Gamma(X)$ is constant on connected components of the moduli space of hyperkähler metrics on X. This essentially follows from the cohomological approach mentioned above.
3. The invariant $b_\Gamma(X)$ depends on the trivalent graph Γ only through its *graph homology* class. Graph homology is the space of rational linear combinations of oriented trivalent graphs modulo the AS and IHX relations. The former says that reversing the orientation of a graph is equivalent to changing its sign, and the latter says that three graphs Γ_I, Γ_H, and Γ_X which are identical except for in a small ball where they look like



and



respectively, are related by

$$\Gamma_I \equiv \Gamma_H - \Gamma_X.$$

In the Rozansky-Witten invariant context, the AS relations follow easily from the definition whereas the IHX relations follow by integrating by parts.
4. The factor

$$\frac{1}{(8\pi^2)^k k!}$$

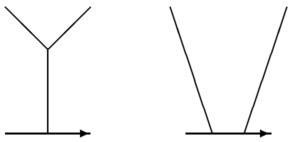
in Equation (1) has been carefully chosen. Firstly, dividing by $k!$ ensures that the invariants satisfy the following multiplicative property

$$b_\Gamma(X \times Y) = \sum_{\gamma \sqcup \gamma' = \Gamma} b_\gamma(X) b_{\gamma'}(Y) \tag{3}$$

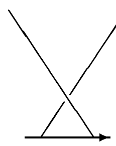
where X and Y are compact hyperkähler manifolds and the sum is over all ways of decomposing Γ into the disjoint union of two trivalent graphs γ and γ'. The additional factors lead to a nice formula for characteristic numbers in terms of Rozansky-Witten invariants which we shall describe in the next section. More importantly, our overall normalization agrees with Rozansky and Witten's.

These properties were known to Rozansky and Witten (for a slightly different presentation see [7] or [12]). We can show that the weights $b_D(X; E_a)$ satisfy similar properties.

1. Since the Atiyah class of E does not depend on the choice of Hermitian structure, we can show via the cohomological approach that neither does $b_D(X; E_a)$ depend on the choices of Hermitian structures on the bundles $E_1, \ldots, E_m$.
2. The weight $b_D(X; E_a)$ depends on D only through its chord diagram equivalence class. In other words, we should consider rational linear combinations of chord diagrams modulo the AS, IHX, and STU relations. The AS and IHX relations are as before, applied to internal vertices, while the STU relations are essentially the IHX relations applied to external vertices. More precisely, let D_S, D_T, and D_U be three chord diagrams which are identical except for in a small ball where they look like



and



respectively. Then they are related by

$$D_S \equiv D_T - D_U.$$

In fact, it can be shown that all of the AS and IHX relations follow from the STU relations. Once again, in the hyperkähler context the STU relations follow by integrating by parts.
3. In the case that all of the vector bundles $E_1, \ldots, E_m$ are trivial, we can choose flat connections and hence the curvatures R_a vanish. The only non-zero weights $b_D(X; E_a)$ will come from chord diagrams which are given by a trivalent graph Γ plus a skeleton consisting of a collection of disjoint circles[1]. Up to the additional factors corresponding to these circles, we simply get $b_\Gamma(X)$, and this is why we have chosen the same factor

$$\frac{1}{(8\pi^2)^k k!}$$

in Equation (2). Note that another way to obtain the Rozansky-Witten invariants $b_\Gamma(X)$ is by letting the vector bundles $E_1, \ldots, E_m$ be the tangent bundle T. In this case, the chord diagram D becomes a trivalent graph, with no distinction between the edges of the unitrivalent graph and the skeleton.

[1]This is analogous to the vanishing of Wilson lines in Chern-Simons theory when we associate the trivial representation to them.

These properties are discussed at greater length in [12]. Whether or not the weights $b_D(X; E_a)$ are also independent of the holomorphic structures on the vector bundles $E_1, \dots, E_m$ is a question requiring further investigation. We expect that in the case of hyperholomorphic bundles (as described in Verbitsky's talk at this meeting) the answer should be in the affirmative.

4. Examples

In this section we shall discuss some specific trivalent graphs and chord diagrams, and the Rozansky-Witten invariants and weights which they lead to. To begin with, suppose we have an *irreducible* hyperkähler manifold X. For such a manifold we know that

$$h^{0,q} = \begin{cases} 0 & \text{if } q \text{ is odd} \\ 1 & \text{if } q \text{ is even} \end{cases}$$

where $h^{p,q}$ are the Hodge numbers of X (see [3] for example). Now suppose we have a trivalent graph γ with $2m < 2k$ vertices. We can still construct

$$\gamma(\Phi) \in \Omega^{0,2m}(X)$$

as before. Furthermore, the Dolbeault cohomology class that this element represents lies in the one-dimensional cohomology group

$$\mathrm{H}^{2m}_{\bar{\partial}}(X).$$

This group is generated by $[\bar{\omega}^m]$ and hence

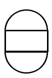
$$[\gamma(\Phi)] = c_\gamma [\bar{\omega}^m] \tag{4}$$

for some constant c_γ. Therefore if Γ is a trivalent graph with $2k$ vertices which decomposes into the disjoint union of the trivalent graphs $\gamma_1, \dots, \gamma_t$, then

$$b_\Gamma(X) = \frac{1}{(8\pi^2)^k k!} c_{\gamma_1} \cdots c_{\gamma_t} \int_X \bar{\omega}^k \omega^k \tag{5}$$

for irreducible X. This formula clearly generalizes to the case that the γ_i may be chord diagrams instead of trivalent graphs, and we introduce a collection of holomorphic vector bundles over X.

For example, if we let Θ_2 denote the trivalent graph



then

$$b_{\Theta^4}(X) b_{\Theta_2^2}(X) = b_{\Theta^2\Theta_2}(X) b_{\Theta^2\Theta_2}(X) \tag{6}$$

for an irreducible sixteen-dimensional manifold X, as both sides equal

$$c_\Theta^4 c_{\Theta_2}^2 \left(\frac{1}{(8\pi^2)^4 4!} \int_X \bar{\omega}^4 \omega^4 \right)^2 .$$

We can also calculate c_Θ explicitly in terms of the $\mathcal{L}^2$-norm of the curvature $\|R\|$ and the volume of X, and this leads to the formula

$$b_{\Theta^k}(X) = \frac{k!}{(4\pi^2 k)^k} \frac{\|R\|^{2k}}{(\mathrm{vol}X)^{k-1}} \tag{7}$$

for an irreducible hyperkähler manifold of real-dimension $4k$ (see [7]).

The other type of trivalent graphs (and chord diagrams) we shall be interested in are those constructed from *wheels*. Wheels are unitrivalent graphs consisting of a circle with attached spokes. We use the notation $w_{2\lambda}$ to denote a wheel with 2λ spokes. In the case of chord diagrams, we use the notation $\mathbf{w}_{2\lambda}$ to denote a wheel whose circle is oriented and part of the skeleton. Figure 1 shows some examples. Note

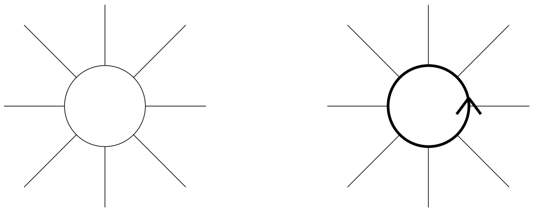


FIGURE 1. The wheels w_8 and $\mathbf{w}_8$

that we are primarily interested in wheels with an even number of spokes. A *polywheel* is obtained by taking the disjoint union of a collection of wheels $w_{2\lambda_1}, \ldots, w_{2\lambda_t}$ and then summing over all possible ways of joining their spokes pairwise, in order to obtain a trivalent graph. We denote this

$$\langle w_{2\lambda_1} \cdots w_{2\lambda_t} \rangle.$$

In the chord diagram case, some of the wheels $w_{2\lambda}$ may be replaced by $\mathbf{w}_{2\lambda}$. Now suppose that

$$\lambda_1 + \ldots + \lambda_t = k$$

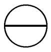
so that the trivalent graphs in the polywheel all have $2k$ vertices. Then for a hyperkähler manifold of real-dimension $4k$

$$b_{\langle w_{2\lambda_1} \cdots w_{2\lambda_t} \rangle}(X) = (-1)^t (2\lambda_1)! \cdots (2\lambda_t)! \int_X \mathrm{ch}_{2\lambda_1} \cdots \mathrm{ch}_{2\lambda_t} \tag{8}$$

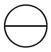
where $\mathrm{ch}_{2\lambda}$ is the $2\lambda^{\text{th}}$ component of the Chern character of X (see [7]). If some of the wheels $w_{2\lambda}$ are replaced by $\mathbf{w}_{2\lambda}$ and we introduce a collection of holomorphic vector bundles over X, then in the above formula $\mathrm{ch}_{2\lambda}$ should be replaced by $\mathrm{ch}_{2\lambda}(E)$, the $2\lambda^{\text{th}}$ component of the Chern character of E, where E is the vector bundle associated to that particular oriented circle in the skeleton of the chord diagram.

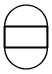
Thus every characteristic number of X can be expressed as a Rozansky-Witten invariant for some choice of linear combination of trivalent graphs. A fundamental question in this theory is "to what extend is the converse true?", ie. can every

k = 1

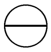
 $= \langle w_2 \rangle$

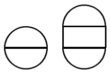
k = 2

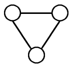
^{2} $= \langle w_2^2 \rangle - \frac{4}{5}\langle w_4 \rangle$

 $= \frac{2}{5}\langle w_4 \rangle$

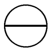
k = 3

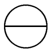
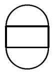
^{3} $= \langle w_2^3 \rangle - \frac{12}{5}\langle w_2 w_4 \rangle + \frac{64}{35}\langle w_6 \rangle$

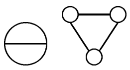
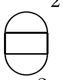
 $= \frac{2}{5}\langle w_2 w_4 \rangle - \frac{16}{35}\langle w_6 \rangle$

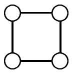
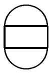
 $= \frac{4}{35}\langle w_6 \rangle$

k = 4

^{4} $= \langle w_2^4 \rangle - \frac{24}{5}\langle w_2^2 w_4 \rangle + \frac{48}{25}\langle w_4^2 \rangle + \frac{256}{35}\langle w_2 w_6 \rangle - \frac{1152}{175}\langle w_8 \rangle$

^{2}  $= \frac{2}{5}\langle w_2^2 w_4 \rangle - \frac{8}{25}\langle w_4^2 \rangle - \frac{32}{35}\langle w_2 w_6 \rangle + \frac{192}{175}\langle w_8 \rangle$

 $= -\frac{1}{2}$ ^{2} $+ \frac{2}{25}\langle w_4^2 \rangle + \frac{4}{35}\langle w_2 w_6 \rangle - \frac{48}{175}\langle w_8 \rangle$

 $= \frac{1}{12}$ ^{2} $- \frac{1}{75}\langle w_4^2 \rangle + \frac{8}{175}\langle w_8 \rangle$

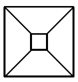
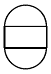
 $= -\frac{41}{96}$ ^{2} $+ \frac{41}{600}\langle w_4^2 \rangle - \frac{16}{175}\langle w_8 \rangle$

TABLE 1. Polywheels and graph homology

Rozansky-Witten invariant be expressed as a linear combination of Chern numbers? We will answer this in the negative in the next section, but first observe Table 1. For $k = 1$, 2, and 3 the graphs given on the left hand side span graph homology and can all be expressed as linear combinations of polywheels. Therefore the Rozansky-Witten invariants *are* all characteristic numbers for $k = 1$, 2, and 3. The first trivalent graph which is not equivalent to a linear combination of polywheels in graph homology is Θ_2^2 which occurs in degree $k = 4$. It is precisely this graph which we will show leads to an invariant which is not a linear combination of Chern numbers.

5. Some calculations

There are two well-known families of irreducible compact hyperkähler manifolds, the Hilbert schemes $S^{[k]}$ of k points on a K3 surface S and the generalized Kummer varieties K_k (see Beauville [3]). Apart from these, the only other known example of an irreducible compact hyperkähler manifold was constructed by O'Grady [9] in real-dimension 20 (as presented at this meeting). For the Hilbert schemes $S^{[k]}$ and generalized Kummer varieties K_k, there are generating sequences for the Hirzebruch χ_y-genuses due to Cheah [5] and Göttsche and Soergel [6]. We can try to use the Riemann-Roch formula to determine the characteristic numbers from this information. For $k = 1$, 2, and 3 this gives us k independent equations in k unknowns (the Chern numbers) which we can invert. Then according to the relations in Table 1, all the Rozansky-Witten invariants may be determined from this information. When $k = 4$ we get four independent equations in five unknowns, and hence we cannot determine all of the Chern numbers, let alone the Rozansky-Witten invariants, from what we know thus far.

Recall Equation (4) which says that

$$[\gamma(\Phi)] = c_\gamma[\bar{\omega}^m] \in \mathrm{H}^{2m}_{\bar{\partial}}(X).$$

The number c_γ may be a constant, but it depends on the manifold X. If we let X run through the family $S^{[k]}$ (respectively K_k), this means a dependence on k. For $\gamma = \Theta$, c_Θ is a linear expression in k (as proved in [12]), and using our calculations for $k = 1$, 2, and 3 we can determine this expression precisely. Substituting into Equation (5) gives us the following results

$$b_{\Theta^k}(S^{[k]}) = 12^k(k+3)^k \tag{9}$$

$$b_{\Theta^k}(K_k) = 12^k(k+1)^{k+1}. \tag{10}$$

From Table 1 we can see that b_{Θ^4} is a characteristic number. Therefore when $k = 4$ we get a fifth equation for the Chern numbers which we can combine with the four independent equations we already have, and this system can then be solved to give all of the Chern numbers. According to Table 1

$$b_{\Theta^2\Theta_2}(X)$$

may also be written in terms of Chern numbers, and hence can now be determined. Then

$$b_{\Theta_2^2}(X)$$

can be calculated from Equation (6), and this allows us to determine all the remaining Rozansky-Witten invariants for $S^{[4]}$ and K_4. For *reducible* compact hyperkähler manifolds in real-dimension sixteen, we merely need to apply the product formula (3).

There is evidence to suggest that c_γ is also linear in k for graphs γ other than Θ (possibly for all trivalent graphs). This would enable us to perform many more calculations, ie. for $k > 4$, though we shall not need such results here. In fact, we already know enough to show that the invariant

$$b_{\Theta_2^2}(X)$$

in real-dimension sixteen is not a linear combination of Chern numbers. We simply take two (disconnected) compact hyperkähler manifolds

$$48K_4 + 294S \times S^{[3]} + 144S^{[2]} \times S^{[2]} + 63S^4$$

and

$$336S^{[4]} + 268S^2 \times S^{[2]}$$

where '+' denotes disjoint union. The coefficients have been chosen so that both of these manifolds have the same Chern numbers. However, our calculations reveal that

$$b_{\Theta_2^2}(48K_4 + 294S \times S^{[3]} + 144S^{[2]} \times S^{[2]} + 63S^4) \neq b_{\Theta_2^2}(336S^{[4]} + 268S^2 \times S^{[2]})$$

and therefore the Rozansky-Witten invariant $b_{\Theta_2^2}$ is *not* a characteristic number. On the other hand, although this Rozansky-Witten invariant cannot be written as a linear combination of Chern numbers, for X irreducible and connected Equation (6) implies that it can be written as a *rational* function of Chern numbers. Hence whether or not the Rozansky-Witten invariants are *really* more general than characteristic numbers is a fairly subtle question.

6. The Wheeling Theorem

The space of equivalence classes of chord diagrams admits two different product structures. The Wheeling Theorem is an isomorphism $\hat{\Omega}$ between the two resulting algebras, which is constructed quite explicitly from a particular linear combination of disjoint unions of wheels

$$\begin{aligned}\Omega &= 1 + \frac{1}{48}w_2 + \frac{1}{2!48^2}(w_2^2 - \frac{4}{5}w_4) + \dots \\ &= \exp_\cup \sum_{m=1}^{\infty} b_{2m} w_{2m}\end{aligned}$$

where

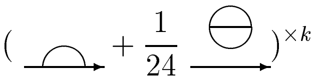
$$\sum_{m=0}^{\infty} b_{2m} x^{2m} = \frac{1}{2}\log\frac{\sinh(x/2)}{x/2}$$

and $\exp_\cup$ means we exponentiate using disjoint union of graphs as our product. This Theorem was recently proved by Bar-Natan, Le, and Thurston [2], and we refer to [1] for a detailed statement of the result. Of course, the isomorphism may be thought of as a family of relations among equivalence classes of chord diagrams

$$\hat{\Omega}(xy) = \hat{\Omega}(x)\hat{\Omega}(y) \tag{11}$$

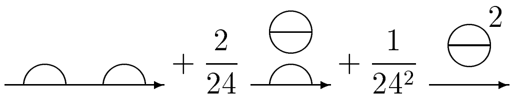
where x and y are chord diagrams, and in this sense it is really a statement about the remarkable properties of Ω. In the Rozansky-Witten context, we wish to investigate the consequences of these relations for our invariants of hyperkähler manifolds and their vector bundles.

The particular relations we are interested in are a special case of (11) and look like



$$= 2^k k! \left(\frac{1}{(2k)!} \langle \Omega_0 \mathbf{w}_{2k} \rangle + \frac{1}{(2k-2)!} \langle \Omega_2 \mathbf{w}_{2k-2} \rangle + \ldots + \langle \Omega_{2k} \mathbf{w}_0 \rangle \right) \tag{12}$$

where Ω_{2m} is the $2m^{\text{th}}$ term of Ω, which consists of wheels and their disjoint unions having $2m$ external legs, and $\times k$ means that we take the k^{th} power where multiplication is given by juxtaposition of skeletons (which are written as directed lines). Note that since we are quotienting by the STU relations, this multiplication is in fact commutative. For example, when $k = 2$ the left hand side of Equation (12) looks like

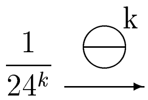


Now suppose we have a compact hyperkähler manifold X of real-dimension $4k$ with a holomorphic vector bundle E over it. Since polywheels give rise to Chern numbers, we expect the weight corresponding to the right hand side of Equation (12) to give us some characteristic number. In fact, the precise form of Ω (in particular, the appearance of $\frac{\sinh(x/2)}{x/2}$ in its generating function) means that we get

$$-2^k k! \int_X \mathrm{Td}^{1/2}(T) \wedge \mathrm{ch}(E) \tag{13}$$

where projection of the integrand to the space of top degree forms before integrating is assumed. The weight corresponding to the left hand side of Equation (12) is less easy to interpret. Let us look at the simplest possible case where E is a trivial vector bundle.

As mentioned earlier, the only weights which do not vanish in this case are those coming from chord diagrams consisting of a trivalent graph plus a skeleton consisting of a disjoint circle. The only such chord diagram in the left hand side of Equation (12) is



and the corresponding weight is

$$\frac{-\mathrm{rank} E}{24^k} b_{\Theta^k}(X).$$

On the other hand, the Chern character

$$\mathrm{ch}(E) = \mathrm{rank} E$$

for E trivial and hence (13) becomes

$$-2^k k!\mathrm{rank}E \int_X \mathrm{Td}^{1/2}(T).$$

Therefore

$$b_{\Theta^k}(X) \quad = \quad 48^k k! \int_X \mathrm{Td}^{1/2}(T). \tag{14}$$

We already have a formula for $b_{\Theta^k}(X)$ when X is irreducible, namely Equation (7), and it follows that in this case the $\mathcal{L}^2$-norm of the curvature $\|R\|$ of X can be expressed in terms of characteristic numbers and the volume of X. This is the main result of [7], where the precise formula may be found. Also, since $\|R\|$ and the volume must be positive, we can conclude that

$$\int_X \mathrm{Td}^{1/2}(T) > 0$$

for irreducible manifolds X. For example, in eight real-dimensions this implies that the Euler characteristic

$$c_4(X) < 3024.$$

In fact, it follows from a result of Bogomolov and Verbitsky (see Beauville [4]) that the sharp upper bound in this case is 324. The author is grateful to Beauville for pointing this out.

Of course the holomorphic vector bundle E has disappeared entirely from Equation (14). To generalize this result to non-trivial vector bundles E we need a better understanding of the weights corresponding to particular chord diagrams, and their relations to standard invariants of vector bundles; we have already seen that characteristic numbers arise - perhaps certain norms of the curvatures of these bundles should also appear. Ultimately one would like a complete interpretation of Equation (11) in the Rozansky-Witten context.

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