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QUATERNIONIC GROUP REPRESENTATIONS AND THEIR CLASSIFICATIONS

G. SCOLARICI AND L. SOLOMBRINO

ABSTRACT. We study quaternionic group representations of finite groups systematically and obtain some basic tools of the theory, such as orthogonality relations and the Clabsch-Gordan series for reducible representations. We also derive all irreducible inequivalent Q-representations of a group G, classifying them according to a suitable generalization of the Wigner and the Frobenius-Schur classification. Some applications to physical problems and to the time reversal symmetry are shown.

1. INTRODUCTION

In the first part of this communication we intend to inquire quaternionic group representations (QGR) directly (i.e. without the detour of transcribing the quaternion operators into complex ones via the symplectic representation) and systematically, going over the basic steps of the theory.

When dealing with this subject the main difficulties come from the non commutativity of Q, which complicates from the very beginning the basic problem of the invertibility of a linear mapping, and the usual form of the character of a representation must be abandoned in favor of a (seemingly) weaker characterization. Moreover the corollary of the Schur's lemma (which is a basic tool for the analysis of representations and for deriving orthogonality relations) fails to be true in its usual form. This notwithstanding, we obtain some orthogonality relations for linear representations and characters in QGR, that can be applied to analyze any reducible Q-representation; in particular we obtain all the (inequivalent) irreducible Q-representations (*Q-irreps*) of a (finite) group G and classify them according to a generalization of the well-known Frobenius-Schur classification [2,8] of C-representations.

The second part of this communication, is devoted to some applications, mainly regarding magnetic groups. The time-reversal symmetry, which is described in complex quantum mechanics by an antiunitary operator, brings out the necessity of introducing the more general concept of *corepresentations* (i.e., representations by unitary and antiunitary operators) whenever the symmetry group contains a time-inversion operator.

In the framework of Quaternionic Quantum Mechanics (QQM) the time-inversion operator is still unitary, with the remarkable property that it anticommutes with the anti-self-adjoint operator which represents the Hamiltonian of the physical system [1]. It follows that one can study the symmetry groups including time-reversal by the same methods adopted in order to study symmetry groups containing spatial symmetries only.

We apply to the magnetic groups a further classification of groups, which in some sense replaces the Wigner classification of corepresentations [9,10] and can be crossed with the generalized Frobenius-Schur classification in order to get a more general classification of these groups. Some physical applications are briefly sketched in the conclusions, from which a suggestion arises to inquire into parity violation from a purely group theoretical point of view.

The main results of this comunication have been exposed in ref.[14,15].

2. Unitary Q-representations

In a (right) n-dimensional vector space Q^n over Q, every linear operator is associated in a standard way [4] to a n×n matrix acting on the left.

In analogy with the case of complex group representations, one can then define the hermitian conjugate $A^\dagger = \overline{A}^T$ of a matrix A (A^T and $\overline{A}$ denote, as usual, the transpose and the quaternionic conjugate of A, respectively), and introduce the concepts of unitarity, hermiticity and so on. The properties of hermitian and unitary matrices in Q^n have been widely investigated [6,7]; moreover if G is a finite (or a compact) group, one can always assume unitarity and complete reducibility of the quaternionic representations [13].

Finally we recall that for Q-irreps Schur's lemma still holds [7] and one obtains as a corollary that: *"If a Hermitian matrix H commutes with an irreducible set D of matrices,it is a (real) multiple of the unit matrix"* [7].

The above corollary allows one to prove the following proposition:

"The equivalence between unitary Q-representations can always be effected by a unitary matrix".

Proof. Let D_1 and D_2 be two equivalent unitary irreducible Q-representations, and let T be the matrix that effects the equivalence between them:

$$D_1 T = T D_2.$$

The conjugate of previous equation reads

$$T^\dagger D_1^\dagger = D_2^\dagger T^\dagger$$

or, recalling the unitarity of D_1 and D_2,

$$T^\dagger D_1 = D_2 T^\dagger.$$

Then, $TT^{\dagger}D_1 = TD_2T^{\dagger} = D_1TT^{\dagger}$, i.e., the hermitian matrix $TT^{\dagger}$ commutes with D_1 and by the Schur Lemma, $TT^{\dagger} = r\mathbf{1}, r \in R$. Moreover, $r \geq 0$ being trivially

$$\forall \left|\phi\right\rangle \in Q^n, \left\langle \phi\right| T^{\dagger}T \left|\phi\right\rangle = r\langle \phi \mid \phi\rangle = \left\|T \left|\phi\right\rangle\right\|^2 .$$

Hence, $T' = \frac{1}{\sqrt{r}}T$ is a unitary matrix such that

$$D_1 = T'D_2T'^{-1}.$$

The proof for reducible representations follows at once, observing that $D_1 \cong D_2$ if and only if the irreducible blocks in their decomposition are both equivalent (see sect.3).

3. Orthogonality relations and analysis of Q-representations

Let $D(G)$ be an n-dimensional irreducible and unitary Q-representation of a finite group G and let us consider the matrix

$$A = \sum_{g\in G} D\left(g^{-1}\right) XD\left(g\right) = \sum_{g\in G} \overline{D}^T\left(g\right) XD\left(g\right) \tag{3.1}$$

with X hermitian; then , trivially, $A = A^{\dagger}$.

Indeed

$$A_{ij} = \sum_{g\in G}\sum_{k,l} \overline{D}_{ki}\left(g\right) X_{kl} D_{lj}\left(g\right) = \overline{A}_{ji}; \tag{3.2}$$

moreover $D\left(g\right) A = AD\left(g\right), \forall g \in G$.

By using the corollary of Schur's lemma [7],

$$A = \lambda^{(X)} I_n \tag{3.3}$$

where $\lambda^{(X)} \in R$ and I_n is the unit $n \times n$ matrix.

Let us now choose in Eq. (3.1) a matrix $X^{(r)}$ in such a way that $X^{(r)}_{kl} = \delta_{kr}\delta_{lr}$, with r fixed, and take the real trace of A. Recalling that the real trace satisfies the cyclic property $ReTrBC = ReTrCB$ [7], we obtain

$$ReTrA = \sum_{g} ReTrX^{(r)} = [G] = \lambda^{(r)}n \tag{3.4}$$

where $[G]$ is the order of G.

By substituting the explicit form of $X^{(r)}_{kl}$ and $\lambda^{(r)}$ in Eq. (3.2), we easily obtain

$$\sum_{g\in G} \overline{D}_{ri}\left(g\right) D_{rj}\left(g\right) = \frac{[G]}{n}\delta_{ij}. \tag{3.5}$$

Analogously, let $D^{(\mu)}(G)$ and $D^{(\nu)}(G)$ $(\mu \neq \nu)$ be two unitary inequivalent Q-irreps of G whose dimensions respectively are n_μ and n_ν; then the matrix

$$A = \sum_{g \in G} D^{(\mu)}\left(g^{-1}\right) X D^{(\nu)}(g) \tag{3.6}$$

for every matrix X , satisfies the condition

$$D^{(\mu)}(h)\, A = A D^{(\nu)}(h) \qquad \forall h \in G.$$

By using Schur' lemma [7], we conclude that A must vanish identically.

Choosing in Eq. (3.6) a matrix $X^{(rs)}$ such that $X_{kl}^{(rs)} = \delta_{kr}\delta_{ls}$ with r, s fixed and writing down the explicit form of A_{ij}, we obtain

$$\sum_{g \in G} \overline{D}_{ri}^{(\mu)}(g)\, D_{sj}^{(\nu)}(g) = 0 \tag{3.7}$$

and finally (expressing Eqs. (3.5) and (3.7) in a more compact form),

$$\sum_{g \in G} \overline{D}_{ri}^{(\mu)}(g)\, D_{rj}^{(\nu)}(g) = \frac{[G]}{n_\mu}\delta_{ij}\delta_{\mu\nu}, \tag{3.8}$$

which is the (weaker) analogue for Q-irreps of the orthogonality relation for C-irreps.

Let us put now $r = i$ and $s = j$ in Eq. (3.8), and let us sum over i and j; then,

$$\sum_{g} \chi^{(\mu)Q}(g)\, \chi^{(\nu)}(g) = 0 \tag{3.9}$$

where $\chi^{(\mu)}(g)$ denotes the (full) trace of $D^{(\mu)}(g)$. Eq. (3.9) express the orthogonality between (quaternionic) characters of two inequivalent Q-irreps of the group G.

On the other hand,the following identity holds:

$$\hat{\chi}^{(\mu)}(g) \equiv Re\chi^{(\mu)}(g) = \frac{1}{4}\left[\chi^{(\mu)}(g) - i\chi^{(\mu)}(g)\, i - j\chi^{(\mu)}(g)\, j - k\chi^{(\mu)}(g)\, k\right]$$

and each term in parentheses, say $-i\chi^{(\mu)}(g)\, i$, can be considered as the character of g in a Q-representation (in our case $-iD^{(\mu)}i$), which is equivalent to the $D^{(\mu)}$ but certainly inequivalent to the $D^{(\nu)}$ [13]. For, we easily get the following relation from Eq. (3.9)

$$\frac{1}{\sum_g \hat{\chi}^{(\mu)2}(g)} \sum_g \hat{\chi}^{(\mu)}(g)\, \hat{\chi}^{(\nu)}(g) = \delta_{\mu\nu}$$

or also (remembering that conjugated elements of a group have the same real character)

$$\frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i^{(\mu)} \hat{\chi}_i^{(\nu)} = \delta_{\mu\nu}, \tag{3.10}$$

where $\hat{\chi}_i^{(\mu)}$ obviously indicates the (real) character of all elements belonging to the i-th conjugation class of G, and k_i is the number of the elements of such a class.

As usual in CGR theory, Eq. (3.10) can be read as an orthogonality relation between vectors in a κ-dimensional space (where κ is the number of the conjugation classes of G),so that we finally obtain that the number r of inequivalent Q-irreps of G must satisfy the following inequality

$$r \leq \kappa$$

(and some cases occur in which strict inequality holds).

The possibility of decomposing any reducible Q-representation follows at once from these results. Indeed, let

$$D(G) = \sum_\mu a_\mu D^{(\mu)}(G)$$

be the Clebsh-Gordan series of a reducible Q-representation $D(G)$. Then, trivially,

$$\hat{\chi}(g) = \sum_\mu a_\mu \hat{\chi}^{(\mu)}(g) \qquad \forall g \in G.$$

By using Eq. (3.10) we obtain

$$a_\mu = \frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i \hat{\chi}_i^{(\mu)}$$

and this decomposition is unique, so that we can finally assert that *two Q-representations are equivalent if and only if their (real) characters coincide.*

4. Q-irreps and the generalized Frobenius-Schur classification

In order to obtain all the Q-irreps, we recall that any C-irrep of a group G can obviously be considered as a (not necessary irreducible) Q-representation and an important theorem (*Main Reduction Theorem*) states that: *"A C-irrep D reduces over Q (into two equivalent Q-irreps D_1 and D_2) if and only if D is equivalent to its complex conjugate D^* by an antisymmetric matrix"* [7].

Moreover we can prove that: *"All the Q-representations found in the sense of Main Reduction Theorem are inequivalent to each other, with the exception of those generated by a pair of complex conjugated representations such that $D \ncong D^*$"* [14],

Recalling that, in the realm of CGR, *"Two inequivalent C-irreps share the same real part of the character if and only if they are complex conjugate of each other"* [14], we can conclude that the choice of characterizing any Q-representation by means of the real part of the trace (due to the necessity of maintaining the cyclic property of this quantity) does not eliminate any relevant information.

Finally we prove that:

"No Q-irrep exists besides those generated (in the sense of the Main Reduction Theorem) by the C-irreps" [14].

Proof. (We give here a more direct proof of this proposition, with respect to ref. [14])

Let $D = D_1 + jD_2$ be a purely quaternionic representation (i.e., D_1 and D_2 are complex matrices and $D_2 \neq \mathbf{0}$ in every basis); if we take the direct sum

$$\begin{pmatrix} D_1 + jD_2 & 0 \\ 0 & D_1 + jD_2 \end{pmatrix},$$

and perform the similarity transformation

$$\frac{1}{2}\begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -j\mathbf{1} & k\mathbf{1} \end{pmatrix}\begin{pmatrix} D_1 + jD_2 & 0 \\ 0 & D_1 + jD_2 \end{pmatrix}\begin{pmatrix} \mathbf{1} & j\mathbf{1} \\ i\mathbf{1} & -k\mathbf{1} \end{pmatrix} = \begin{pmatrix} D_1 & -D_2^* \\ D_2 & D_1^* \end{pmatrix},$$

we obtain a complex representation which is equivalent to its complex conjugate by an antisymmetric matrix:

$$\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} D_1 & -D_2^* \\ D_2 & D_1^* \end{pmatrix}\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} D_1^* & -D_2 \\ D_2^* & D_1 \end{pmatrix}.$$

One can easily verify that the complex commutant of such representation is a complex multiple of identity operator, therefore, by the corollary of Schur's Lemma for C-irreducible representations, this representation is irreducible over the complex field. The Main Reduction Theorem, cited above, ensures on the other hand that to any C-representation with the previous properties is associated an purely quaternionic irreducible Q-representation and the theorem is prooved.

We have shown elsewhere [14] that all irreducible linear (inequivalent) Q-representations of a finite group G fall into three classes: *potentially real or of type R, potentially complex or of type C, (purely) quaternionic or of type Q (generalized Frobenius-Schur classification).* The generalized irreducibility criterion reads

$$\sum_g \hat{\chi}^{(\mu)2}(g) = \frac{[G]}{c^{(\mu)}} \tag{4.1}$$

where $c^{(\mu)} = \begin{cases} 1 \\ 2 \\ 4 \end{cases}$ when the representation $D^{(\mu)}$ is of type $\begin{cases} R \\ C \\ Q \end{cases}$.

Let us recall [14] that the following relation occur between the character $\chi_C^{(\mu)}$ of a complex representation and the character $\hat{\chi}^{(\mu)}$ of the corresponding quaternionic representation:

$$\hat{\chi}^{(\mu)} = \begin{cases} \chi_C^{(\mu)} \\ Re\chi_C^{(\mu)} \\ \frac{1}{2}\chi_C^{(\mu)} \end{cases} \text{ when } D^{(\mu)} \text{ of type } \begin{cases} R \\ C \\ Q \end{cases}. \tag{4.2}$$

Then, we also obtain, by using the classical Frobenius-Schur criterion:

$$\sum_g \hat{\chi}^{(\mu)}\left(g^2\right) = d^{(\mu)}[G] \tag{4.3}$$

where $d^{(\mu)} = \begin{cases} +1 \\ 0 \\ -\frac{1}{2} \end{cases}$ if $D^{(\mu)}$ is a Q-irrep of type $\begin{cases} R \\ C \\ Q \end{cases}$.

We conclude that *the couple of values of* $\sum_g \hat{\chi}^{(\mu)2}(g)$ *and* $\sum_g \hat{\chi}^{(\mu)}\left(g^2\right)$ *uniquely identifies all Q-irreps and their class.*

5. Magnetic groups and their classification

Color groups are defined in the literature[12] by

$$G' = G + aG, \qquad a \notin G, \tag{5.1}$$

where a is an operator which switches color (or, even, a product of such operator with a spatial symmetry which does not belong to G) and G is a (normal) subgroup of G' of index 2, whose elements represent spatial symmetries. In the CGR theory, the same equation defines the *magnetic groups* [3,11], where the elements of the coset aG are antiunitary operators. We call magnetic group in the following any group defined by Eq. (5.1), without entering into the physical interpretation of the elements of aG . We only characterize algebraically these elements by requiring that *all elements in G commute with a given operator, say H, while all elements in aG anticommute with it.*

We are now ready to study and possibly classify the representations of magnetic groups in the spaces Q^n. Let X be a finite dimensional vector space and let $D(G')$ be an irreducible (unitary) representation of a magnetic group G' in X. Whenever the restriction of $D(G')$ to G is reducible, let X_1 be an irreducible G-invariant subspace of X and let $\{\mid e_i\rangle\}$ be a basis in it. Then,

$$\langle e_i \mid D(g) \mid e_j\rangle = D_{ij}(g) \doteq \Delta_{ij}(g) \qquad \forall g \in G; \tag{5.2}$$

moreover, if $\mid f_i\rangle \doteq D(a) \mid e_i\rangle$, we get

$$\bar{\Delta}_{ij}(g) \doteq \langle f_i \mid D(g) \mid f_j\rangle = \langle e_i \mid D\left(a^{-1}\right) D(g) D(a) \mid e_j\rangle = \Delta_{ij}\left(a^{-1}ga\right). \tag{5.3}$$

Since $a^{-1}ga \in G$, the set of matrices $\bar{\Delta}(G)$ coincides with $\Delta(G)$ which is supposed irreducible in X_1 (then, they share the same global properties); for, $\bar{\Delta}(G)$ too is an irreducible representation of G in $X_2 = D(a) X_1$. Furthermore, we note that

$$D(a) X_2 = D\left(a^2\right) X_1 = X_1.$$

Now, let us observe that both the subspaces $X_1 \cap X_2$ and $X_1 \oplus X_2$ are G'-invariant; being by hypothesis $D(G')$ irreducible, we easily obtain $X_1 \cap X_2 = \emptyset$ and $X_1 \oplus X_2 = X$.

Choosing as a basis in X the set $\{| e_i\rangle\} \cup \{| f_j\rangle\}$, we get

$$D(G) = \begin{pmatrix} \Delta(G) & 0 \\ 0 & \bar{\Delta}(G) \end{pmatrix}, D(a) = \begin{pmatrix} 0 & \Delta(a^2) \\ \mathbf{1} & 0 \end{pmatrix}, \tag{5.4}$$

and two cases arise, according to whether Δ is equivalent to $\bar{\Delta} : \Delta \cong \bar{\Delta}$ or not.

We have thus obtained a threefold classification of the irreducible representations of magnetic groups :

I- *the restriction $D(G)$ of D to the subgroup G is irreducible;*

II- *$D(G)$ is reducible and has the above form, with $\Delta \ncong \bar{\Delta}$;*

III- *$D(G)$ is reducible and has the above form, with $\Delta \cong \bar{\Delta}$.*

This classification makes no reference to the scalar field of the vector space X, so that it generalizes the Wigner classification of corepresentations [18] in CGR theory and can replace it in the framework of QGR theory.

Thus, the idea arises to cross this new classification with the generalized FS classification discussed in Sect.(4), so as to obtain a more detailed description of Q-irreps of magnetic groups.

By using the orthogonality relations we can prove [15] that case I splits into five subcases:

I-R $D(G') \backsim R, D(G) \backsim R$ (i.e., $D(G)$ and $D(G')$ both of type R)

I-C/R $D(G') \backsim C, D(G) \backsim R$

I-C/C $D(G') \backsim C, D(G) \backsim C$

I-Q/C $D(G') \backsim Q, D(G) \backsim C$

I-Q/Q $D(G') \backsim Q, D(G) \backsim Q$,

case II splits into three subcases [15]:

II-R $D(G') \backsim R, D(G) \backsim R + R$

II-C $D(G') \backsim C, D(G) \backsim C + C$

II-Q $D(G') \backsim Q, D(G) \backsim Q + Q$.

(We denote by $R + R, C + C, Q + Q$ a decomposition of $D(G)$ in two inequivalent representations of type R, C, Q respectively.)

Finally, case III splits in two subcases [15] only:

III-R $D(G') \backsim R, D(G) \backsim 2C$;

III-C $D(G') \backsim C, D(G) \backsim 2Q$.

(We denote by $2C, 2Q$ here a decomposition of $D(G)$ in two equivalent representations of type C, Q respectively.) We observe that the above crossed classification is not trivial, because some of the nine cases that one could in principle obtain split in subcases, whereas one of them cannot occur, so that it provides a valuable insight into the properties of magnetic groups and their Q-irreps.

A remarkable role is played among the magnetic groups by the *factorizable* groups, the physical interest of which has been widely outlined [5,10].

We recall that a magnetic group $G' = G + aG$ is said to be factorizable if the automorphism

$$g \to g' = a^{-1}ga \qquad \forall g \in G \tag{5.5}$$

is an inner automorphism, i.e., an element $w \in G$ exists such that $g' = w^{-1}gw, \forall g \in G$. It is easy to see that G' is factorizable if and only if an element $t = aw^{-1} \in aG$ exists which commutes with all elements in G (hence with a, that is with all elements in G').

In many physical applications, when such an operator t exists, it is interpreted as a *time-inversion operator.* Indeed Adler [1] has shown that in the realm of QQM all spatial symmetries commute with the Hamiltonian H of the system, whereas the time-inversion operator anticommutes with H and commutes with all spatial symmetries; thus, in this framework, every symmetry group containing the time-inversion operator is a factorizable group.

We therefore studyed magnetic groups of the form

$$G' = G + tG, \qquad [t, G] = 0, \tag{5.6}$$

and determined that only the cases *I-R, I-C/C, I-Q/Q,I-C/R, III-C* of the crossed classification actually occur for such groups. In the cases *I-R, I-C/C, I-Q/Q* results $D(t^2) = \mathbf{1}$, and in the cases *I-C/R, I-C/C, III-C* we obtain $D(t^2) = -\mathbf{1}$.

If one now recalls that the squared time-inversion operator in QQM [1] is equal to the identity for fermionic systems, it has opposite sign for bosonic systems, we conclude that:

i) whenever a fermionic system is considered, a magnetic factorizable group falls into one of the cases I-R, I-C/C, I-Q/Q of the previous classification and $D(t^2) = \mathbf{1}$;

ii) whenever a bosonic system is considered, a magnetic factorizable group falls into one of the cases I-C/R, I-C/C, III-C of the previous classification and $D(t^2) = -\mathbf{1}$.

6. Conclusions

We conclude the discussion recalling that the mathematical methods and results developed in this communication have been applied to quantum physical problems, such as the study of degeneracy of energy levels in QQM whenever a time-reversal symmetry exists [15] (Kramers degeneracy). Kramers theorem applies in the context of CQM [17] and states that all energy levels of a fermionic system must be at least doubly degenerate, as really happens. Of course, Kramers degeneracy must appear in all attempts of modifying or generalizing ordinary quantum mechanics; our results perfectly agree with the experimental data.

Secondly we obtained the Q-representations of the quaternionic complete symmetry group [16] (obtained by extending the connected Poincare' group and the internal symmetry group by means of the CPT (Θ_0) and the generalized parity ($\mathcal{P}$) operators), in order to classify the particle multiplets.

Further investigations are suggested by an examination of the explicit forms of the Q-representations of the complete group. For instance, one of the possible forms of such extensions is, in a suitable basis:

$$(6.1)\quad \mathcal{D}(G)=\begin{pmatrix}\Delta^*(G) & \mathbf{0}\\ \mathbf{0} & \Delta(G)\end{pmatrix},\mathcal{D}(\Theta_0)=\begin{pmatrix}\mathbf{0} & k\mathbf{1}\\ k\mathbf{1} & \mathbf{0}\end{pmatrix},\mathcal{D}(\mathcal{P})=\begin{pmatrix}\mathbf{0} & -S_1\\ S_1^* & \mathbf{0}\end{pmatrix}$$

and the Hamiltonian is

$$H=ih_0\mathbf{1},\quad h_0\in R, \tag{6.2}$$

where $\Delta(G)$ is a Q-irrep of the internal symmetry group G, and Θ_0 and $\mathcal{P}$ denote the CPT and the parity operators, respectively.

On the other hand, if we consider a physical theory which is not invariant with respect to the (generalized) parity operator and then study the extension of the same representation $\Delta(G)$ of G obtained by means only of Θ_0, the case I-C/C arises. Performing again a suitable change of basis, we obtain:

$$D(G)=\begin{pmatrix}\Delta^*(G) & \mathbf{0}\\ \mathbf{0} & \Delta(G)\end{pmatrix},D(\Theta_0)=\begin{pmatrix}\mathbf{0} & k\mathbf{1}\\ k\mathbf{1} & \mathbf{0}\end{pmatrix} \tag{6.3}$$

and the Hamiltonian is:

$$H'=ih_0\mathbf{1}+jh_1\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix},h_0,h_1\in R, \tag{6.4}$$

It follows at once that the representations of G and Θ_0 are identical in both cases: but the presence in the former case of a further symmetry, namely $\mathcal{P}$, forces the cancellation in the form of H of the genuinely quaternionic term in j, to which we can then ascribe the parity violation (in perfect accordance with some arguments due to Adler [1] in a very different context).

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G. Scolarici: Università di Lecce and INFN sez. di Lecce, via per Arnesano, 73100 Lecce - Italy

E-mail address: `Scolarici@le.infn.it`

L. Solombrino: Università di Lecce and INFN sez. di Lecce, via per Arnesano, 73100 Lecce - Italy

E-mail address: `Solombrino@le.infn.it`