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# ON STOCHASTIC EVOLUTION OF A GENERAL RELATIVISTIC QUANTUM PARTICLE WITHOUT CONSERVATION OF ENERGY

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Abstract. We prove existence of solutions for Newton-Nelson equation on manifolds with connections for systems with non-conservative force fields and as a corollary validate a certain extension of the procedure of quantization for particles motion in General Relativity.

## 1. Introduction.

Here we deal with the so-called Newton-Nelson equation describing the evolution of a quantum particle in stochastic terms. It is well-defined for a broader class of forces, than the Schrödinger equation and (in relativistic case) Klein-Gordon or Dirac equations, and it is equivalent to those if they exist. The main problem of our interest in this paper is to show the existence of solutions of the Newton-Nelson equation for forces for which classical (Schrödinger, etc.) methods of quantization are ill-posed. We consider the general case of systems on manifolds with connections and, as a corollary, on Lorentz manifolds, with non-conservative forces and with both random and deterministic initial data. For such systems we show the existence of a solution of the Newton-Nelson equation and so validate a certain extension of the procedure of quantization. The case of deterministic initial data leads to a picture looking similar to a description of Big Bang.

The material is organized as follows. In Section 2 we describe the notion of Nelson's mean derivative for stochastic processes in $R^n$. Section 3 develops methods for studying systems on manifolds with connections. This forms the main tool for further considerations. In Section 4 we prove the existence results and Section 5 is devoted to the general relativistic systems.

We refer the reader to [9] and [10] for details of the involved machinery and for some previous results in the themes.

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## 2. Mean derivatives in $R^n$.

Let $x(t)$ and $y(t)$ be $L^1$-stochastic processes in $R^n$ defined on a certain probability space $(\Omega, F, P)$. Introduce $y$-forward derivative of $x(t)$ by the formula

$$D^y x(t) = \lim_{\Delta t \to +0} E^y_t\left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right) \quad (1)$$

and $y$-backward derivative of $x(t)$ by the formula

$$D^y_* x(t) = \lim_{\Delta t \to +0} E^y_t\left(\frac{x(t) - x(t-\Delta t)}{\Delta t}\right) \quad (2)$$

where, $E^y_t$ is the conditional expectation with respect to the $\sigma$-subalgebra of $F$ , generated by preimages of Borel sets in $R^n$ with respect to the map $y(t) : \Omega \to R^n$. $\Delta t \to +0$ means that $\Delta t$ tends to 0 and $\Delta t > 0$. If $x(t)$ coincides with $y(t)$ we shall denote the mean derivatives by $Dx(t)$ and $D_*x(t)$, respectively.

Normally the limits above are assumed to exist in $L^1(\Omega, F, P)$. But according to the original Nelson's definition (see, e.g., [13]-[15]) we can consider the limits in any sense in which they exist.

From the properties of the conditional expectation it follows that for a process $\xi(t)$ the derivatives $D\xi(t)$ and $D_*\xi(t)$ are expressed as compositions of $\xi(t)$ and the Borel measurable vector fields, namely the regressions

$$Y^0(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \mid \xi(t) = x\right) \quad (3)$$

and

$$Y^0_*(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t)) - \xi(t-\Delta t)}{\Delta t} \mid \xi(t) = x\right) \quad (4)$$

on $R^n$, i.e., $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y^0_*(t, \xi(t))$.

**2.1. Definition** . *The derivative $D_S = \frac{1}{2}(D+D_*)$ is called the symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called the antisymmetric mean derivative.*

Consider the vector fields $v^\xi(t,x) = \frac{1}{2}(Y^0(t,x) + Y^0_*(t,x))$ and $u^\xi(t,x) = \frac{1}{2}(Y^0(t,x) - Y^0_*(t,x))$.

**2.2. Definition**. *$v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called the current velocity of the process $\xi(t)$; $u^\xi(t) = u^\xi(t, \xi(t)) = D_A\xi(t)$ is called the osmotic velocity of the process $\xi(t)$.*

In what follows we shall mainly consider Itô processes of diffusion type of the form

$$\xi(t) = \int_0^t \beta(s)ds + \sigma w(t) \quad (5)$$

where $\sigma > 0$ is a real constant, $w(t)$ is a Wiener process adapted to $\xi(t)$ and $\beta(t)$ is measurable with respect to the $\sigma$-algebra $P^\xi_t$ generated by $\xi(s)$ for $0 \le s \le t$. For the sake of simplicity and without loss of generality we shall usually set $\sigma = 1$, i.e., $\xi(t)$ will take the form:

$$\xi(t) = \int_0^t \beta(s)ds + w(t). \quad (6)$$

Let $Z(t,x)$ be a $C^2$-smooth vector field on $F$, and $\xi(t)$ be a stochastic process in $F$.

**2.3. Definition**. *The forward $DZ(t,\xi(t))$ and the backward $D_*Z(t,\xi(t))$ mean derivatives of $Z$ along $\xi(\cdot)$ at $t$ are the limits of the form*

$$DZ(t,\xi(t)) = \lim_{\Delta t\to +0} E_t^{\xi}\Big(\frac{Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t}\Big)$$

$$D_*Z(t,\xi(t)) = \lim_{\Delta t\to +0} E_t^{\xi}\Big(\frac{Z(t,\xi(t)) - Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t}\Big) \tag{7}$$

Of course $DZ(t,\xi(t))$ and $D_*Z(t,\xi(t))$ can be presented as compositions of $\xi(t)$ with certain Borel vector fields (regressions) on $R^n$. These vector fields will be also denoted by $DZ$ and $D_*Z$, respectively.

**2.4. Lemma**. [13]-[15] *For the process (5) in $R^n$ the following formulas hold*

$$DZ = \frac{\partial}{\partial t}Z + (Y^0\cdot\nabla)Z + \frac{\sigma^2}{2}\nabla^2 Z,$$

$$D_*Z = \frac{\partial}{\partial t}Z + (Y^0_*\cdot\nabla)Z - \frac{\sigma^2}{2}\nabla^2 Z, \tag{8}$$

*where $\nabla = (\frac{\partial}{\partial x^1}), \frac{\partial}{\partial x^2},...,\frac{\partial}{\partial x^n}$, $\nabla^2$ is the Laplacian, the dot denotes the scalar product in $R^n$ and $Y^0(t,X)$ and $Y^0_*(t,X)$ are introduced above.*

## 3. Mean derivatives and Itô processes on a manifold with connection

Consider an $n$-dimensional manifold $M$ and specify a Wiener process $w(t)$, defined on $(\Omega,F,P)$ and taking values in $R^n$. Let a certain vector field $a(t,m)$ and a certain field of linear operators $A(t,m): R^n\to T_mM$ be given on $M$. Such a pair will be called an Itô vector field and denoted by $(a,A)$. For the sake of simplicity of presentation we assume that the fields $a(t,m)$ and $A(t,m)$ are smooth in both variables $t\in[0,l]$ and $m\in M$.

**3.1. Definition**. [1], [9] *The forward stochastic differential $(a(t,m)dt + A(t,m)dw(t))$ at $m\in M$, $t\in[0,l]$, determined by the field $(a,A)$, is a class of stochastic processes in the tangent space $T_mM$ that consists of the solutions of all Itô stochastic differential equations in $T_mM$ of the type*

$$X(t+r) = \int_t^{t+r}\tilde a(s,X(s))ds + \int_t^{t+r}\tilde A(s,X(s))dw(s) \tag{9}$$

*where $\tilde a(s,X)$ is a vector field on $T_mM$ and $\tilde A(s,X): R^n\to T_mM$ is a field of linear operators depending on $s\in[t,l], X\in T_mM$; $\tilde a(s,X)$ and $\tilde A(s,X)$ are Lipschitz continuous and satisfy the following conditions:*

*(i) they are equal to zero outside a certain neighbourhood of the origin in $T_mM$,*

*(ii) for all $s\in[t,l]$ we have $\tilde a(s,0) = a(s,m)$ and $\tilde A(s,0) = A(s,m)$.*

Note that since $\tilde a$ and $\tilde A$ are Lipschitz continuous the solutions to (9) are strong, i.e., they are well-defined for any realization of the Wiener process $w(t)$.

Specify a connection $H$ on the manifold $M$ and consider it's exponential map *exp*.

**3.2. Definition**. *The Itô equation in the form of Belopolskaya-Dalecky is an expression of the form*

$$d\xi(t) = exp_{\xi(t)}(a(t,\xi(t))dt + A(t,\xi(t))dw(t)). \tag{10}$$

*It means that for any $t \in [0,l]$ (assuming that $[0,l]$ is the domain of $\xi(t)$ ) the process $\xi(t+r)$ a.s. coincides with some process of the class $exp_{\xi(t)}(a(t,\xi(t))dt + A(t,\xi(t))dw(t))$ until $\xi(t)$ leaves a certain neighbourhood of $\xi(t)$ in $M$.*

It is shown (see, e.g., [1], [9]) that in a local chart (10) takes the form

$$d\xi(t) = a(t,\xi(t))dt - \frac{1}{2}tr\Gamma_{\xi(t)}(A,A)dt + A(t,\xi(t))dw(t). \tag{11}$$

where $\Gamma_m(\cdot,\cdot)$ is the local connector (connection coefficient) of $H$ in the chart.

**3.3. Theorem**. *Let $M$ and $Q$ be manifolds, equipped with connections; let $f : M \to Q$ be a smooth map such that for any geodesic $r(t)$ on $M$ the curve $f(r(t))$ is a geodesic on $Q$, i.e., $f(exp^M(Z)) = exp^Q(Tf(Z))$ for any $Z \in TM$ and $exp^M$ and $exp^Q$ are the exponential maps corresponding to the connections on $M$ and $Q$, respectively. Then for a solution $\xi(t)$ of (10) on $M$ the process $f(\xi(t))$ on $Q$ satisfies the equation*

$$df(\xi(t)) = exp^Q_{f(\xi(t))}(Tf \circ a(t,\xi(t))dt + Tf \circ A(t,\xi(t))dw(t)).$$

Let, as above, a certain connection $H$ be given on the manifold $M$. Let $\xi(t)$ be a stochastic process on $M$. According to formulas (1) and (2) we can introduce mean forward and mean backward derivatives of $\xi(t)$, if they exist, in any chart. But from formula (11) it obviously follows that doing this, e.g., for solutions of (10) we shall obtain the mean derivatives depending on the local connector of the connection $H$ in the chart and even on $A$, while for physical reasons the derivatives should be vectors. That is why the determination of mean derivatives has a standard modification as follows.

Consider the Borel fields $Y^0(t,\cdot)_\alpha$ and $Y^0_*(t,\cdot)_\alpha$ on a chart $U_\alpha$ calculated by formulas (3) and (4), respectively. Now construct the vector field $Y^0(t,\cdot)$ (and $Y^0_*(t,\cdot)$) on $M$ whose vector at any $m \in M$ coincides with $Y^0(t,m)_n$ (with $Y^0_*(t,m)_n$, respectively), where the $Y^0(t,m)_n$ ($Y^0_*(t,m)_n$, respectively) is calculated in the normal chart $U_n(m)$ of $H$ at $m$. Obviously the fields $Y^0$ and $Y^0_*$ are Borel measurable sections of the tangent bundle $TM$.

**3.4. Definition**. *$D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y^0_*(t,\xi(t))$ are called the forward and backward, respectively, mean derivatives of $\xi(\cdot)$ at $t$ on $M$ with respect to $H$; $D_S\xi(t) = v^\xi(t,\xi(t))$ and $D_A\xi(t) = u^\xi(t,\xi(t))$ are called the current and osmotic velocities, respectively, of $\xi(\cdot)$ with respect to $H$ where $v^\xi(t,m) = \frac{1}{2}(Y^0(t,m)+Y^0_*(t,m))$ and $u^\xi(t,m) = \frac{1}{2}(Y^0(t,m)-Y^0_*(t,m))$.*

In the same manner we modify the definitions of $D^y x(t)$ and $D^y_* x(t)$ (see (1) and (2)).

Since $H$ is specified, we do not indicate it in the notations of mean derivatives.

Denote by $K : TTM \to TM$ the connector of the connection $H$ (see, e.g., [2], [9], [10]). Let $Z(t,m)$ be a vector field on M. Introduce the covariant mean derivatives of $Z(t,m)$ along $\xi(t)$ by analogy with the ordinary covariant derivatives via the formulas

$$DZ(t,\xi(t)) = K \circ \lim_{\Delta t \to +0} E^\xi_t(\frac{Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t})$$

$$D_*Z(t,\xi(t)) = K \circ \lim_{\Delta t\to +0} E^\xi_t\Big(\frac{Z(t,\xi(t)) - Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t}\Big) \qquad (12)$$

Directly from (12) we obtain the following analogue of formulae (8):

$$DZ = \frac{\partial}{\partial t}Z + \nabla_{Y^0}Z + \frac{\sigma^2}{2}\nabla^2 Z,$$

$$D_*Z = \frac{\partial}{\partial t}Z + \nabla_{Y^0_*}Z - \frac{\sigma^2}{2}\nabla^2 Z, \qquad (13)$$

where $\nabla_X Z$ is the covariant derivative, $\nabla^2$ is the Laplace-Beltrami operator and the vectors $Y^0(t,X)$ and $Y^0_*(t,X)$ are introduced above.

Let $\pi : GM \to M$ be a certain subbundle of the frame bundle over $M$ whose structure group is $G$, a subgroup of $GL(n,R)$. Specify a connection $H$ on $GM$. As usual we denote by the same symbol $H$ the corresponding connection on $TM$. Let $V$ be the vertical distribution on $GM$. Recall (see, e.g., [2]) that the bundles $H$ and $V$ over $GM$ are trivial. In particular $H$ is trivialized by the so-called basic vector fields on $GM$, i.e., by the $C^\infty$-smooth bundle isomorphism $E : GM\times R^n \to H$, where the vector $E_b(x)\in H_b$ for $b\in GM$, $x\in R^n$ is defined by the equality $E_b(x) = T\pi^{-1}(bx)_{|H_b}$; here the frame $b = (e_1,...,e_n)$ is considered as the linear operator $b : R^n \to T_m M$ sending $x = (x^1,...,x^n)\in R^n$ into the vector $b\circ x = x^1e_1 + ... + x^ne_n$. The trivialization of $V$ is given by the so-called fundamental vector fields on $GM$ (see, e.g., [2] for details). Thus the tangent bundle $TGM = H + V$ is trivial.

Recall that the integral curves of autonomous basic vector fields and they alone are horizontal lifts of the geodesics of $H$ from $M$.

Let $\gamma_y(t)$ be the integral curve of the basic vector field $E(x)$ with the initial condition $\gamma(0) = b^0$, where $E_{b^0}(x) = y \in H_{b^0}$. Determine the map $e : H_{b^0} \to GM$ by the formula $e(y) = \gamma_y(1)$. From the above-mentioned properties of integral curves of basic vector fields follows

**3.5. Lemma**. *(i) The map $e$ is the restriction on $H_{b^0}$ of the exponential map of a certain connection on the manifold $GM$; (ii) $\pi e(y) = exp(T\pi y)$, where $exp$ is the exponential map of the connection $H$ on $M$.*

Consider a probability space $(\Omega,F,P)$, a non-decreasing family $B_t$ of complete $\sigma$-subalgebras of $F$ and a Wiener process $w(t)$ in $R^n$ adapted to $B_t$. Let a stochastic process $a(t)$, $t\in[0,l]$, with values in $R^n$, and a stochastic process $A(t)$, $t\in[0,l]$, with values in the space of linear operators $L(R^n)$, be defined on $(\Omega,F,P)$ so that they are non-anticipative with respect to $B_t$ and the Itô process $z(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ is well defined. Consider on $GM$ the following equation

$$dy(t) = e_{y(t)}(E_{y(t)}(a(t))dt + E_{y(t)}(A(t))dw(t)) \qquad (14)$$

From Lemma 3.5 and the other above arguments it follows that (14) is an equation of type (10). Moreover, for (14) and its projection on $M$ Theorem 3.3 is valid.

Specify a certain random variable $\eta$ with values in $M$ and the density $\rho$. Let $\beta$ be a Borel measurable section of $GM$ over $M$ (recall that continuous sections may not exist while Borel measurable ones obviously do exist).

Consider the solution $y^0(t)$ of (14) with the initial condition $y^0(0) = \beta(\eta)$, defined on a certain interval $t \in [0, l]$ and the process $\pi y^0(t)$ on $M$. (The existence problem for solutions of (14) is discussed, e.g., in [9], [10]. We are using some existence theorems below.)

**3.6. Definition**. *(i) The process $\pi y^0(t)$ is denoted by $R_I z(t)$ and called the Itô development of $z(t)$ with initial condition $\eta$. The processes being Itô developments of Itô processes in $R^n$ are called Itô processes in $M$. (ii) The solution $y^0(t)$ is called the horizontal lift of the process $\pi y^0(t)$ with the initial condition $\beta(\eta)$ or the parallel translation of $\beta$ along $R_I z(t)$. For a specified vector $x = (x^1, ..., x^n)$ the vector field $y(t)^0 \circ x$ is called a parallel vector field along $R_I z(t)$.*

Note that from the invariance of the connection with respect to the right action of the group $G$ on $GM$ it follows that both an Itô development, starting from $\eta$, and parallel translations of vectors along it do not depend on the initial condition $\beta$ over $\eta$ in $GM$ for the horizontal lift.

In what follows we shall denote by $\Gamma_{t,s}$ the operator of parallel translation of a vector $v(t, m)$ along a certain Itô process $\xi(t)$ from the time instant $s$ to the time instant $t$. The case of $\Gamma_{0,t}$ and $\Gamma_{t,0}$ should be described in a special manner. Specify a certain horizontal lift $b(t)$ of $\xi(t)$ in $GM$ and use it for parallel translation of vectors and tensors along $\xi(t)$ to $\xi(0) = \eta$. Applying $b(0)^{-1}$ we send them into $R^n$ and denote the result by the symbol $\Gamma_{0,t} v(t, x(t))$.

For any random vector $v$ on $R^n$ we can apply $b(0)$ to it in order to get the vector in $T_{\xi(0)}M$ and after that we can apply to the latter the parallel translation along $\xi$ (see above) to the point $\xi(t)$. The result in $T_{\xi(t)}M$ is denoted by $\Gamma_{t,0} v$. Obviously the same construction is valid for $v$ being a general tensor.

Specify $\eta$ independent of a Wiener process $w(t)$.

**3.7. Definition**. *The Itô development $R_I w(t)$, $R_I z(0) = \eta$, of a Wiener process $w(t)$ in $R^n$ is called a Wiener process in $M$ with initial condition $\eta$ at $t = 0$.*

**3.8. Remark**. The definition of a Wiener process in $M$ as a development, treated through Stratonovich approach, was suggested by Eells and Elworthy (see details, e.g., in [7]).□

In general a Wiener process in $M$ is defined on a random time interval $[0, \tau(\omega))$ where $\tau(\omega)$ is called the explosion time.

**3.9. Definition**. *If any Wiener process in $M$ is well defined for $t \in [0, \infty)$, the specified connection $H$ on $M$ is called stochastically complete.*

**3.10. Theorem**. *Consider $z(t)$ in $R^n$ of the form (6). Let the connection $H$ be stochastically complete and let the process $\beta(t)$ in $R^n$ satisfy the condition*

$$P(\int_0^l \beta(s)^2 ds < \infty) = 1. \tag{15}$$

*Then the Itô development $R_I z(t)$, $R_I z(0) = \eta$, is well defined for $t \in [0, l]$.*

**Proof**. Consider the probability space $\breve{\Omega} = (M \times C^0([0, l], R^n), B \times F, \rho \times \nu)$ where $B$ is the Borel $\sigma$-algebra on $M$, $F$ is the $\sigma$-algebra on $C^0([0, l], R^n)$ generated by cylinder sets, $\rho$ is the density of $\eta$ and $\nu$ is the Wiener measure. The coordinate process on $\breve{\Omega}$ is a Wiener process $W(t)$ on $R^n$ and its Itô development $W_M(t)$ is a Wiener process on $M$ started from $\eta$, which exists for all $t \in [0, l]$ by the stochastic completeness of $M$. From (15) it follows that the measure $\mu$ on $(C^0([0, l], R^n), F)$, corresponding to $z(t)$, is absolutely continuous with respect to $\nu$ (see [12]). Replace $\nu$ by the measure $\mu$. The process $W_M(t)$ on the new probability space turns to the Itô development $\xi(t) = R_I z(t)$. The arbitrariness of $l$ completes the proof.□

**3.11. Corollary**. *The Itô development of an Itô process $z(t)$ of type (6), with $\beta(t)$ satisfying (15), on stochastically complete Riemannian manifolds is an extension of the operator, converse to Cartan's development, from the class of piecewise smooth curves in $R^n$ onto almost all continuous curves in $R^n$ with respect to $\mu$, where $\mu$ is the measure on $(C^0([0,l], R^n), F)$, corresponding to $z(t)$.*

**3.12. Corollary**. *Under the condition of Theorem 3.10 the parallel translation along $R_I z(\cdot)$ in the sense of Definition 3.6 is a.s. the uniform limit of classical parallel translations along a sequence of processes with piecewise smooth sample paths, a.s. uniformly for $t \in [0,l]$ converging to $R_I z(t)$.*

Both corollaries follow from Theorem 3.10 and from the fact, proved in [9], that a Wiener process on $M$ is the limit of a certain sequence of piece-wise geodesic approximations.

## 4. Newton-Nelson equation on a manifold with connection

Let $M$ be a manifold with a specified connection $H$ and let $F(t,m,X)$, $t \in [0,l]$, $m \in M$, $X \in T_mM$, be a vector field on $M$. The Newton-Nelson equation is an expression of the form

$$\frac{1}{2}(DD_* + D_*D)\xi(t) = F(t, \xi(t), v^\xi(t, \xi(t))) \tag{16}$$

$F$ plays the role of a force field. We consider $F$ of the form

$$F(t,m,X) = \alpha_0(t,m) + \alpha_1(t,m)X \tag{17}$$

where $\alpha_0$ is a vector field on $M$ and $\alpha_1$ is a (1,1)-tensor field, i.e., a field of linear operators $\alpha_1(t,m) : T_mM \to T_mM$. Note that generally speaking the force $F$ (17) may not be potential or even gyroscopic. With such $F$ (16) takes the form

$$\frac{1}{2}(DD_* + D_*D)\xi(t) = \alpha_0(t, \xi(t)) + \alpha_1(t, \xi(t))v^\xi(t, \xi(t)) \tag{18}$$

Below in this section we suppose that the following condition is satisfied:

**4.1. Condition.** *(i) The connection $H$ is stochastically complete; (ii) the vector field $\alpha_0$ is continuous and has compact support; (iii) the tensor field $\alpha_1$ is $C^1$-smooth and has compact support.*

We introduce Condition 4.1 for the sake of simplicity: it can be easily formulated and verified. Conditions of less restrictive form can be formulated in analogy to [9].

For (18) we obtain existence of solutions for two types of initial conditions: "absolutely" random (i.e., the density of initial value is nowhere zero) and deterministic.

**4.2 Theorem.** *Let $\eta$ be a random variable with values in $M$ such that its density $\rho$ with respect to a natural Lebesgue measure on $M$ is nowhere equal to zero. Let $\beta$ be a continuous vector field with compact support. If Condition 4.1 is fulfilled, there exists an Itô process $\xi(t)$, $t \in [0,l]$, satisfying (18) such that $\xi(0) = \eta$ and $D_S\xi(0) = \beta(\eta)$.*

**Proof.** Consider the probability space $C^0([0,l], R^n), F, \nu)$ where $F$ is the $\sigma$-algebra of cylinder sets and $\nu$ is the Wiener measure. The coordinate process $W(t)$ on $C^0([0,l], R^n), F, \nu)$

is the standard Wiener process on $R^n$. Consider its Itô development $W_M$ on $M$ with initial condition $W_M(0) = \eta$. For $W_M(t)$ denote its osmotic velocity vector field (see Definition 3.4) by $u^W(t,m)$ and by $D^W u^W$ its forward mean derivative along $W_M(t)$, calculated according to formula (12). We need also the linear operator $\nabla u^W(t,m)$ of covariant derivative of $u^W(t,m)$ and the bilinear operator $\nabla\alpha_1(t,m)(\cdot,\cdot)$ of covariant derivative of $\alpha_1$. Note that by the construction for a vector field $X$ one obtains $\nabla u^W(t,m)(X) = \nabla_X u^W(t,m)$ – the covariant derivative of $u^W(t,m)$ along $X$. Consider the parallel translation $\Gamma_{0,s}\nabla\alpha_1(t,m)(\alpha_1(t,m)\cdot,\cdot)$ (see Section 3) of the bilinear operator $\nabla\alpha_1(t,m)(\alpha_1(t,m)\cdot,\cdot)$ into $R^n$ and define its trace $\Gamma_{0,s}\mathrm{tr}\nabla\alpha_1(t,m)(\alpha_1)$ by the formula

$$\Gamma_{0,s}\mathrm{tr}\nabla\alpha_1(t,m)(\alpha_1) = \sum_{i=1}^{n}\Gamma_{0,s}\mathrm{tr}\nabla\alpha_1(t,m)(\alpha_1 e_i, e_i)$$

where $e_i,\ i = 1,\dots,n$, is an orthogonal frame in $R^n$.

Now denote by $\beta_0$ the vector $b(0)^{-1}(\beta(\eta) + u^W(0,\eta))$ and consider in $R^n$ the following stochastic differential equation

$$\beta(t) = \beta_0 + \int_0^t \Gamma_{0,s}\alpha_0(s,W(s))ds +$$

$$\int_0^t \Gamma_{0,s}\alpha_1(s,W(s))dW(s) + \frac{1}{2}\int_0^t \Gamma_{0,s}\mathrm{tr}\nabla\alpha_1(s,W(s))(\alpha_1)ds +$$

$$\int_0^t \Gamma_{0,s}D^W u^W(s,W(s))ds + \int_0^t (\Gamma_{0,s}\nabla u^W(s,W(s)))(\beta(s))ds \tag{19}$$

Since (19) is linear in $\beta$, it has a unique solution $\beta(t)$. Set

$$\theta(t) = \exp\{-\frac{1}{2}\int_0^t [\beta(s)]^2\,ds + \int_0^t \beta(s)\cdot dW(s)\}$$

Under the above assumptions $\theta(t)$ is a martingale with $E\theta(t) = 1$, i.e., (15) is satisfied. So we can introduce a new probability measure $\mu$ on $(C^0([0,l],R^n),F)$ connected with $\nu$ by the relation $d\mu = \theta(l)d\nu$. Denote by $\zeta(t)$ the coordinate process on $(C^0([0,l],R^n),F,\mu)$. By Girsanov theorem we have $\zeta(t) = \int_0^t \beta(s)ds + w(t)$ where $w(t)$ is a Wiener process on $C^0([0,l],R^n),F,\mu)$ adapted to $\zeta(t)$,and $\beta(t)$ is measurable with respect to $P_t^\zeta$, i.e., $\zeta$ is a process of type (6).

Since $M$ is stochastically complete and, as mentioned above, $\beta(t)$ obviously satisfies (15), the Itô development $\xi(t) = R_I\zeta(t)$ with the initial condition $\eta$ exists as a process on $(M\times C^0([0,l],R^n), B\times F, \rho\times\mu)$.

**4.3. Lemma.** *The process $\xi(t)$ is a solution we are searching for.*

Lemma 4.3 follows from direct verification (using the list of mean derivatives obtained in [9]) and properties of the parallel translation and Itô development. This completes the proof of Theorem 4.2.□

**4.4. Theorem.** *Let Condition 4.1 be fulfilled. For any point $m_0 \in M$, any vector $\beta_0 \in T_{m_0}M$ and any $t_0 \in (0,l)$ there exists an Itô process $\xi(t)$, $t\in[0,l]$, such that $\xi(0) = m_0$, $D\xi(0) = \beta_0$ and for all $t \in (t_0,l)$ the process $\xi(t)$ satisfies (18).*

**Proof.** Introduce the following real-valued function:

$$t_0(t) = \begin{cases} \frac{1}{t} \text{ if } t \in (t_0, l] \\ \\ \frac{1}{t_0} \text{ if } t \in [0, t_0] \end{cases}$$

Identify $T_mM$ with $R^n$ and realize the scheme of arguments analogous to that in the proof of Theorem 4.2 with replacement of (19) by the equation

$$\beta(t) = \beta_0 + \int_0^t \Gamma_{0,s}\alpha_0(s, W(s))ds - \frac{1}{2}\int_0^t t_0(s)\beta(s)ds + \frac{1}{2}\int_0^t \Gamma_{0,s}\mathrm{tr}\nabla\alpha_1(s, W(s))(\alpha_1)ds +$$
$$\int_0^t \Gamma_{0,s}\alpha_1(s, W(s))dW(s) + \frac{1}{2}\int_0^t t_0(s)dW(s) - \frac{1}{2}t_0(t)W(t) \tag{20}$$

Having obtained the process $\xi(t)$ as a development, we see that by construction it satisfies (18) only for $t > t_0$.□

**4.5. Remark.** Note that we can move $t_0$ as close to 0 as we want but cannot set it equal to zero since (20) would become ill-posed.□

## 5. Solutions of Newton-Nelson equation on Lorentz manifolds

In this section we apply the above construction of solution of Newton-Nelson equation to the case of equations on space-times of General Relativity following the scheme, developed in Section 15.B of [9]. We use a modfication of the notion of mean derivative, suggested by Guerra and Ruggiero in [6] and [11], that makes it Lorentz-covariant, and Zastawniak's idea of transition from stochastic processes in Minkowski spaces to those in the underlying Euclidean spaces (see, e.g., [17], [18]). The Levi-Civitá connection on the bundle of Lorentz-orthonormal frames is involved here under the supposition that this connection is stochastically complete.

Denote by $R^4_M$ the Minkowski space and consider a certain stochastic process $\xi(\tau)$ in it. Here $\tau$ is an invariant parameter that plays the role of proper time (see [9], [17] for the concept of the proper time in solutions to the Newton-Nelson equations). For such $\xi(\tau)$, according to the idea of Guerra and Ruggiero [6], [11] we define relativistic mean forward $D_+\xi(\tau)$ and mean backward $D_-\xi(\tau)$ derivatives as follows:

$$D_+\xi(\tau) = \lim_{\Delta\tau\to+0} E(\xi(\frac{\xi(\tau+\Delta\tau)-\xi(\tau)}{\Delta\tau}|N^\xi_t, (\xi(\tau+\Delta\tau)-\xi(\tau))^2 \le 0)+$$
$$\lim_{\Delta\tau\to+0} E(\xi(\frac{\xi(\tau)-\xi(\tau-\Delta\tau)}{\Delta\tau}|N^\xi_t, (\xi(\tau+\Delta\tau)-\xi(\tau))^2 \ge 0) \tag{21}$$

$$D_-\xi(\tau) = \lim_{\Delta\tau\to+0} E(\xi(\frac{\xi(\tau)-\xi(\tau-\Delta\tau)}{\Delta\tau}|N^\xi_t, (\xi(\tau+\Delta\tau)-\xi(\tau))^2 \le 0)+$$
$$\lim_{\Delta\tau\to+0} E(\xi(\frac{\xi(\tau+\Delta\tau)-\xi(\tau)}{\Delta\tau}|N^\xi_t, (\xi(\tau+\Delta\tau)-\xi(\tau))^2 \ge 0) \tag{22}$$

It should be noted that (21) and (22) are covariant under the Lorentz transformations. Of course, $D_+\xi(\tau) = Y_+(\tau, \xi(\tau))$ and $D_-\xi(\tau) = Y_-(\tau, \xi(\tau))$ where $Y_+$ and $Y_-$ are the corresponding regressions (See Section 2).

One can easily give the definitions of $D_+Z$ and $D_-Z$ for a vector field $Z$ analogous to Definition 2.3 with relativistic mean derivatives instead of ordinary ones. Also the analogues of formulae (8) hold where $\nabla^2$ is evidently replaced by the wave operator $\Box$.

For the sake of simplicity, in this section we work in a system of units in which the speed of light $c = 1$ and $\hbar = 1$. We also consider particles with rest mass 1.

Let $\mathbf{M}$ be a 4-dimensional Lorentz manifold with signature $(-,+,+,+)$. For the sake of simplicity we assume from now on that $\mathbf{M}$ is orientable and oriented in time. In other words, the time direction 'to the future' is fixed in every tangent space $T_m\mathbf{M}$, $m \in \mathbf{M}$.

Consider the principal bundle $L(\mathbf{M})$ with structural group $L^+_-$, the proper orthochronous Lorentz group (see [8]). The action of $L^+_-$ on Minkowski space preserves the standard and time orientation. The bundle $L(\mathbf{M})$ is a subbundle of the principal bundle of Lorentz-orthonormal frames. Denote by $\mathbf{H}$ the restriction of the Levi-Civitá connection to $L(\mathbf{M})$, and by $V$ the vertical distribution on $L(\mathbf{M})$. As well as above both bundles are trivial. In particular $\mathbf{H}$ is trivialized by the basic vector fields on $L(\mathbf{M})$, etc.

In the Minkowski space $R^4_M$ pick a Lorentz orthonormal frame $b$. Introduce the Euclidean structure in $R^4_M$ by setting $b$ to be orthonormal in the Euclidean sense. We may now consider a Wiener process $w(\tau)$ in $R^4_M$ as well as Itô processes with this $w(\tau)$. One can easily show that the entire construction of Itô developments on manifolds is clearly applicable to the case the above processes on Lorentz $\mathbf{M}$ by using connections on $L(\mathbf{M})$. Those developments will be called Itô processes on the Lorentz manifold $\mathbf{M}$. We say that the above connection on $L(\mathbf{M})$ is stochastically complete if all above-mentioned Itô developments of Wiener processes are well-defined on $[0,\infty)$. In what follows we suppose the connection to be stochastically complete.

Itô processes on $\mathbf{M}$, which are the developments of Itô processes of diffusion type of the form $z = \int_0^\tau \beta(s)ds + w(\tau)$, are of particular interest to us. Let $\xi(\tau)$ be such a process in $\mathbf{M}$. Using (21) and (22) and corresponding regressions in normal charts of $\mathbf{H}$ we define $D_+\xi(\tau)$ and $D_-\xi(\tau)$ in the same manner as in Section 3 above. The derivative $\bar{D}_S = \frac{1}{2}(D_+ + D_-)$ is called the relativistic symmetric mean derivative, $\bar{D}_A = \frac{1}{2}(D_+ - D_-)$ is called the relativistic antisymmetric mean derivative. Denote the relativistic current velocity by $\bar{v}^\xi(\tau) = \bar{D}_S\xi(\tau)$ and the relativistic osmotic velocity by $\bar{u}^\xi(\tau) = \bar{D}_A\xi(\tau)$. The relativistic mean derivatives $D_+Z(\tau,\xi(\tau))$ and $D_-Z(\tau,\xi(\tau))$ of a vector field $Z$ along $\xi(\tau)$ are defined by formulae (12) modified as formulae of (21) and (22) type.

We also can apply the Levi-Civitá connection on $L(\mathbf{M})$ (i.e., its normal charts on $\mathbf{M}$, connector, etc.) to define non-relativistic mean derivatives $D\xi(\tau)$, $D_*\xi(\tau)$, $D^\xi\eta(\tau)$, $D^\xi_*\eta(\tau)$, $DZ(\tau,\xi(\tau))$ and $D_*Z(\tau,\xi(\tau))$ as well as in Section 3. We use the old notations without any ambiguity.

Let us pick a Lorentz orthonormal frame in $T_m\mathbf{M}$ at some $m$ and represent a vector in terms of its coordinates with respect to the frame: $X = (X^0, \bar{X})$ where $X^0$ denotes the 'time-like' component and $\bar{X}$ the 3-dimensional 'space-like' component. Note that this coordinate decomposition is covariant with respect to Lorentz transformations in $T_m\mathbf{M}$ for $D_+\xi(\tau)$ and $D_-\xi(\tau)$ and not covariant for $D\xi(\tau)$ and $D_*\xi(\tau)$. Nevertheless $v^\xi(\tau)$ is covariant since one can easily see that $\bar{v}^\xi(\tau) = v^\xi(\tau)$. Indeed, direct calculations show that both $\bar{v}^\xi(\tau)$ and $v^\xi(\tau)$ have the same coordinate decomposition: $(D^\xi_S\xi(\tau)^0, D^\xi_S\bar{\xi}(\tau))$. It is not the case for the osmotic velocity $\bar{u}^\xi(\tau)$: $\bar{u}^\xi(\tau) = (D^\xi_A\xi^0(\tau), -D^\xi_A\bar{\xi}(\tau))$ while $u^\xi(\tau) = (D^\xi_A\xi^0(\tau), D^\xi_A\bar{\xi}(\tau))$.

The main technical result here can be easily proved by direct calculation:

**5.1. Theorem.** [9] *For an Itô process $\xi(\tau)$ in $\mathbf{M}$ as above the following equality holds*

$$\frac{1}{2}(DD_* + D_*D)\xi(\tau) = \frac{1}{2}(D_+D_- + D_-D_+)\xi(\tau) \tag{23}$$

The relativistic Newton-Nelson equation is an equality of the form

$$\frac{1}{2}(D_+D_- + D_-D_+)\xi(\tau) = \alpha(\xi(\tau))(\bar{v}^{\xi}(\tau)) \tag{24}$$

Recall that in relativistic dynamics the 4-force necessarily depends on the 4-velocity and should not depend on the proper time $\tau$. That is why we consider the force field $\alpha(m)X$ to be a linear operator $\alpha(m) : T_m\mathbf{M} \to T_m\mathbf{M}$, i.e., (1,1)-tensor. As comparing with the general case of manifolds with connections (See Section 4 above), $\alpha$ is autonomous and has no $\alpha_0$ term (and so it coincides with $\alpha_1$).

There is a natural relation between the Newton-Nelson equation (24) and the Klein-Gordon equation (see, e.g., [6], [11], [17]). In particular, if both equations are well-posed, their predictions are equivalent. But (24) is well-posed for a broader class of 4-forces than the Klein-Gordon equation. The relation between (24) and Dirac equation can be seen in [4], [18].

**5.2. Theorem.** *Let the Levi-Civitá connection on $L(\mathbf{M})$ be stochastically complete. Let also the (1,1)-tensor field $\alpha(m)$ be continuous and has compact support. Then for $\eta$ and $\beta$ as in Theorem 3.2 there exists an Itô process $\xi(\tau)$, satisfying (24), $\tau \in [0, l)$, such that $\xi(0) = \eta$ and $\bar{D}_S\xi(0) = \beta(\eta)$.*

Taking into account Theorem 5.1 and the fact that $\bar{v}^{\xi}(\tau) = v^{\xi}(\tau)$ (see above), one can easily see that Theorem 5.2 follows from Theorem 4.2.

**5.3. Theorem.** *Let the Levi-Civitá connection on $L(\mathbf{M})$ be stochastically complete. Let also the (1,1)-tensor field $\alpha(m)$ be continuous and has compact support. For any point $m_0 \in M$, any vector $\beta_0 \in T_{m_0}M$ and any $\tau_0 \in (0, l)$ there exists an Itô process $\xi(\tau)$, $\tau \in [0, l]$, such that $\xi(0) = m_0$, $D\xi(0) = \beta_0$ and for all $\tau \in (\tau_0, l)$ the process $\xi(\tau)$ satisfies (24).*

This theorem follows from Theorems 4.4 and 5.1.

**5.4. Remark.** Recall that $\xi(\tau)$ from Theorem 5.3 can be interpreted as a stochastic world line of a particle (solution of (24)) not starting from the initial point of $\xi(\tau)$ but only for $\tau > \tau_0$ where $\tau_0$ can be specified as close to 0 as we want but cannot be equal to 0 (see Remark 4.5). We suggest the hypothesis that this looks like a model for the Big Bang, the initial point for all world lines. All physical laws are ill-posed at the Big Bang and we can really start observing a world line at any time greater than the instant of Big Bang but cannot do that for the Big Bang itself.□

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