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COURANT ALGEBROIDS AND STRONGLY HOMOTOPY LIE ALGEBRAS

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Abstract. Courant algebroids are structures which include as examples the doubles of Lie bialgebras and the bundle $TM \oplus T^*M$ with the bracket introduced by T. Courant for the study of Dirac structures. Within the category of Courant algebroids one can construct the doubles of Lie bialgebroids, the infinitesimal objects for Poisson groupoids. We show that Courant algebroids can be considered as strongly homotopy Lie algebras.

1. Introduction

Dirac structures on manifolds were introduced and studied by T. Courant in his 1990 paper [4]. These structures provide a geometric setting for Dirac's theory of constrained mechanical systems [5] in the same way as symplectic or Poisson structures do for unconstrained ones. Given a manifold M, a Dirac structure on M is by definition a subbundle $L \subset TM \oplus T^*M$ that is maximally isotropic with respect to the canonical symmetric bilinear form on $TM \oplus T^*M$ and satisfies a certain integrability condition. To formulate the integrability condition, Courant introduced a skew-symmetric bracket operation on sections of $TM \oplus T^*M$; the integrability condition is then that the sections of L be closed under this bracket. The Courant bracket is natural, in the sense that it does not depend on any additional structure on M for its definition, but it has anomalous properties. In particular, it does not satisfy the Jacobi identity or the Leibniz rule with respect to multiplication by functions. The "defects" in both cases are differentials of certain expressions depending on the bracket and the bilinear form, hence they disappear upon restriction to a Dirac subbundle because of the isotropy condition. Particular cases of Dirac subbundles are (graphs of) 2-forms and bivector fields

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on M, in which case the integrability condition turns out to coincide with the form being closed (resp. the bivector field being Poisson).

The nature of the Courant bracket itself remained uninvestigated until several years later when it was observed by Liu, Weinstein and Xu [15] that $TM \oplus T^*M$ endowed with the Courant bracket plays the role of a "double" object, in the sense of Drinfeld (cf.[6] or the book [3]), for pairs of *Lie algebroids* over M. Lie algebroids are structures on vector bundles that combine the features of both Lie algebras and vector fields. They, as well as the corresponding global objects, Lie groupoids, have recently found many applications in differential geometry [16], symplectic and Poisson geometry [20],[21],[22], and also algebraic geometry and representation theory [2]. Many constructions in the category of Lie algebras carry over to Lie algebroids. Thus, in complete analogy with Drinfeld's Lie bialgebras, in the category of Lie algebroids there also exist "bi-objects", *Lie bialgebroids*, introduced by Mackenzie and Xu [17] as linearizations of Poisson groupoids. Besides their role in Poisson geometry and quantization [20],[21],[22],[13], Lie bialgebroids and Poisson groupoids have turned out to be the geometric structures behind the classical dynamical Yang-Baxter equation [7].

It is well known ([6],[3]) that every Lie bialgebra has a double which is also a Lie bialgebra. This is not so for general Lie bialgebroids. Instead, Liu, Weinstein and Xu [15] show that the double of a Lie bialgebroid is a more complicated structure they call a *Courant algebroid*, $TM \oplus T^*M$ with the Courant bracket being a special case. On the other hand, when the base manifold is a point, all anomalies disappear and we get a Lie algebra with an invariant inner product.

The purpose of this work is to understand the anomalies of Courant algebroids. From the very beginning, the observation that these anomalies were, in some sense, differentials of certain expressions suggested a homological/homotopical algebraic approach. This approach has turned out to be correct: we show that Courant algebroids can be viewed as *strongly homotopy Lie algebras* (SHLA's). Homotopy Lie algebras play an important role in deformation theory as deformations of (differential, graded) Lie algebras ([18],[10]), and they have also arisen in the latest developements in theoretical physics, in particular in string theory [23] (this is not a coincidence, since it has been shown ([19]) that what physicists are dealing with is precisely deformation theory).

Given a Courant algebroid, we realize it as a (finite) homological resolution of a Lie algebra. From general considerations it is natural to expect the resolution spaces of Lie algebras to carry homotopy Lie algebra structures (as a resolution of an object is an equivalent object in the derived category), and in [1] it is shown that a Lie algebra structure on a vector space can indeed be lifted to a SHLA structure on the total space of its homological resolution. However, the lifting depends on many choices, and it is not clear in advance how to make the "best" choice. In case of a Courant algebroid, we find explicit expressions for the structure maps which are natural and simple (in fact, most of them vanish), and prove that they satisfy the structure identities of a strongly homotopy Lie algebra.

The paper is organized as follows. In Section 2 we recall the basic notions related to Lie algebroids and bialgebroids; in Section 3 we introduce Courant algebroids and recall the construction of [15] of doubles of Lie bialgebroids; in Section 4 we introduce homotopy Lie algebras and prove the main theorem; Section 5 is devoted to the proofs of some technical lemmas; and lastly, Section 6 contains a few concluding remarks.

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2. Lie algebroids and bialgebroids

To begin, we recall the definition of Lie algebroid [16]:

Definition 2.1 *A* Lie algebroid *is a vector bundle $A \to M$ together with a Lie algebra bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$ and a bundle map $\rho : A \to TM$, called the* anchor, *satisfying the following conditions:*

1. *For any $a_1, a_2 \in \Gamma(A)$, $\rho[a_1, a_2] = [\rho a_1, \rho a_2]$*
2. *For any $a_1, a_2 \in \Gamma(A), f \in C^\infty(M)$, $[a_1, f a_2] = f[a_1, a_2] + (\rho(a_1)f)a_2$*

In other words, the sections of the bundle act on smooth functions by derivations via the anchor in such a way that the brackets act as commutators, and the behavior of the bracket with respect to multiplication by functions is governed by the Leibniz rule. Thus, Lie algebroids are a straightforward generalization of the tangent bundle. They are also the infinitesimal objects corresponding to Lie groupoids [16]: when the base manifold is a point, a Lie groupoid reduces to a Lie group, while a Lie algebroid is just a Lie algebra.

A Lie algebroid structure on a vector bundle $A \to M$ gives rise to the following structures, dual to one another: first, the Lie bracket on $\Gamma(A)$ and the action of $\Gamma(A)$ on functions can be uniquely extended to a graded Lie algebra bracket of degree -1 on $\Gamma(\wedge^* A)$ which is a derivation of the exterior multiplication in each argument. This bracket is called the *Schouten bracket*, by analogy with the well-known bracket of multivector fields, and the resulting structure is a type of graded Poisson algebra called a *Gerstenhaber algebra*. Dually, one gets a differential d_A on the graded commutative algebra $\Gamma(\wedge^* A^*)$, defined by the same formula as the usual de Rham differential and satisfying similar properties. The space $\Gamma(\wedge^* A^*)$ thereby acquires the structure of a differential graded commutative algebra.

Now suppose that we are given a pair (A, A^*) of Lie algebroids over M which are in duality as vector bundles. Then the Lie algebroid structure of A induces a Schouten bracket on $\Gamma(\wedge^* A)$ and a differential d_A on $\Gamma(\wedge^* A^*)$; on the other hand, from A^* we get a Schouten bracket on $\Gamma(\wedge^* A^*)$ and a differential d_{A^*} on $\Gamma(\wedge^* A)$.

Definition 2.2 *A pair (A, A^*) of Lie algebroids in duality is a* Lie bialgebroid *if the induced differential d_* is a derivation of the Schouten bracket on $\Gamma(\wedge^* A)$.*

It can be shown that this notion is self-dual (cf. Corollary 3.5 of the next section). Lie bialgebroids correspond to *differential Gerstenhaber algebras* [11].

Example 2.3 *Let M be a Poisson manifold with Poisson tensor π and the corresponding bundle map $\tilde{\pi} : T^*M \to TM$ given by $\langle \tilde{\pi}\alpha, \beta\rangle = \pi(\alpha, \beta)$. Let $A = TM$, the tangent bundle Lie algebroid, $A^* = T^*M$ with anchor $\tilde{\pi}$ and the bracket of 1-forms given by*

$$[\alpha, \beta] = \mathcal{L}_{\tilde{\pi}\alpha}\beta - \mathcal{L}_{\tilde{\pi}\beta}\alpha - d(\pi(\alpha, \beta)) \tag{1}$$

Then d is the usual deRham differential of forms, $d_* = [\pi, \cdot]$, and it is straightforward to verify that (A, A^*) is a Lie bialgebroid.

A detailed discussion and more examples of Lie bialgebroids and Gerstenhaber algebras from geometry and physics can be found in [11],[12] and [13].

3. Courant algebroids

Definition 3.1 *Given a bilinear, skew-symmetric operation $[\cdot,\cdot]$ on a vector space V, its* Jacobiator *J is given by*

$$J(e_1,e_2,e_3)=[[e_1,e_2],e_3]+[[e_2,e_3],e_1]+[[e_3,e_1],e_2],$$

$e_1,e_2,e_3\in V$.

The Jacobiator is obviously trilinear and skew-symmetric. Of course, in a Lie algebra $J\equiv 0$.

Definition 3.2 *A* Courant algebroid *is a vector bundle $E\longrightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$ and a bundle map $\rho: E\longrightarrow TM$ such that the following properties are satisfied:*

1. *For any $e_1,e_2,e_3\in\Gamma(E)$, $J(e_1,e_2,e_3)=\mathcal{D}T(e_1,e_2,e_3)$;*
2. *for any $e_1,e_2\in\Gamma(E)$, $\rho[e_1,e_2]=[\rho e_1,\rho e_2]$;*
3. *for any $e_1,e_2\in\Gamma(E)$ and $f\in C^\infty(M)$, $[e_1,fe_2]=f[e_1,e_2]+(\rho(e_1)f)e_2-\langle e_1,e_2\rangle\mathcal{D}f$;*
4. *$\rho\circ\mathcal{D}=0$, i.e., for any $f,g\in C^\infty(M)$, $\langle\mathcal{D}f,\mathcal{D}g\rangle=0$;*
5. *for any $e,h_1,h_2\in\Gamma(E)$, $\rho(e)\langle h_1,h_2\rangle=\langle[e,h_1]+\mathcal{D}\langle e,h_1\rangle,h_2\rangle+\langle h_1,[e,h_2]+\mathcal{D}\langle e,h_2\rangle\rangle$,*

where $T(e_1,e_2,e_3)$ is the function on the base M defined by:

$$T(e_1,e_2,e_3)=\frac{1}{3}\langle[e_1,e_2],e_3\rangle+c.p., \tag{2}$$

("c.p." denotes the cyclic permutations of the e_i's) and $\mathcal{D}: C^\infty(M)\longrightarrow\Gamma(E)$ is the map defined by $\mathcal{D}=\frac{1}{2}\beta^{-1}\rho^* d_0$, where β is the isomorphism between E and E^* given by the bilinear form and d_0 is the deRham differential. In other words,

$$\langle\mathcal{D}f,e\rangle=\frac{1}{2}\rho(e)f. \tag{3}$$

In a Courant algebroid E, a *Dirac structure*, or *Dirac subbundle*, is a subbundle L that is maximally isotropic under $\langle\cdot,\cdot\rangle$ and whose sections are closed under $[\cdot,\cdot]$. It is immediate from the definition that a Dirac subbundle is a Lie algebroid under the restrictions of the bracket and anchor.

Suppose now that both A and A^* are Lie algebroids over the base manifold M, with anchors a and a_* respectively. Let E denote their vector bundle direct sum: $E=A\oplus A^*$. On E, there exist two natural nondegenerate bilinear forms, one symmetric and another antisymmetric, which are defined as follows:

$$(X_1+\xi_1, X_2+\xi_2)_\pm = \frac{1}{2}(\langle \xi_1, X_2\rangle \pm \langle \xi_2, X_1\rangle). \tag{4}$$

On $\Gamma(E)$, we introduce a bracket by

$$[e_1,e_2] = ([X_1,X_2]+L_{\xi_1}X_2 - L_{\xi_2}X_1 - d_*(e_1,e_2)_-)+([\xi_1,\xi_2]+L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1,e_2)_-), \tag{5}$$

where $e_1 = X_1+\xi_1$ and $e_2 = X_2+\xi_2$.

Finally, we let $\rho : E \longrightarrow TM$ be the bundle map defined by $\rho = a + a_*$. That is,

$$\rho(X+\xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A) \text{ and } \xi \in \Gamma(A^*) \tag{6}$$

It is easy to see that in this case the operator $\mathcal{D}$ as defined by Equation (3) is given by

$$\mathcal{D} = d_* + d,$$

where $d_* : C^\infty(M) \longrightarrow \Gamma(A)$ and $d : C^\infty(M) \longrightarrow \Gamma(A^*)$ are the usual differential operators associated to Lie algebroids (cf. Sec. 2 and [17] for more details).

The following results, proved in [15], indicate that the notion of Courant algebroid does yield a correct generalization of the double construction to Lie bialgebroids:

Theorem 3.3 *If (A, A^*) is a Lie bialgebroid, then $E = A \oplus A^*$ together with $([\cdot,\cdot], \rho, (\cdot,\cdot)_+)$ is a Courant algebroid.*

Conversely, we have

Theorem 3.4 *In a Courant algebroid $(E, \rho, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$, suppose that L_1 and L_2 are Dirac subbundles transversal to each other, i.e., $E = L_1 \oplus L_2$. Then, (L_1, L_2) is a Lie bialgebroid, where L_2 is considered as the dual bundle of L_1 under the pairing $2\langle\cdot,\cdot\rangle$.*

This gives the equivalence between Drinfeld doubles and Manin triples for Lie bialgebroids. An immediate consequence of the theorems above is the following duality property of Lie bialgebroids, which was first proved in [17] and then by Kosmann-Schwarzbach [11] using a simpler method.

Corollary 3.5 *If (A, A^*) is a Lie bialgebroid, so is (A^*, A).*

Example 3.6 *Given a manifold M, consider TM with its standard Lie algebroid structure and T^*M with zero anchor and bracket. Then (TM, T^*M) is obviously a Lie bialgebroid, and the double bracket (5) reduces to*

$$[X_1+\xi_1, X_2+\xi_2] = [X_1,X_2] + (L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(\tfrac{1}{2}(\xi_1(X_2) - \xi_2(X_1)))).$$

This is the bracket originally introduced by Courant in [4]. The anchor ρ in this case is the projection to TM, and $\mathcal{D} = d$, the deRham differential.

Example 3.7 *When M is a point, (A, A^*) is a Lie bialgebra and E is the usual Drinfeld double.*

4. Strongly homotopy Lie algebras and Courant algebroids

Let V be a graded vector space. Let $T(V)$ denote the tensor algebra of V in the category of graded vector spaces, and let $\bigwedge(V)$ denote its exterior algebra in the same category ($\bigwedge(V) = T(V)/ < v \otimes w + (-1)^{\tilde{v}\tilde{w}} w \otimes v >$, where $\tilde{v}$ denotes the degree of v). $T(V)$ (resp. $\bigwedge(V)$) is not only an associative algebra, but also a coassociative coalgebra: if V is of finite type, the comultiplication on $T(V)$ (resp. $\bigwedge(V)$) is the adjoint of the multiplication on $T(V^*)$ (resp. $\bigwedge(V^*)$), but in fact one does not need the dual space to define the comultiplication (see [14] for details).

Definition 4.1 *A* strongly homotopy Lie algebra *is a graded vector space V together with a collection of linear maps $l_k : \bigwedge^k V \to V$ of degree $k-2$, $k \geq 1$, satisfying the following relation for each $n \geq 1$ and for all homogeneous $x_1, \ldots, x_n \in V$:*

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma) l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{7}$$

where $\epsilon(\sigma)$ is the Koszul sign *(the fundamental convention of supermathematics saying that a minus sign is introduced whenever two consecutive odd elements are permuted) and σ runs over all $(i, n-i)$-unshuffles (permutations satisfying $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$) with $i \geq 1$.*

For $n = 1$ this means simply that l_1 is a differential on V; for $n = 2$, l_2 is a superbracket on V of which l_1 is a derivation (equivalently, $l_2 : \bigwedge^2(V) \to V$ is a chain map of complexes); $n = 3$ gives the Jacobi identity for l_2 satisfied up to chain homotopy given by l_3, and higher l_k's can be interpreted as higher homotopies. The algebraic theory of L_∞-algebras is studied in [9] and [14] .

We shall write the equation (7) in a more succinct equivalent form:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} l_j l_i = 0, \tag{8}$$

where we have extended each l_i to all of $\bigwedge(V)$ uniquely as a coderivation of the coalgebra structure on $\bigwedge(V)$. This accounts for the permutations and signs in (7).

We are interested in L_∞-algebras for the following reason: it is shown in [1] that, given a homological resolution (X_*, d) of a vector space H (graded or not), any Lie algebra structure on H can be lifted to an L_∞-algebra structure on the total resolution space X with $l_1 = d$. The strating point of this construction is the observation that Lie brackets on H correspond to bilinear skew-symmetric brackets on X_0 for which the boundaries form an ideal and the Jacobi identity is satisfied up to a boundary. This correspondence is in no way unique or canonical as it requires a choice of a homotopy inverse to the quasi-isomorphism $(X_*, d) \to (H, 0)$). But it is this latter bracket on X_0 that provides the starting point for constructing the SHLA structure on X, hence, if it is given, no choice is required at this stage, and we need never mention H. We shall presently see that with Courant algebroids we are in precisely this situation.

Let E be a Courant algebroid over a manifold M. We know from the definition that the Courant bracket on $\Gamma(E)$ satisfies Jacobi up to a $\mathcal{D}$-exact term. It turns out that, moreover,

$Im(\mathcal{D})$ is an ideal in $\Gamma(E)$ with respect to the bracket. More precisely, the following identity holds:

Proposition 4.2 *For any $e \in \Gamma(E)$, $f \in C^\infty(M)$ one has*

$$[e, \mathcal{D}f] = \mathcal{D}\langle e, \mathcal{D}f\rangle$$

Proof. Use axiom 5 in the definition of Courant algebroid with $e = \mathcal{D}f$ and arbitrary h_1 and h_2, and then cyclically permute e, h_1 and h_2:

$$\begin{aligned}
\rho(\mathcal{D}f)\langle h_1, h_2\rangle &= \langle [\mathcal{D}f, h_1] + \mathcal{D}\langle \mathcal{D}f, h_1\rangle, h_2\rangle + \langle h_1, [\mathcal{D}f, h_2] + \mathcal{D}\langle \mathcal{D}f, h_2\rangle\rangle \\
\rho(h_1)\langle h_2, \mathcal{D}f\rangle &= \langle [h_1, h_2] + \mathcal{D}\langle h_1, h_2\rangle, \mathcal{D}f\rangle + \langle h_2, [h_1, \mathcal{D}f] + \mathcal{D}\langle h_1, \mathcal{D}f\rangle\rangle \\
\rho(h_2)\langle \mathcal{D}f, h_1\rangle &= \langle [h_2, \mathcal{D}f] + \mathcal{D}\langle h_2, \mathcal{D}f\rangle, h_1\rangle + \langle \mathcal{D}f, [h_2, h_1] + \mathcal{D}\langle h_2, h_1\rangle\rangle
\end{aligned}$$

Now add the first two identities and subtract the third. Using Courant algebroid axioms 2, 4 and the definition of $\mathcal{D}$, we get:

$$\frac{1}{2}\rho([h_1, h_2])f = \langle \mathcal{D}f, 2[h_1, h_2]\rangle + \langle h_1, 2[\mathcal{D}f, h_2]\rangle + \langle h_2, 2\mathcal{D}\langle \mathcal{D}f, h_1\rangle\rangle$$

Using the definition of $\mathcal{D}$ again, we can rewrite this as:

$$\begin{aligned}
0 &= \frac{1}{2}\rho([h_1, h_2])f + \langle h_1, 2[\mathcal{D}f, h_2]\rangle + \rho(h_2)\langle h_1, \mathcal{D}f\rangle = \\
&= \frac{1}{2}\rho([h_1, h_2])f + \langle h_1, 2[\mathcal{D}f, h_2]\rangle + \frac{1}{2}\rho(h_2)(\rho(h_1)f) = \\
&= \frac{1}{2}\rho(h_1)(\rho(h_2)f) + \langle h_1, 2[\mathcal{D}f, h_2]\rangle = \\
&= \langle h_1, \mathcal{D}(\rho(h_2)f) + 2[\mathcal{D}f, h_2]\rangle = \\
&= \langle h_1, 2(\mathcal{D}\langle h_2, \mathcal{D}f\rangle - [h_2, \mathcal{D}f])\rangle.
\end{aligned}$$

The statement follows from the nondegeneracy of $\langle\cdot,\cdot\rangle$.

□

It will follow that we can extend the Courant bracket to an L_∞-structure on the total space of the following resolution of $H = \text{coker } \mathcal{D}$:

$$\cdots \longrightarrow 0 \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow H \longrightarrow 0, \tag{9}$$

where $X_0 = \Gamma(E)$, $X_1 = C^\infty(M)$, $X_2 = \ker \mathcal{D}$, $d_1 = \mathcal{D}$ and d_2 is the inclusion $\iota : \ker \mathcal{D} \hookrightarrow C^\infty(M)$. Remarkably, it turns out that, owing to the properties of Courant algebroids, the choices in the extension procedure can be made in a natural and simple way.

Let us fix some notation: we will denote elements of X_0 by e, elements of X_1 by f or g, and elements of X_2 by c.

Theorem 4.3 *A Courant algebroid structure on a vector bundle $E \longrightarrow M$ gives rise naturally to a SHLA structure on the total space X of (9) with $l_1 = d$ and the higher structure maps given by the following explicit formulas:*

$$\begin{array}{lcll} l_2(e_1 \wedge e_2) & = & [e_1, e_2] & \textit{in degree 0} \\ l_2(e \wedge f) & = & \langle e, \mathcal{D} f\rangle & \textit{in degree 1} \\ l_2 & = & 0 & \textit{in degree} > 1 \\ l_3(e_1 \wedge e_2 \wedge e_3) & = & -T(e_1, e_2, e_3) & \textit{in degree 0} \\ l_3 & = & 0 & \textit{in degree} > 0 \\ l_n & = & 0 & \textit{for } n > 3 \end{array}$$

Proof. Starting with the Courant bracket on X_0, we shall, following [1], extend it to an l_2 on all of X satisfying (8) for $n = 2$. The extension will proceed, essentially, by induction on the degree of the argument: for each degree l_2 will be a primitive of a certain cycle depending on the values of l_2 on elements of lower degree. Higher l_k's will be introduced and extended in a similar fashion, as primitives of cycles (using the acyclicity of (9)). The main body of work will consist in calculating these cycles, in particular, showing that most of them vanish; these computations are mostly relegated to the technical lemmas of the next section.
Step 1: $n = 2$. In degree 0, we are given $l_2(e_1 \wedge e_2) = [e_1, e_2]$. Consider now an element $e \wedge f$ of degree 1. Then $l_2 l_1(e \wedge f) \in X_0$ is defined and is, in fact, a boundary by Proposition 4.2:

$$l_2 l_1(e \wedge f) = l_2(l_1 e \wedge f + e \wedge l_1 f) = [e, \mathcal{D} f] = \mathcal{D}\langle e, \mathcal{D} f\rangle,$$

so we set $l_2(e \wedge f) = \langle e, \mathcal{D} f\rangle$ so that the SHLA identity (8) for $n = 2$,

$$l_1 l_2 - l_2 l_1 = 0, \tag{10}$$

holds in degree 1.

Now, $\bigwedge^2(X)_2$ is spanned by elements of the form $f \wedge g$ or $c \wedge e$. As above, $l_2 l_1$ is defined on elements of degree 2, and is, in fact, a cycle (cf. [1]). We have

$$l_2 l_1(f \wedge g) = l_2(l_1 f \wedge g - f \wedge l_1 g) = l_2(\mathcal{D} f \wedge g - f \wedge \mathcal{D} g) = \langle \mathcal{D} f, \mathcal{D} g\rangle + \langle \mathcal{D} g, \mathcal{D} f\rangle = 0$$

by Courant algebroid axiom 4, whereas

$$l_2 l_1(c \wedge e) = l_2(l_1 c \wedge e + c \wedge l_1 e) = l_2(\iota c \wedge e) = -\langle e, \mathcal{D}\iota c\rangle = 0,$$

so we set $l_2(f \wedge g) = l_2(c \wedge e) = 0$. Now observe that, since $l_2 = 0$ in degree 2, we can define l_2 to be zero on elements of degree higher than 2 as well and still have (10). We have thus defined an l_2 that satisfies (10) by construction.
Step 2: $n = 3$. In degree 0, by Courant algebroid axiom 1 we have

$$l_2 l_2(e_1 \wedge e_2 \wedge e_3) = J(e_1, e_2, e_3) = \mathcal{D} T(e_1, e_2, e_3),$$

where J is the Jacobiator. So we set $l_3(e_1 \wedge e_2 \wedge e_3) = -T(e_1, e_2, e_3)$, so that the homotopy Jacobi identity identity (8) for $n = 3$,

$$l_1 l_3 + l_2 l_2 + l_3 l_1 = 0, \tag{11}$$

holds on $\bigwedge^3(X)_0$ (as $l_1(X_0) = 0$).

Consider now an element $e_1 \wedge e_2 \wedge f \in \bigwedge^3(X)_1$. The expression $(l_2 l_2 + l_3 l_1)(e_1 \wedge e_2 \wedge f)$ is defined and is a cycle in X_1 (cf. [1]), hence we can define $l_3(e_1 \wedge e_2 \wedge f)$ to be some primitive of this cycle, so that (11) holds. But in our particular situation we in fact have (see the next section for a proof)

Lemma 4.4 $(l_2 l_2 + l_3 l_1)(e_1 \wedge e_2 \wedge f) = 0 \ \forall e_1, e_2, f$.

Therefore, we can define $l_3(e_1 \wedge e_2 \wedge f) = 0$. Now observe that on elements of degree > 1 l_3 has to be 0 because $\deg(l_3) = 1$, whereas $X_k = 0$ for $k > 2$. We now have l_3 defined on all of $\bigwedge^3(X)$ and satisfying (11) by construction.
Step 3: $n = 4$ and higher. Proceeding in a similar fashion, we look at the expression $(l_3 l_2 - l_2 l_3)(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$ (always a cycle in X_1) and define $l_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$ to be its primitive in X_2, so as to satisfy (8). However, it turns out that (see the next section for a proof)

Lemma 4.5 $(l_3 l_2 - l_2 l_3)(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 0 \ \forall e_1, e_2, e_3, e_4$.

Hence we can set $l_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 0$ and observe that l_4 has to vanish on elements of degree > 0 as $\deg(l_4) = 2$, while $X_k = 0$ for $k > 2$. By similar degree counting, all l_n, $n > 4$, have to vanish identically. This finishes the proof modulo Lemmas 4.4 and 4.5.

□

5. Proofs of technical lemmas

Let $(E, \langle , \rangle, [\cdot, \cdot], \rho)$ be a Courant algebroid over M. Given $e \in \Gamma(E)$, $f \in C^\infty(M)$, we will denote $\rho(e)f$ simply by ef, for short. Let us first prove two auxilliary lemmas.

Lemma 5.1 *The identity*

$$T(e_1, e_2, \mathcal{D}f) = \frac{1}{4}[e_1, e_2]f$$

holds in any Courant algebroid.

Proof. Using Courant algebroid axiom 2 and Proposition 4.2, we have

$$\begin{aligned}
T(e_1, e_2, \mathcal{D}f) &= \frac{1}{3}(\langle [e_1, e_2], \mathcal{D}f\rangle + \langle [\mathcal{D}f, e_1], e_2\rangle + \langle [e_2, \mathcal{D}f], e_1\rangle) = \\
&= \frac{1}{3}(\langle [e_1, e_2], \mathcal{D}f\rangle - \langle \mathcal{D}\langle e_1, \mathcal{D}f\rangle, e_2\rangle + \langle \mathcal{D}\langle e_2, \mathcal{D}f\rangle, e_1\rangle) = \\
&= \frac{1}{3}(\frac{1}{2}[e_1, e_2]f - \frac{1}{4}e_2(e_1 f) + \frac{1}{4}e_1(e_2 f)) = \\
&= \frac{1}{3}(\frac{1}{2}[e_1, e_2]f + \frac{1}{4}[e_1, e_2]f) = \frac{1}{4}[e_1, e_2]f.
\end{aligned}$$

□

Lemma 5.2 *Given* $e_1, e_2, e_3, e_4 \in \Gamma(E)$*, let*

$$\begin{aligned}\mathbf{J} &= \langle J(e_1,e_2,e_3),e_4\rangle - \langle J(e_1,e_2,e_4),e_3\rangle + \langle J(e_1,e_3,e_4),e_2\rangle - \langle J(e_2,e_3,e_4),e_1\rangle \\ \mathbf{K} &= \langle [e_1,e_2],[e_3,e_4]\rangle - \langle [e_1,e_3],[e_2,e_4]\rangle + \langle [e_1,e_4],[e_2,e_3]\rangle,\end{aligned}$$

where J *is the Jacobiator (cf. Def. 3.1). Then* $\mathbf{K} + 2\mathbf{J} = 0$.

Proof. Using Courant algebroid axioms 1 and 5, we can rewrite $\mathbf{J}$ as follows:

$$\begin{aligned}\langle J(e_1,e_2,e_3),e_4\rangle &= \langle \mathcal{D}T(e_1,e_2,e_3),e_4\rangle = \frac{1}{2}e_4T(e_1,e_2,e_3) = \frac{1}{6}e_4(\langle [e_1,e_2],e_3\rangle + c.p.) = \\ &= \frac{1}{6}(\langle [e_4,[e_1,e_2]] + \mathcal{D}\langle e_4,[e_1,e_2]\rangle, e_3\rangle + \langle [e_1,e_2],[e_4,e_3] + \mathcal{D}\langle e_4,e_3\rangle\rangle) + c.p.\end{aligned}$$

Expressing the other summands of $\mathbf{J}$ in this form and collecting the like terms in the parentheses, we find that the terms of the form $\langle [e_i,e_j],\mathcal{D}\langle e_k,e_l\rangle\rangle$ cancel out, terms of the form $\langle [e_i,e_j],[e_k,e_l]\rangle$ add up to $-4\mathbf{K}$, those of the form $\langle [e_i,[e_j,e_k]],e_l\rangle$ add up to $\mathbf{J}$, and finally, terms of the form $\langle \mathcal{D}\langle e_i,[e_j,e_k]\rangle,e_l\rangle$ add up to $-3\mathbf{J}$ after we've used Courant algebroid axiom 1. Thus,

$$\mathbf{J} = \frac{1}{6}(\mathbf{J} - 3\mathbf{J} - 4\mathbf{K}),$$

and the statement of the lemma follows immediately.

□

Proof of Lemma 4.4. In the notations of the previous section, we have, using Lemma 5.1 and Courant algebroid axiom 2:

$$\begin{aligned}& (l_2l_2 + l_3l_1)(e_1\wedge e_2\wedge f) = \\ =\ & l_2(l_2(e_1\wedge e_2)\wedge f + l_2(e_2\wedge f)\wedge e_1 + l_2(f\wedge e_1)\wedge e_2) + \\ +\ & l_3(l_1e_1\wedge e_2\wedge f + e_1\wedge l_1e_2\wedge f + e_1\wedge e_2\wedge l_1 f) = \\ =\ & l_2([e_1,e_2]\wedge f + \langle e_2,\mathcal{D}f\rangle\wedge e_1 - \langle \mathcal{D}f,e_1\rangle\wedge e_2) + l_3(e_1\wedge e_2\wedge \mathcal{D}f) = \\ =\ & \langle [e_1,e_2],\mathcal{D}f\rangle - \langle e_1,\mathcal{D}\langle e_2,\mathcal{D}f\rangle\rangle + \langle e_2,\mathcal{D}\langle e_1,\mathcal{D}f\rangle\rangle - T(e_1,e_2,\mathcal{D}f) = \\ =\ & \frac{1}{2}[e_1,e_2]f - \frac{1}{4}e_1(e_2f) + \frac{1}{4}e_2(e_1f) - \frac{1}{4}[e_1,e_2]f = 0\end{aligned}$$

□

Proof of Lemma 4.5. In the notations of the previous chapter we have

$$\begin{aligned}l_2l_3(e_1\wedge e_2\wedge e_3\wedge e_4) &= l_2(l_3(e_1\wedge e_2\wedge e_3)\wedge e_4 \pm (3,1) - unshuffles) = \\ &= -l_2(T(e_1,e_2,e_3)\wedge e_4 \pm (3,1) - unshuffles) = \\ &= \langle \mathcal{D}T(e_1,e_2,e_3),e_4\rangle \pm (3,1) - unshuffles = \\ &= \langle J(e_1,e_2,e_3),e_4\rangle \pm (3,1) - unshuffles = \mathbf{J}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
l_3 l_2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= l_3(l_2(e_1 \wedge e_2) \wedge e_3 \wedge e_4) \pm (2,2) - unshuffles = \\
&= -T([e_1, e_2], e_3, e_4) \mp (2,2) - unshuffles = \\
&= -\frac{1}{3}(\langle [[e_1, e_2], e_3], e_4\rangle + \langle [e_3, e_4], [e_1, e_2]\rangle + \langle [e_4, [e_1, e_2]], e_3\rangle) \pm \cdots = \\
&= -\frac{1}{3}(\mathbf{J} - 2\mathbf{K}),
\end{aligned}$$

after collecting the like terms. An application of Lemma 5.2 immediately yields $l_2 l_3 = l_3 l_2$.
□

6. Concluding remarks

L_∞-algebras occur in physics in the framework of the Batalin-Vilkovisky procedure for quantizing gauge theories. On the other hand, the Courant bracket seems to provide a geometric framework for constrained Hamiltonian systems. It is known ([8]) that gauge Lagrangians lead to constrained theories in the Hamiltonian formalism. This suggests that homotopy Lie algebras arising in the Batalin-Vilkovisky formalism and those in the Courant formalism might be somehow related. Our current investigations are aimed in this direction.

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