International Conference on
Secondary calculus and cohomological Physics
Moscow, August 1997

CONSERVATION LAWS AND MAIN BOUNDARY PROBLEMS OF PLASTICITY

S.I. Senashov, A.N. Iakhno.

Krasnoyarsk State University.
79, Prospect Svobodniy,
Krasnoyarsk, 660041, Russia.

Abstract. An application of the conservation laws of 2-dimensional plasticity equations for finding the analytical solutions of the main boundary problems is the main idea of this article.

Until recently we have no method for exact solving of boundary problems of 2-dimensional plasticity. In this paper we describe the method of analytical solving of main boundary problems of plasticity. The fundamental part of these procedures are conservation laws. They make it possible to linearize the problems without using nonsingular transformations and to obtain an explicit form of exact solutions for fundamental problems of plasticity.

The use of conservation laws of two-dimensional plasticity equations for the finding of analytical solutions of the main boundary problems is the aim of this article.

One of the well-known way of solving of plasticity equations is closely connected with the linearization of them. On this way there was obtained a lot of results, and some theorems about equation's and solution's properties were proved. But the linearization of equations doesn't make it possible to solve the boundary-value problems of plasticity, because of two problems arising. First, there is a difficulty, and sometimes an impossibility of the setting the corresponding boundary problems for linearized equations. Second, the Jacobian of transformation may become zero, therefore it's necessary to assume in advance that it is not the case, thus coming to a contradiction prior the solving of the problem.

The construction of solutions by means of semi-inverse methods (the group analysis method is among them) does not permit to solve the concrete boundary problems. Typically, these methods give explicit solutions, and afterwards the bounding conditions are selected for them.

In other words, until now there was no way of construction solutions of boundary-value problems of plane plasticity theory in an analytical form.

Key words and phrases. Conservation Laws, Plasticity, Symmetries.

In this paper, new method of solving of the main boundary plasticity problems is proposed. Its main point consists of linearization of equations that are used for determination of the conservation laws; in this case all the transformations used are non-singular.

1. The Setting of Main Boundary Problems

Let us consider the system of equations

$$\begin{aligned} &\sigma_x - 2k(\theta_x \cos 2\theta + \theta_y \sin 2\theta) = 0, \\ &\sigma_y - 2k(\theta_x \sin 2\theta - \theta_y \cos 2\theta) = 0, \end{aligned} \tag{1.1}$$

where σ - is the hydrostatic pressure, θ - is the angle between first main direction of stress tensor and axis ox, k - is the constant of plasticity; indices mean the derivation with respect to corresponding variables.

This system is of the hyperbolic type. The equations

$$\frac{dy}{dx} = tg\theta, \quad \frac{dy}{dx} = -ctg\theta, \tag{1.2}$$

define two families of characteristics α, β.

The full statement of mathematical problems of plasticity theory includes one of the following main boundary-value problems [1].

Cauchy problem. Let L be some smooth curve in the plane xy, which is transversal to characteristics (1.2) of the system (1.1) and intersects with them only once.

Let σ and θ are the smooth known functions defined along L. The problem is to construct the solution of equations (1.1) such that this solution has given values on L.

Let's take two points $P(p_1, p_2)$ and $Q(q_1, q_2)$ on L and consider two characteristics α, β respectively at these points, with the relations $(\xi = \xi_0 = const)|_\alpha, (\eta = \eta_0 = const)|_\beta$. Denote by $M(m_1, m_2)$ the point of their intersection. Then the solution is defined in the interior of the curvilinear triangle PQM.

Riemann problem. Let σ and θ be the functions defined along the segments OP and OQ of characteristics α, β respectively. These functions are connected by relations

$\frac{\sigma}{2k} - \theta = \xi_1 = const$, along OP,

$\frac{\sigma}{2k} + \theta = \eta_1 = const$, along OQ.

Consider two characteristics $\xi = \xi_0 = const, \eta = \eta_0 = const$ containing Q and P . Denote by $C(c_1, c_2)$ the point of their intersection. Then the solution is defined on the interior of curvilinear quadrangle $OQCP$.

2. Conservation Laws of System of Equations (1.1)

All conservation laws of the system (1.1) are described, see [2]. It's however necessary for us to deduce conservation laws of the special form directly. We seek conservation laws of system (1.1) in the following form

$$\partial_x(-A) + \partial_y(B) = 0, \tag{2.1}$$

where A, B are the functions depending on σ and θ only.

The equality (2.1) must be satisfied only on the solutions of system (1.1).

The analysis (2.1) yields

$$\begin{gathered}-A_\sigma\sigma_x - A_\theta\theta_x + B_\sigma\sigma_y - B_\theta\theta_y = \\ = \theta_x(-A_\theta - 2kA_\sigma\cos 2\theta + 2kB_\sigma\sin 2\theta) + \\ +\theta_y(B_\theta - 2kA_\sigma\sin 2\theta - 2kB_\sigma\cos 2\theta) = 0.\end{gathered}$$

From this expression we obtained equations on functions A and B

$$\begin{aligned}A_\theta + 2k(A_\sigma\cos 2\theta - B_\sigma\sin 2\theta) = 0, \\ B_\theta - 2k(A_\sigma\sin 2\theta + B_\sigma\cos 2\theta) = 0.\end{aligned} \tag{2.2}$$

System (2.2) is of the hyperbolic type, and the equations on their characteristics are

$$\frac{d\sigma}{d\theta} = 2k, \quad \frac{d\sigma}{d\theta} = -2k. \tag{2.3}$$

If we introduce new variables in the following way $\sigma = k(\eta + \xi), 2\theta = \eta - \xi$, then the system (2.2) comes to

$$\begin{aligned}A_\xi + B_\xi ctg\theta = 0, \\ A_\eta - B_\eta tg\theta = 0.\end{aligned} \tag{2.4}$$

Let us introduce new unknown functions ϕ, ψ

$$A = \phi\sin\theta + \psi\cos\theta, \quad B = \phi\cos\theta - \psi\sin\theta.$$

Now the system (2.4) takes the form

$$\phi_\xi + \frac{1}{2}\psi = 0, \quad \psi_\eta + \frac{1}{2}\phi = 0. \tag{2.5}$$

3. Solving The Boundary Problems

Cauchy problem. Put the conservation law (2.1) in a form of an integral taken over the closed path PQM. We have

$$\oint\limits_{PQM} A\,dy + B\,dx = 0. \tag{3.1}$$

Decompose this integral into the three components

$$\begin{aligned}\oint\limits_{PQM} A\,dy + B\,dx = \int\limits_{\eta=\eta_0} A\,dy + B\,dx + \int\limits_{\xi=\xi_0} A\,dy + B\,dx + \\ + \int\limits_{PQ} A\,dy + B\,dx.\end{aligned} \tag{3.2}$$

Integrating (3.2) by parts and taking into account (1.2), we conclude that

$$\int\limits_{\xi=\xi_0} A\,dy + B\,dx = \int\limits_{\xi=\xi_0} (Atg\theta + B)\,dx =$$
$$= x(Atg\theta + B)|_{m_1}^{p_1} - \int\limits_{\xi=\xi_0} x\partial_\eta(Atg\theta + B)\,d\eta,$$
$$\int\limits_{\eta=\eta_0} A\,dy + B\,dx = \int\limits_{\eta=\eta_0} (-Actg\theta + B)\,dx =$$
$$= x(-Actg\theta + B)|_{q_1}^{m_1} - \int\limits_{\eta=\eta_0} x\partial_\xi(-Actg\theta + B)\,d\xi.$$

Let

$$\partial_\eta((Atg\theta + B)|_{\xi=\xi_0}) = 0, \quad \partial_\xi((-Actg\theta + B)|_{\eta=\eta_0}) = 0. \tag{3.3}$$

Then

$$\int\limits_{PQ} A\,dy + B\,dx = -x(Atg\theta + B)|_{m_1}^{p_1} - x(-Actg\theta + B)|_{q_1}^{m_1}. \tag{3.4}$$

Taking into account that

$$Atg\theta + B = \frac{\phi}{\cos\theta}, \quad -Actg\theta + B = -\frac{\psi}{\sin\theta},$$

and using (2.5) and (3.3), we obtain the problem for ϕ:

$$\phi_{\xi\eta} - \frac{1}{4}\phi = 0,$$
$$\partial_\eta\left(\frac{\phi}{\cos\theta}\bigg|_{\xi=\xi_0}\right) = 0, \quad \partial_\xi\left(\frac{\phi_\xi}{\sin\theta}\bigg|_{\eta=\eta_0}\right) = 0.$$

The following function is a solution of this problem

$$\phi(\xi,\eta) = CI_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)\cos\left(\frac{\eta_0-\xi_0}{2}\right) -$$
$$-\frac{C}{2}\int\limits_{\eta_0}^{\eta} I_0\left(\sqrt{(\xi-\xi_0)(\eta-\tau)}\right)\sin\left(\frac{\tau-\xi_0}{2}\right)d\tau. \tag{3.5}$$

Here I_0 is a Bessel function of the first genus of order zero of an imaginary argument, $I_0(0) = 1, I_0'(0) = 0, C$ — is an arbitrary constant.

The equation (3.4) is homogeneous with respect to A and B. Therefore, without loss of generality, we put $C = 1$.

Back to (3.4), we obtain

$$\int A\,dy + B\,dx = m_1 - p_1. \tag{3.6}$$

On the other hand, from (3.2) it follows

$$\int\limits_{\xi=\xi_0} (A + Bctg\theta)\, dy = y(A + Bctg\theta)|_{m_2}^{p_2} - \int\limits_{\xi=\xi_0} x\partial_\eta (A + Bctg\theta)\, d\eta,$$

$$\int\limits_{\eta=\eta_0} (A - Btg\theta)\, dy = y(A - Btg\theta)|_{q_2}^{m_2} - \int\limits_{\eta=\eta_0} x\partial_\xi (A - Btg\theta)\, d\xi.$$

Let

$$\partial_\eta((A + Bctg\theta)|_{\xi=\xi_0}) = 0, \quad \partial_\xi((A - Btg\theta)|_{\eta=\eta_0}) = 0.$$

Then

$$\int\limits_{PQ} A\, dy + B\, dx = -y(A + Bctg\theta)|_{m_2}^{p_2} - y(A - Btg\theta)|_{q_2}^{m_2}. \tag{3.7}$$

The function

$$\phi(\xi,\eta) = I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)\sin\left(\frac{\eta_0-\xi_0}{2}\right) + \\ +\frac{1}{2}\int\limits_{\eta_0}^{\eta} I_0\left(\sqrt{(\xi-\xi_0)(\eta-\tau)}\right)\cos\left(\frac{\tau-\xi_0}{2}\right) d\tau \tag{3.8}$$

is a solution of the problem on ϕ,

$$\phi_{\xi\eta} - \frac{1}{4}\phi = 0,$$

$$\partial_\eta\left(\frac{\phi}{\sin\theta}\bigg|_{\xi=\xi_0}\right) = 0, \partial_\xi\left(\frac{\phi_\xi}{\cos\theta}\bigg|_{\eta=\eta_0}\right) = 0.$$

Then, from (3.7) it follows that

$$\int\limits_{PQ} Ady + Bdx = m_2 - p_2. \tag{3.9}$$

The equations (3.6), (3.9) defines the coordinates of the point M, at which the values of functions σ and θ are evaluated.

Riemann problem.

We put the conservation laws (2.1) in a form of an integral, taken over the closed path $POQC$.

$$\oint\limits_{POQC} A\, dy + B\, dx = 0.$$

Proceding by analogy with subsection 1, we obtain the system of equations

$$\int\limits_{POQ} Ady + Bdx = c_1 - q_1,$$

$$\int\limits_{POQ} Ady + Bdx = c_2 - q_2,$$

for coordinates of the point C. Solving this system, we can obtain the values of σ and θ at the point C.

Supported by Russian Foundation for Basic Research (grant N 96-01-01830)

References

1. Kachanov L.M., *The foundation of plasticity theory*, Nauka, Moscow, 1969.(In Russian).
2. Senashov S. I., Vinogradov A. M., *Symmetries and conservation laws of 2-dimensional ideal plasticity*, //Proc. Edinburg Math. Soc. vol.31, 1988.
3. Senashov S. I., *The foundations of group analysis for mecanics*, Krasnoyarsk university, Krasnoyarsk, 1993.(In Russian).