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A DIRECT METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS

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Abstract. An effective method is proposed for solving boundary-value problems for linear equations using complete systems of conservation laws, (i.e., the systems uniquely determining the solution by the boundary conditions). The ideas of the principal method are illustrated by one of the equations of the filtration theory.

1. Introduction

A number of classical methods for solving boundary-value problems for equationsof mathematical physics are based on the concept of the conservation law, with the Green identity being a background of this. In particular, along this way the Riemann method and notions of the Riemann and Green functions as families (subject to certain conditions on generating functions of the conservation laws) appeared. However, in its general form the problem of systematic employment of the conservation laws for solving boundary-value problems for differential equations seems to be first formulated by A.M.Vinogradov and published by him in conjunction with S.I.Senashov in [1]. Later on S.I.Senashov solved the Gauchy problem for ideal plasticity equations using the conservation laws in [2].

This paper proposes an effective method for solving boundary-value problems for linear equations using complete systems of conservation laws, (i.e., the systems uniquely determining the solution by the boundary conditions). The ideas of the principal method are illustrated by one of the equations of the filtration theory. First, the procedure for construction of the complete systems of the conservation laws is described. These laws are applied to setting appropriate boundary conditions and construction on this base of solutions (as series) of a number of boundary-value problems for the equation for an arbitrary spatial dimension. Second, it is shown how this procedure can be used to construct efficient algorithms for solving the boundary-value problems under consideration in the one-dimension space.

All the above is immediately applicable to the heat equation. A typical boundary-value problems for the wave equation can be solved similarly (with some special features).

Compared to the Fourier method, the proposed method is more economical and has a higher flexibility.

Key words and phrases: generating function of conservation law, complete system of conservation laws, Sturm-Liouville problem

2. General construction

Consider the equation

$$u_t - \eta\Delta u_t = \kappa\Delta u,$$
$$\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2; \quad \eta, \kappa = \text{const}, \qquad \eta^2 + \kappa^2 \neq 0 \tag{1}$$

for the parameters η and κ; n is a number of independent variables. At $n = 3$ and positive η and κ, Eq. (1) describes the process of fluid filtration in cleaved-porous medium. In this case u is fluid pressure in cracks, η and κ depend on the fluid and medium properties [3].

Below we describe a method for construction of complete systems of the conservation laws for Eq.(1) allowing to solve the boundary-value problems for it with the boundary conditions of the following three types:

$$1.\ u\mid_{\partial G}= h, \quad 2.\ \tfrac{\partial u}{\partial \mathbf{n}}\mid_{\partial G}= h, \quad 3.\ \left(fu + \tfrac{\partial u}{\partial \mathbf{n}}\right)\mid_{\partial G}= h.$$

Here G is a bounded domain in $\mathbf{R}^n(x)$ with piecewise smooth boundary ∂G.

For this purpose it is sufficient to restrict our consideration to the conservation laws with the generating functions $v = v(x,t)$, $x = (x_1, ..., x_n)$.

The conservation law corresponding to a function v can be obtained from the Green identity for the operators $l_G = D_t - \eta D_t \circ \mathit{\Delta} - \kappa\mathit{\Delta}$ and l_G^*:

$$D_t\left((v - \eta\Delta v)u\right) + \mathrm{Div}\left((\eta u_t + \kappa u)\nabla v - (\eta\nabla u_t + \kappa\nabla u)v\right) = 0. \tag{2}$$

In all the three cases the complete systems of the conservation laws are constructed similarly, so we dwell in detail only on the problem with the boundary conditions of the first kind:

$$\begin{array}{lll} u(x,0) = \varphi(x), & x \in \overline{G}, & \\ u(x,t) = h(x,t), & x \in \partial G, & t \geq 0, \\ \varphi(x) = h(x,0), & x \in \partial G. & \end{array} \tag{3}$$

If

$$D_t S + \mathrm{Div}\mathbf{P} = 0, \quad \mathbf{P} = (P_1, ..., P_n)$$

is an arbitrary conservation law for Eq.(1), then, according to the general Stokes formula, for each solution of this equation we have

$$\int_{G_t} S dx = \int_{G_0} S dx - \int_0^t \left(\int_{\partial G} \mathbf{P}\cdot\mathbf{n} d\sigma\right) d\tau, \tag{4}$$

where $G_t = G \times \{t\}$, $d\sigma$ is the surface element ∂G, $\mathbf{n}$ is the unit outer normal to ∂G. In case of conservation law (2) formula (4) becomes

$$\begin{aligned} I_v[u](t) &\equiv \int_{G_t} (v - \eta\Delta v)u dx = \\ &= \int_{G_0} (v - \eta\Delta v)u dx - \\ &- \int_0^t \left(\int_{\partial G} \left[(\eta u_t + \kappa u)\frac{\partial v}{\partial \mathbf{n}} - \left(\eta\frac{\partial u_t}{\partial \mathbf{n}} + \kappa\frac{\partial u}{\partial \mathbf{n}}\right)v\right] d\sigma\right) d\tau \end{aligned} \tag{5}$$

For the integral characteristics of solution we take the values of the functionals $I_v[u](t)$ on this solution. At given v in the right-hand side of formula (5) all the values are defined by the initial and boundary conditions, except for those depending on $\frac{\partial u}{\partial \mathbf{n}}$. Hence, to find the functional value at an arbitrary time, it is necessary to require that the following equality holds for v:

$$v(x,t)\mid_{\partial G}=0, \quad t\geq 0. \tag{6}$$

As v is the generating function of the conservation law for Eq.(1), we have

$$-l_G^*(v)=v_t-\eta\Delta v_t+\kappa\Delta v=0. \tag{7}$$

In case of $\kappa\neq 0$ we make use of separation of variables to solve problem (6), (7). Finally, for all the values of the parameter η satisfying the relations $1+\eta\lambda_{1,k}\neq 0$; $k=1,2,3,...$, we obtain the following system of generating functions subject to condition (6):

$$\left\{\ v_k=e^{\lambda_k t}X_k(x);\quad k=1,2,3,...\ \right\} \tag{8}$$

Here $\{X_k\}$ is a complete orthonormal (in $L_2(G)$ metrics) system of linearly independent eigenfunctions with eigenvalues $\lambda_{1,k}$for the Sturm-Liouville problem for the operator $-\Delta$ and a given domain G

$$\Delta X+\lambda_1 X=0,$$

$$X\mid_{\partial G}=0, \qquad \lambda_k=\tfrac{\kappa\lambda_{1,k}}{1+\eta\lambda_{1,k}}.$$

Note that $\lambda_{1,k}>0$, see [4]. In case of $\kappa=0$ and for any η we take the following as a system of generating functions satisfying condition (6)

$$\left\{\ v_k=X_k(x);\quad k=1,2,3,...\ \right\}. \tag{9}$$

For each fixed t the set $\{v_k\}$ (both (8) and (9)) is complete orthogonal systems in $L_2(G)$.

The equality (6) and boundary conditions (3) are used to reduce formula (5) on the functions $v=v_k$ to the form:

$$\begin{aligned}(1+\eta\lambda_{1,k})\int_{G_t}v_k u\,dx \;&=\; (1+\eta\lambda_{1,k})\int_{G_0}v_k\varphi(x)\,dx- \\ &-\int_0^t\left(\int_{\partial G}(\eta h_t+\kappa h)\frac{\partial v_k}{\partial \mathbf{n}}d\sigma\right)d\tau.\end{aligned} \tag{10}$$

For each k, such that $1+\eta\lambda_{1,k}\neq 0$, formula (10) determines the Fourier coefficient $c_k(t)=\frac{1}{\|v_k\|^2}\int_{G_t}v_k u\,dx$, $\|v_k\| = e^{\lambda_k t}$ of the function $u=u(x,t)$ over the ortogonal system $\{v_k\}$. When $1+\eta\lambda_{1,k}\neq 0$; $k=1,2,3,...$, by virtue of completeness (for any t) of the system $\{v_k\}$ in $L_2(G)$ the function u is uniquely determined by the sequence $\{c_k(t)\}$. Hence, the system of the conservation laws with the generating functions v_k is complete. It immediately yields

Theorem (of uniqueness). *If the parameter η of Eq.(1) satisfies $1+\ \eta\lambda_{1,k} \neq 0$; $k = 1, 2, 3, ...$, then the boundary-value problem (1), (3) has no more than one solution. In particular, this is always true for the heat equation ($\eta = 0,\ \kappa > 0$) and for the equation of filtration ($\eta > 0,\ \kappa > 0$).*

The issues of solution existence and dependence on boundary conditions are more subtle and will not be discussed here.

Let $\kappa = 0$ and $1+\eta\lambda_{1,k} = 0$; $k = k_0, ..., k_0+r_0-1$, r_0 is multiplicity of the eigenvalue λ_{1,k_0}. Then $\eta \neq 0$ and $v_k = X_k(x)$. Hence, the necessary solvability conditions of boundary-value problem (1), (3) follow from (10):

$$\begin{aligned} &\int_{\partial G} h_t(x,t)\, \tfrac{\partial X_k(x)}{\partial \mathbf{n}} d\sigma = 0, \\ &k = k_0, ..., k_0 + r_0 - 1. \end{aligned} \tag{11}$$

Under the assumption $\varphi \in C^2(G)$ it is easy to show that these conditions are also sufficient.

Note that conditions (11) are similar to the solvability conditions of the boundary-value problems for elliptic equations, see [5].

3. The case of one spatial dimension

For a one spatial dimension: $n = 1$, $G = (0, l)\,the$ boundary-value problem(1), (3) with $\eta > 0, \kappa \neq 0$, all the previous constructions are fully applicable. In this case

$$\begin{aligned} &v_k = e^{\lambda_k t} \sin \tfrac{k\pi}{l} x, \\ &\lambda_k = \tfrac{\kappa\lambda_{1,k}}{1+\eta\lambda_{1,k}}, \qquad \lambda_{1,k} = \left(\tfrac{k\pi}{l}\right)^2; \quad k = 1, 2, 3, ... \end{aligned}$$

Substituting v_k into (10) we obtain the computation formula for Fourier coefficients of the expansion of a solution $u(x,t)$ in sines in the interval $(0, l)$.

Example. Consider a special case of one-dimensional problem:

$$\begin{aligned} &u_t - \eta u_{txx} = \kappa u_{xx}, && 0 < x < l, \quad 0 < t < \infty, \\ &\ \eta > 0, \quad \kappa \neq 0, && \\ &u(x,0) = 0, && 0 \leq x \leq l, \\ &\begin{cases} u(0,t) = 0, \\ u(l,t) = 1 - e^{-\frac{\kappa t}{\eta}}, \end{cases} && 0 \leq t < \infty. \end{aligned} \tag{12}$$

From the general formula it follows that solution of problem (12) can be represented as a series

$$\begin{aligned} &u(x,t) = \tfrac{x}{l} + \sum_{k=1}^{\infty} (-1)^k \tfrac{2}{k\pi} e^{-\lambda_k t} \sin \tfrac{k\pi}{l} x, \\ &\lambda_k = \tfrac{\kappa\left(\frac{k\pi}{l}\right)^2}{1+\eta\left(\frac{k\pi}{l}\right)^2}; \qquad\qquad k = 1, 2, 3, ... \end{aligned} \tag{13}$$

In order to prove that function (13) possesses required differentiability properties, we proceed as follows. Transform conservation law (2) into the form

$$D_t\left(v\left(u-\eta\Delta u\right)\right)+\mathrm{Div}\left(\left(\eta v_t-\kappa v\right)\nabla u-\left(\eta\nabla v_t-\kappa\nabla v\right)u\right)=0. \tag{14}$$

Now, following the above scheme, to solve problem (12), we use conservation laws (14) with generating functions $v=v_k$ instead of (2). Then we obtain the equation on $u\left(x,t\right)$

$$\begin{aligned}&u-\eta u_{xx}=\tfrac{2}{l}\sum_{k=1}^{\infty}f_k\left(t\right)\sin\tfrac{k\pi}{l}x,\\&f_k\left(t\right)=\left(-1\right)^{k+1}\left\{\tfrac{l}{k\pi}\left(1-e^{-\lambda_k t}\right)+\tfrac{k\pi}{l}\eta\left(e^{-\frac{\kappa t}{\eta}}-e^{-\lambda_k t}\right)\right\},\\&k=1,2,3,\ldots\end{aligned} \tag{15}$$

The sum of series in the right-hand side of Eq.(15) is a continuous function in $[0,l)\times[0,\infty)$ by the Dirichlet criterion of uniform convergence. Solve formally Eq.(15) with boundary conditions (12) to obtain the following representation for $u\left(x,t\right)$:

$$u\left(x,t\right)=\left(1-e^{-\frac{\kappa t}{\eta}}\right)\frac{\mathrm{sh}\frac{x}{\sqrt{\eta}}}{\mathrm{sh}\frac{l}{\sqrt{\eta}}}+\frac{2}{l}\sum_{k=1}^{\infty}\frac{f_k\left(t\right)}{1+\left(\frac{k\pi}{l}\right)^2\eta}\sin\frac{k\pi}{l}x. \tag{16}$$

Using the Dirichlet and Abel criteria of uniform convergence, it is easy to verify that the function (16) possesses required differentiability properties and, hence, it is a solution of boundary-value problem (12).

Remark. The one-dimensional boundary-value problem can be solved by the standard scheme (based on the Fourier method of variables separation). However, it should be noted that the approach proposed here is more effective and brings the final result much faster. Moreover, it does not require any transformations of the initial setting-up of the boundary-value problem (required by the Fourier method) and therefore it may be referred to as the direct method. Finally, in contrast to the Fourier method, the example considered here shows that the proposed method allows to partially sum the obtained series and, thus, study their differentiability properties.

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References

[1] S.I.Senashov, *Unsolved problems of group analysis. Part 1,* Krasnoyarsk, Krasnoyarsk State University Publishers, 1989 (in Russian).
[2] S.I.Senashov // Dokl. Akad. Nauk (Russia), 1995, V.345, No.5, pp.619-620.
[3] G.I.Barenblatt, Yu.P.Zheltov, I.N.Kochina // Prikl. Matem. i Mekhanika, 1960, V.24, pp.852-864 (in Russian).
[4] V.S.Vladimirov, *Mathematical physics equations*, Moscow, Nauka Publishers, 1988 (in Russian).
[5] N.S.Koshlyakov, E.B.Gliner, M.M.Smirnov, *Mathematical physics partial differential equations*, Moscow, Vysshaya Shkola Publishers, 1970 (in Russian).