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DIVERGENCES AS A GRADING OF THE FORMAL VARIATIONAL CALCULUS

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Abstract. It is shown that the new formula for the field theory Poisson brackets arise naturally in the extension of the formal variational calculus incorporating divergences. The linear spaces of local functionals, evolutionary vector fields, functional forms, multi-vectors and differential operators become graded with respect to divergences. The bilinear operations, such as the action of vector fields on functionals, the commutator of vector fields, the interior product of forms and vectors and the Schouten-Nijenhuis bracket are compatible with the grading. A definition of the adjoint graded operator is proposed and antisymmetric operators are constructed with the help of boundary terms. The fulfilment of the Jacobi identity for the new Poisson brackets is shown to be equivalent to vanishing of the Schouten-Nijenhuis bracket of the Poisson bivector with itself. It is demonstrated, as an example, that the second structure of the Korteweg-de Vries equation is not Hamiltonian with respect to the new brackets until special boundary conditions are prescribed.

1. Introduction

The Hamiltonian formalism of classical mechanics may serve as the ideal model illustrating the harmony of physics and mathematics. During the last 20 years it was realized that a number of its mathematical constructions, for example, the Schouten-Nijenhuis bracket [1], could be extrapolated to field theory [2, 3]. This made the search for new nonlinear integrable models much easier. Even more general constructions uniting the Schouten-Nijenhuis bracket with the Frolicher-Nijenhuis bracket were considered by A. Vinogradov [4].

But the methods mentioned above (and usually called the formal variational calculus [5]) have some restrictions arising from boundary conditions which should allow free integration by parts. As a rule, the periodic boundary conditions or the rapid decay of fields at spatial infinity are necessary. Of course, these are not all the physically interesting cases. For example, the Coulomb potential in electrodynamics does not tend to zero rapidly enough. The similar behaviour is typical for Yang-Mills and gravitation fields. Nontrivial boundary problems arise also in the motion of material continua.

Key words and phrases: hamiltonian formalism, field theory, Poisson brackets, boundary terms

We consider the Hamiltonian treatment of nontrivial boundary problems as an interesting field of research where there is some place for new approaches and results. The field theory Poisson brackets which fulfil the Jacobi identity under arbitrary boundary conditions have been proposed in [6]. Here we extend the formal variational calculus to the most general case when no one boundary term arising in integration by parts can be discarded. We hope to present some physical applications of the methods developed here in the future. Let us also say that the interest to the role of divergences in field theory is vivid now as can be seen from related papers [7, 8, 9].

As an example, illustrating the nonstandard nature of the problems to be considered, let us remind the history of the long discussion on the role of surface integrals in the canonical formulation of General Relativity. During about 15 years Arnowitt, Deser and Misner [10], Dirac [11], Higgs [12], Schwinger [13], DeWitt [14], Regge and Teitelboim [15] were involved in it. The solution obtained in the last work [15] serves as the paradigm for the treatment of similar problems up to now. It has been proposed to work with the special class of the so-called differentiable or admissible functionals. These functionals are defined by the requirement that their variation should not have surface contributions under the prescribed boundary conditions. The Poisson brackets for these functionals are the standard ones, i.e., they are just the same as given in the formal variational calculus

$$\{F, G\} = \int\limits_{\Omega} \left(\frac{\delta F}{\delta q^A(x)} \frac{\delta G}{\delta p_A(x)} - \frac{\delta G}{\delta q^A(x)} \frac{\delta F}{\delta p_A(x)} \right) d^n x,$$

but now nonzero surface contributions are allowed.

Here a natural question to ask is: do these brackets fulfil the standard axiomatic requirements, i.e., the Jacobi identity and the existence of the Poisson algebra on this space of admissible functionals? For the infinite domain of integration and the asymptotic boundary conditions the affirmative result for the second requirement was obtained by Brown and Henneaux [16]. The first requirement was partially analysed by us and in the case treated above the answer is also positive.

It is more difficult to study the finite domain case. Let us take as a second example the motion of a fluid or plasma. It was shown by Lewis, Marsden, Montgomery and Ratiu [17] that the Jacobi identity for the standard Poisson bracket can be violated even in the case of fixed boundary, and so the Poisson brackets should be modified by surface terms. In the free boundary case it turns out to be natural to extend the space of admissible functionals so that their variation could include nonzero surface contributions. But according to [17] the presence of nonzero term with δq^A in the boundary integral requires the absence of the corresponding term with δp_A and vice versa. A new formula for Poisson brackets arises as a result of a generalization of the variational derivative which is now allowed to contain a surface contribution

$$\delta H = \int\limits_{\Omega} \frac{\delta^{\wedge} H}{\delta q^A} \delta q^A d^n x + \oint\limits_{\partial\Omega} \frac{\delta^{\vee} H}{\delta q^A} \delta q^A|_{\partial\Omega} dS + \int\limits_{\Omega} \frac{\delta^{\wedge} H}{\delta p_A} \delta p_A d^n x + \oint\limits_{\partial\Omega} \frac{\delta^{\vee} H}{\delta p_A} \delta p_A|_{\partial\Omega} dS.$$

Unfortunately, it is not quite clear whether the Poisson bracket of the two functionals, admissible in this new sense, will be admissible functional itself.

As a third example, we would like to attract reader's attention to consequences of the noncommutativity of the standard variational derivatives, i.e., the Euler-Lagrange derivatives. This point was discussed formally in publications by I. Anderson [19, 20] and Aldersley [21]. We faced with the problem independently, in the course of studying surface terms in the Poisson algebra of the Ashtekar formalism of canonical gravity [18]. There it was found that transformations of the type

$$q^A(x) \to q^A(x), \qquad p_A(x) \to p_A(x) + \frac{\delta F[q]}{\delta q^A(x)},$$

were noncanonical ones if surface terms were not ignored. Tracing the connection of this calculation with the standard calculations with δ-functions [22], we have found that the correspondence could be restored by introduction of θ_Ω — the characteristic function of the domain Ω

$$\theta_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then the standard relations

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) \delta(x, y) = 0,$$

should be modified as

$$\left(\theta_\Omega(x)\frac{\partial}{\partial x^i} + \theta_\Omega(y)\frac{\partial}{\partial y^i}\right) \delta(x, y) = -\frac{\partial \theta_\Omega}{\partial x^i}\delta(x, y),$$

where the usually discarded surface term is preserved.

All the above examples lead us to the necessity to extend the formal variational calculus onto total divergences. This extension consists in the introduction of a new grading for the linear spaces of local functionals, vector fields, functional forms, multi-vectors and differential operators. To come back to the standard case one should put $\theta_\Omega(x) \equiv 1$ in $\mathbb{R}^n$.

The extension of the formal variational calculus naturally incorporates the new definition of local functionals (not modulo divergences) and of their differentials (as a full variation, not fixed on the boundary). The Poisson bivectors are now defined in a more general way and they can include boundary contributions. The definition of pairing (interior product) is also revised and now the trace of two differential operators is used for it, so the pairing becomes compatible with the grading. The Poisson bracket found more or less heuristically in [6] arises now on the base of the geometrical constructions as

$$\{F, G\} = \mathsf{d}G \,\lrcorner\, \mathsf{d}F \,\lrcorner\, \Psi,$$

where Ψ is the Poisson bivector.

We show here that the Jacobi identity for the new Poisson brackets can be verified without the long calculations of binomial sums used in [6]. Its fulfilment is equivalent to the vanishing of the Schouten–Nijenhuis bracket of the Poisson bivector with itself. And in its turn this condition can be easily tested along with the procedure proposed by Olver [2] with a minimal modification of it. More attention than in [6] is paid here to nonultralocal Hamiltonian operators with nonconstant coefficients, because now the technical obstacles are removed. It turns out to be that not all operators which are Hamiltonian with respect to the standard brackets remain Hamiltonian in relation to the new brackets. For example, the second structure of the Korteweg–de Vries equation is not automatically Hamiltonian with respect to the new formalism. In this respect it strongly differs from the first KdV structure.

The content of this work is as follows. In Section 2 we introduce the grading for the local functionals, and the evolutionary vector fields. In Section 3 the differential, the functional m-forms, the interior product of vector fields onto m-forms and Lie derivative are defined. Section 4 deals with graded differential operators and their adjoints. In Section 5 we discuss multi-vectors and the Schouten-Nijenhuis bracket. It is shown how 1-vectors and evolutionary vector field spaces are isomorphic. Section 6 contains the general definition of the Poisson bracket, its realization in this formalism, the definition of Hamiltonian vector fields and the theorem on connection between the commutator of two Hamiltonian vector fields and the Poisson bracket of corresponding Hamiltonians. All constructions are illustrated by an example: the first Hamiltonian structure of the Korteweg-de Vries equation. The proof of Jacobi identity is postponed until Section 7. This new proof is applicable for all local Poisson brackets and so supersedes the proofs given earlier in [6]. In the same time it is easy to compare this proofs because notations are the same. At last, in Section 8 we consider two examples of the non-ultralocal Poisson brackets with field dependent coefficients (this class of brackets we were unable to study by the methods of [6]). We show that the second structure of Korteweg-de Vries equation is not Hamiltonian if surface terms are not ignored, whereas the 2-dimensional flow of the ideal fluid is described by Hamiltonian structure. This points to nontrivial character of the generalization of the formal variational calculus suggested here. A short summary is given in Conclusion and the points remaining unclear are mentioned. We hope to continue this research by the detailed treatment of the boundary conditions role and applications to some physical problems.

As a rule, we use below the same notations as in [6] except a change of the notation for the Fréchet derivative from D_f to f' and omitting the sign of summation according to the Einstein rule. We find convenient to represent integrals over finite domain Ω through integrals over the infinite space $\mathbb{R}^n$ by inserting into all integrands the characteristic function θ_Ω. Then the formalism seems closer to the standard formal variational calculus where local functionals and functional forms are defined modulo divergences. But the formal divergences that we discard here are integrated to zero under arbitrary conditions on the boundary of the finite domain, whereas real divergences are incorporated into graded structures. All the operations introduced below are compatible both with discarding formal total divergences (if one object is a formal divergence than the result of operation is also formal divergence) and with the grading (i.e., the same is valid for real divergences). Extension of the space of differential operators by admitting their grading permits to use the concept of adjoint operator. So, antisymmetric operators can now be constructed and the Poisson bracket formulas become more compact, than in [6] though their content is the same. Nevertheless, in the proof of the Jacobi identity we prefer to use the old notations to make easier the comparison with the not so general proofs of [6].

Henceforth we consider the space $\mathbb{R}^n$ and the multi-index notations $J = (j_1, ..., j_n)$ where

$j_i \geq 0$

$$\phi_A^{(J)} = \frac{\partial^{|J|}\phi_A}{\partial^{j_1}x^1 \dots \partial^{j_n}x^n}, \qquad |J| = j_1 + \dots + j_n.$$

The Fréchet derivative is defined as

$$f'_A = \frac{\partial f}{\partial \phi_A^{(J)}} D_J, \tag{1}$$

where

$$D_i = \frac{\partial}{\partial x^i} + \phi_A^{(J+i)} \frac{\partial}{\partial \phi_A^{(J)}}, \quad D_J = D_1^{j_1} \dots D_n^{j_n}, \quad D_i^0 = 1, \quad D_i^{-1} = 0.$$

Binomial coefficients for multi-indices are

$$\binom{J}{K} = \binom{j_1}{k_1} \cdots \binom{j_n}{k_n},$$

$$\binom{j}{k} = \begin{cases} j!/(k!(j-k)!) & \text{if } 0 \leq k \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

With the help of them we introduce the so-called higher Eulerian operators [2, 21, 23]

$$E_A^J(f) = (-1)^{|K|+|J|} \binom{K}{J} D_{K-J} \frac{\partial f}{\partial \phi_A^{(K)}}. \tag{2}$$

2. Local functionals and evolutionary vector fields

Let us start with notions from the theory of graded spaces as they are given in Ref. [3]. A *grading* in linear space L is a decomposition of it into direct sum of subspaces, with a special value of some function p (grading function) assigned to all the elements of any subspace.

Below the function p takes its values in the set of all positive multi-indices $J = (j_1, \dots, j_n)$ and so,

$$L = \bigoplus_{J=0}^{\infty} L^{\langle J \rangle}.$$

Elements of each subspace are called homogeneous.

A bilinear operation $x, y \mapsto x \circ y$, defined on L, is said to be *compatible with the grading* if the product of any homogeneous elements is also homogeneous, and if

$$p(x \circ y) = p(x) + p(y).$$

Now let us turn to the concrete structures.

There are two ways to write a local functional: as the integral of a smooth function $f^{\langle 0 \rangle}\left(\phi_A^{(K)}(x)\right)$ of fields and their derivatives up to some finite order over the prescribed domain Ω in $\mathbb{R}^n$, or as the integral over all the space $\mathbb{R}^n$ but with the characteristic function of the domain θ_Ω included into the integrand

$$F = \int_\Omega f^{\langle 0 \rangle}\left(\phi_A^{(K)}(x)\right) d^n x \equiv \int_{\mathbb{R}^n} \theta_\Omega f^{\langle 0 \rangle} d^n x. \tag{3}$$

As in [6], let us denote the space of local functionals as $\mathcal{A}$. Here we shall call the expression given above the *canonical form of a local functional.* We formally extend that definition by allowing local functionals to be written as follows

$$F=\int\limits_{\mathbb{R}^n} D_J\theta_\Omega f^{\langle J\rangle}\left(\phi_A^{(K)}(x)\right)d^nx\equiv\int\theta^{(J)}f^{\langle J\rangle}d^nx\equiv\int fd^nx,$$

where only a finite number of terms is allowed. Here and below we simplify the notation for derivatives of θ and remove Ω. All the integrals without the domain of integration shown explicitly are integrals over $\mathbb{R}^n$, below we shall omit d^nx. Of course, any functional can be transformed to the above form (3), exclusively used in [6], through integration by parts

$$F=\int\theta\tilde f^{\langle 0\rangle}\equiv\int\limits_\Omega\tilde f^{\langle 0\rangle},$$

where

$$\tilde f^{\langle 0\rangle}=(-1)^{|J|}D_Jf^{\langle J\rangle}.$$

Evidently, the formal integration by parts over infinite space $\mathbb{R}^n$ changes the grading. It will be clear below that the general situation is the following: from one side we have the compatibility of all the bilinear operations with the grading and from the other side — with the formal integration by parts. So, basic objects (local functionals etc.) are defined as equivalence classes modulo formal divergences (i.e., divergences of expressions containing θ-factors) and the unique decomposition into the homogeneous subspaces with the fixed grading function can be made only for representatives of these classes.

We call expressions of the form

$$\Psi=\int\theta^{(J)}D_K\psi_A^{\langle J\rangle}\frac{\partial}{\partial\phi_A^{(K)}}\equiv\int\theta^{(J)}\psi^{\langle J\rangle}\equiv\int\psi,$$

the *evolutionary vector fields.* The action of the evolutionary vector field on a local functional is given by the expression

$$\Psi F=\int\theta^{(I+J)}D_K\psi_A^{\langle J\rangle}\frac{\partial f^{\langle I\rangle}}{\partial\phi_A^{(K)}}\equiv\int\theta^{(I+J)}\psi^{\langle J\rangle}f^{\langle I\rangle}\equiv\int\psi f. \tag{4}$$

It is a straightforward calculation to check that this operation is compatible with the formal integration by parts, i.e.

$$\psi Df=D(\psi f),$$

as it is in the standard formal variational calculus. This relation is, of course, valid for integrands.

It is easy to see that the evolutionary vector field with coefficients

$$\psi_A^{\langle J\rangle}=D_L\xi_B^{\langle I\rangle}\frac{\partial\lambda_A^{\langle J-I\rangle}}{\partial\phi_B^{(L)}}-D_L\lambda_B^{\langle I\rangle}\frac{\partial\xi_A^{\langle J-I\rangle}}{\partial\phi_B^{(L)}}$$

can be considered as the *commutator of the evolutionary vector fields* Ξ and Λ

$$\Psi F = [\Xi, \Lambda] F = \int \Big(\xi(\lambda f) - \lambda(\xi f) \Big),$$

with the Jacobi identity fulfilled for the commutator operation, and so these vector fields form a Lie algebra.

Let us comment upon the representation of the evolutionary vector fields as integrals, which is different from the traditional notations.

The formal variational calculus [5] operates with the local functionals which are expressed by single integrals of functions of the specified class, for example, infinitely differentiable ones. The functional forms and multi-vectors are expressed by similar integrals. The pairing of two such objects gives us a single integral again.

At the same time, another notations are widespread, especially in physical literature, which use δ-function and its derivatives. Then a result of the pairing of two single integrals is understood as a double integral. But as this double integral contains the δ-function, it always can be converted into the single one.

This convertation of a double integral into the single one with the help of δ-function is trivial when no boundary terms could arise. The subject of this work is just a study of the opposite case. The new rule is necessary here and it have been proposed in [6] as Rule 4.2

$$\int_\Omega \int_\Omega f(x) g(y) D_J^{(x)} D_K^{(y)} \delta(x,y) = \int_\Omega D_K f D_J g. \tag{5}$$

In this article we give really a new and equivalent form of these rules which help to avoid the usage of double integrals at all.

The concept of vector field appeared initially in the course of studying the evolutionary differential equations and their symmetries. In the formal variational calculus [5] functionals are, in fact, replaced by equivalence classes of functions, and so the action of evolutionary vector fields onto local functionals is replaced by their action on functions

$$\psi f = D_K \psi_A \frac{\partial f}{\partial \phi_A^{(K)}}.$$

However, to represent functionals by integrals and to require that the result of the action of an evolutionary vector field onto a local functional is a local functional, i.e. an integral, it is absolutely natural to represent the evolutionary vector fields also as integrals

$$\Psi = \int_\Omega D_K \psi_A(x) \frac{\partial}{\partial \phi_A^{(K)}(x)} \equiv \int \psi,$$

in combination with the standard rule

$$\frac{\partial \phi_A^{(J)}(y)}{\partial \phi_B^{(K)}(x)} = \delta(x,y) \delta_{AB} \delta_{JK}. \tag{6}$$

Another argument supporting our notations is the equivalence between evolutionary vector fields and 1-vectors, which is demonstrated for the standard formal variational calculus in book [2] and also for the graded case in Section 5 of this article. 1-vectors as a partial case of multi-vectors are always written as integrals.

Apart from the notational revision we would like to mention a new feature in our treatment of the vector fields: now they are not derivations when applied to standard functions, but only to the graded ones. Of course, in the traditional approach the vector fields are not derivations when applied to functionals, because their multiplication is not defined. But these vector fields, traditionally written without the integral sign, are derivations of functions. This property is partially lost here. It can be restored formally if we consider integrands containing θ as functions and take a relation

$$D_I\theta \times D_J\theta = D_{I+J}\theta. \tag{7}$$

as a definition of their multiplication.

In this context, formula (4), introduced as a definition, can be interpreted also as a consequence of the standard relation (6) and a new definition (7).

Therefore, it is evident that our "rule for multiplication of distributions" taken from [6], i.e., equation (7) is nothing more than another way to define the pairing compatible with the introduced grading.

At last, let us mention the possibility to use other notations in this formalism. It is, of course, possible to avoid θ-functions and to use integrals over the domain Ω only. Then any local functional can be given as

$$F = \int_\Omega D_J f^{\langle\langle J\rangle\rangle},$$

where

$$f^{\langle\langle J\rangle\rangle} = (-1)^{|J|} f^{\langle J\rangle},$$

with analogous rewriting of the other objects. Correspondingly, equation (4) will be written as

$$\Psi F = \int_\Omega D_{I+J}\left(D_K\psi_A^{\langle\langle J\rangle\rangle}\frac{\partial f^{\langle\langle I\rangle\rangle}}{\partial\phi_A^{(K)}}\right).$$

3. Differentials and functional forms

The *differential of a local functional* is simply the first variation of it

$$\mathsf{d}F = \int \theta^{(J)}\frac{\partial f^{\langle J\rangle}}{\partial\phi_A^{(K)}}\delta\phi_A^{(K)} \equiv \int \theta^{(J)}\mathsf{d}f^{\langle J\rangle} \equiv \int \mathsf{d}f,$$

here and below $\delta\phi_A^{(K)} = D_K\delta\phi_A$. It can also be expressed through the Fréchet derivative (1) or through the higher Eulerian operators (2)

$$\mathsf{d}F = \int \theta^{(J)} f^{\langle J\rangle\prime}(\delta\phi) = \int \theta^{(J)} D_K\left(E_A^K(f^{\langle J\rangle})\delta\phi_A\right).$$

This differential is a special example of functional 1-form. A general functional 1-form can be written as

$$\Sigma = \int \theta^{(J)} \sigma^{\langle J \rangle}_{AK} \delta\phi^{(K)}_A \equiv \int \theta^{(J)} \sigma^{\langle J \rangle} \equiv \int \sigma.$$

Of course, the coefficients $\sigma^{\langle J \rangle}_{AK}$ are not unique since we can make formal integration by parts. Let us call the following expression the *canonical form of a functional 1-form*

$$\Sigma = \int \theta^{(J)} \sigma^{\langle J \rangle}_{A} \delta\phi_A.$$

Analogously, we can define *functional m-forms* as integrals or equivalence classes modulo formal divergences of vertical m-forms

$$\Sigma = \frac{1}{m!} \int \theta^{(J)} \sigma^{\langle J \rangle}_{A_1 K_1, \dots, A_m K_m} \delta\phi^{(K_1)}_{A_1} \wedge \dots \wedge \delta\phi^{(K_m)}_{A_m} = \int \theta^{(J)} \sigma^{\langle J \rangle} = \int \sigma.$$

Define the *pairing* (or the *interior product*) of an evolutionary vector field and 1-form as

$$\Sigma(\Xi) = \Xi \rfloor \Sigma = \int \theta^{(I+J)} \sigma^{\langle J \rangle}_{AK} D_K \xi^{\langle I \rangle}_A = \int \theta^{(I+J)} \sigma^{\langle J \rangle}(\xi^{\langle I \rangle}) = \int \sigma(\xi). \tag{8}$$

The interior product of an evolutionary vector field and a functional m-form will be given as follows

$$\Xi \rfloor \Sigma = \frac{1}{m!}(-1)^{i+1} \int \theta^{(I+J)} \sigma^{\langle J \rangle}_{A_1 K_1, \dots, A_m K_m} D_{K_i} \xi^{\langle I \rangle}_{A_i} \delta\phi^{(K_1)}_{A_1} \wedge \dots$$
$$\dots \wedge \delta\phi^{(K_{i-1})}_{A_{i-1}} \wedge \delta\phi^{(K_{i+1})}_{A_{i+1}} \wedge \dots \wedge \delta\phi^{(K_m)}_{A_m}.$$

Then a value of the m-form on the m evolutionary vector fields will be defined by the formula

$$\Sigma(\Xi_1, \dots, \Xi_m) = \Xi_m \rfloor \dots \Xi_1 \rfloor \Sigma.$$

It can be checked by straightforward calculation that

$$(D\sigma)(\xi_1, \dots, \xi_m) = D\left(\sigma(\xi_1, \dots, \xi_m)\right).$$

The *differential of the m-form* which is given as

$$\mathsf{d}\Sigma = \frac{1}{m!} \int \theta^{(J)} \frac{\partial \sigma^{\langle J \rangle}_{A_1 K_1, \dots, A_m K_m}}{\partial \phi^{(K)}_A} \delta\phi^{(K)}_A \wedge \delta\phi^{(K_1)}_{A_1} \wedge \dots \wedge \delta\phi^{(K_m)}_{A_m} = \int \theta^{(J)} \mathsf{d}\sigma^{\langle J \rangle} = \int \mathsf{d}\sigma,$$

satisfies standard properties

$$\mathsf{d}^2 = 0$$

and

$$\mathsf{d}\Sigma(\Xi_1, \dots, \Xi_{m+1}) = \sum_i (-1)^{i+1} \Xi_i \Sigma(\Xi_1, \dots, \hat{\Xi}_i, \dots, \Xi_{m+1}) +$$
$$+ \sum_{i<j} (-1)^{i+j} \Sigma([\Xi_i, \Xi_j], \Xi_1, \dots, \hat{\Xi}_i, \dots, \hat{\Xi}_j, \dots, \Xi_{m+1}).$$

The *Lie derivative* of a functional form Σ along the evolutionary vector field Ξ can be introduced by the standard formula

$$L_\Xi \Sigma = \Xi \rfloor \mathsf{d}\Sigma + \mathsf{d}\Big(\Xi \rfloor \Sigma\Big).$$

4. Graded differential operators and their adjoints

We call linear matrix differential operators of the form

$$\hat{I}_{AB} = \theta^{(J)} I_{AB}^{\langle J\rangle N} D_N$$

the *graded differential operators.*

Let us call the linear differential operator $\hat{I}^*$ the *adjoint* to $\hat{I}$ if for an arbitrary set of smooth functions f_A, g_A

$$\int f_A \hat{I}_{AB} g_B = \int g_A \hat{I}^*_{AB} f_B.$$

For coefficients of the adjoint operator we can derive the expression

$$I_{AB}^{*\langle J\rangle M} = (-1)^{|K|} \binom{K}{L} \binom{K-L}{M} D_{K-L-M} I_{BA}^{\langle J-L\rangle K}. \tag{9}$$

It is easy to check that the relation

$$\hat{I}_{AB}(x)\delta(x,y) = \hat{I}^*_{BA}(y)\delta(x,y)$$

follows from Rule 4.2 of [6]. For example, we have

$$\left(\theta(x)\frac{\partial}{\partial x^i} + \theta(y)\frac{\partial}{\partial y^i}\right)\delta(x,y) = -\theta^{(i)}\delta(x,y). \tag{10}$$

In one of our previous publications [18] we tried to connect the appearance of surface terms in Poisson brackets and the standard manipulations with the δ-function. The ansatz used there for the above simplest example coincided with (10) up to the sign. The reason for this difference laid in the other choice made there instead of Rule 4.2 of [6]. That ansatz lead us to the standard Poisson brackets which are not appropriate for nontrivial boundary problems.

Operators satisfying the relation

$$\hat{I}^* = -\hat{I}$$

will be called the *antisymmetric* ones. With the help of them it is possible to express 2-forms (and also 2-vectors to be defined below) in the canonical form

$$\Sigma = \frac{1}{2}\int \delta\phi_A \wedge \hat{I}_{AB}\delta\phi_B.$$

It is clear that we can consider representations of functional forms as decompositions over the basis derived as a tensor product of $\delta\phi_A$, with the totally antisymmetric multilinear operators

$$\hat{\sigma} = \theta^{(J)} \sigma^{\langle J\rangle}_{A_1K_1,\ldots,A_mK_m}\left(D_{K_1}\cdot, \ldots, D_{K_m}\cdot\right)$$

as coefficients of these decompositions.

5. Multi-vectors, mixed tensors and Schouten-Nijenhuis bracket

Let us introduce dual basis to $|\delta\phi_A\rangle$ by the relation

$$\left\langle \frac{\delta}{\delta\phi_B(y)}, \delta\phi_A(x) \right\rangle = \delta_{AB}\delta(x,y) \tag{11}$$

and construct by means of the tensor product a basis

$$\frac{\delta}{\delta\phi_{B_1}(y)} \otimes \frac{\delta}{\delta\phi_{B_2}(y)} \otimes \ldots \otimes \frac{\delta}{\delta\phi_{B_m}(y)}.$$

Then by using totally antisymmetric multilinear operators described in the previous Section we can define the *functional m-vectors* (or *multi-vectors*)

$$\Psi = \frac{1}{m!}\int \theta^{(J)}\psi^{\langle J\rangle}_{B_1L_1,\ldots,B_mL_m} D_{L_1}\frac{\delta}{\delta\phi_{B_1}} \wedge \ldots \wedge D_{L_m}\frac{\delta}{\delta\phi_{B_m}} = \int \theta^{(J)}\psi^{\langle J\rangle}.$$

Here a natural question on the relation between evolutionary vector fields and 1-vectors arises. Evidently, evolutionary vector fields lose their form when being integrated by parts whereas 1-vectors preserve it. Let us make a partial integration in the expression of a general evolutionary vector field

$$\Xi = \int \theta^{(J)} D_K \xi^{\langle J\rangle}_A \frac{\partial}{\partial\phi^{(K)}_A}$$

by removing D_K from $\xi^{\langle J\rangle}_A$, then we get

$$\Xi = \int \xi^{\langle J\rangle}_A \theta^{(J+L)}(-1)^{|K|}\binom{K}{L} D_{K-L}\frac{\partial}{\partial\phi^{(K)}_A}.$$

It is easy to see that by using (7), i.e., Rule 5.4 from [6], in the backward direction we can write

$$\Xi = \int \left(\theta^{(J)}\xi^{\langle J\rangle}_A\right)\left(\theta^{(L)}(-1)^{|L|}E^L_A\right) = \int \theta^{(J)}\xi^{\langle J\rangle}_A \frac{\delta}{\delta\phi_A},$$

where the higher Eulerian operators (2) and the full variational derivative (Definition 5.1 of [6])

$$\frac{\delta F}{\delta\phi_A} = \sum (-1)^{|J|}E^J_A(f)D_J\theta,$$

are consequently used. Therefore, we arrive at the following Statement.

Statement 5.1 *There is a one-to-one correspondence between the evolutionary vector fields and the functional 1-vectors. The coefficients of 1-vector in the canonical form $\xi^{\langle J\rangle}_A$ are equal to the characteristics of the evolutionary vector field.*

It is not difficult to show that we can deduce the pairing (interior product) of 1-forms and 1-vectors and this pairing preserves the identification. Really, the definition of the dual basis (11) and (7), i.e., Rule 5.4 of [6], permits us to derive that

$$\Sigma(\Xi) = \Xi \,\lrcorner\, \Sigma = \int\int \theta^{(I)}(x)\theta^{(J)}(y)\sigma^{\langle I\rangle}_{AK}(x)\xi^{\langle J\rangle}_{BL}(y)\left\langle D_L\frac{\delta}{\delta\phi_B(y)}, D_K\delta\phi_A(x)\right\rangle =$$

$$= \int \theta^{(I+J)} D_L \sigma^{\langle I \rangle}_{AK} D_K \xi^{\langle J \rangle}_{AL} = \int \theta^{(I+J)} \sigma^{\langle I \rangle}(\xi^{\langle J \rangle}) = \int \sigma(\xi) = \int \theta^{(I+J)} \mathrm{Tr}(\sigma^{\langle I \rangle}\xi^{\langle J \rangle}),$$

and when 1-vector is in the canonical form (only $L = 0$ term is nonzero) this result coincides with Eq.(8).

This formula for the pairing will be exploited below also for interior product of 1-vectors and m-forms or 1-forms and m-vectors. Its importance comes from the fact that it is invariant under the formal partial integration both in forms and in vectors, i.e.,

$$(\mathrm{D}\sigma)(\xi) = \mathrm{D}(\sigma(\xi)) = \sigma(\mathrm{D}(\xi)).$$

Evidently, it is the trace construction for convolution of differential operators (as coefficients of tensor objects in the proposed basis) that guarantees this invariance.

The interior product of 1-vector onto m-form and, analogously, of 1-form onto m-vector is defined as

$$\begin{aligned} \Xi \rfloor \Sigma \quad = \quad & \frac{1}{m!}(-1)^{(i+1)} \int \theta^{(I+J)} D_{K_i} \xi^{\langle I \rangle}_{A_i L} D_L \Big(\sigma^{\langle J \rangle}_{A_1 K_1, \ldots, A_m K_m} \delta\phi^{(K_1)}_{A_1} \wedge \ldots \\ \ldots \quad & \wedge \delta\phi^{(K_{i-1})}_{A_{i-1}} \wedge \delta\phi^{(K_{i+1})}_{A_{i+1}} \wedge \ldots \wedge \delta\phi^{(K_m)}_{A_m} \Big) = (-1)^{(i+1)} \int \theta^{(I+J)} \xi^{\langle I \rangle} \rfloor \sigma^{\langle J \rangle}. \end{aligned} \tag{12}$$

Then we also can define the value of m-form on m 1-vectors (or, analogously, m-vector on m 1-forms)

$$\Sigma(\Xi_1, \ldots, \Xi_m) = \Xi_m \rfloor \ldots \Xi_1 \rfloor \Sigma = \int \theta^{(J+I_1+\ldots+I_m)} \mathrm{Tr}\left(\sigma^{\langle J \rangle} \xi^{\langle I_1 \rangle}_1 \cdots \xi^{\langle I_m \rangle}_m\right),$$

where each entry of multilinear operator σ acts only on the corresponding ξ, whereas each derivation of the operator ξ acts on the product of σ and all the rest of ξ's.

It is possible to define the *differential of m-vector*

$$\mathsf{d}\Psi = \frac{1}{m!} \int \theta^{(J)} \frac{\partial \psi^{\langle J \rangle}_{A_1 K_1, \ldots, A_m K_m}}{\partial \phi^{(L)}_B} \delta\phi^{(L)}_B D_{K_1} \frac{\delta}{\delta\phi_{A_1}} \wedge \ldots \wedge D_{K_m} \frac{\delta}{\delta\phi_{A_m}},$$

as an example of a mixed $\binom{m}{1}$ object. Evidently, $\mathsf{d}^2\Psi = 0$.

With the help of the previous constructions we can define the *Schouten-Nijenhuis bracket*

$$[\Xi, \Psi]_{SN} = \mathsf{d}\Xi \rfloor \Psi + (-1)^{pq} \mathsf{d}\Psi \rfloor \Xi$$

for two multi-vectors of orders p and q. The result of this operation is $p + q - 1$-vector and it is analogous to the Schouten-Nijenhuis bracket in tensor analysis [1]. Its use in the formal variational calculus is described in Refs. [2, 3]. However, in cited references this bracket is usually defined for operators. We can recommend Ref. [24] as an interesting source for the treatment of the Schouten-Nijenhuis bracket of multi-vectors. Our construction of this bracket guarantees a compatibility with the equivalence modulo divergences

$$[D\xi, \psi]_{SN} = D[\xi, \psi]_{SN} = [\xi, D\psi]_{SN}.$$

Statement 5.2 *The Schouten-Nijenhuis bracket of functional 1-vectors up to a sign coincides with the commutator of the corresponding evolutionary vector fields.*

Proof. Let us take the two 1-vectors in canonical form without loss of generality

$$\Xi = \int \theta^{(J)} \xi_A^{\langle J\rangle} \frac{\delta}{\delta\phi_A}, \qquad \Psi = \int \theta^{(K)} \psi_B^{\langle K\rangle} \frac{\delta}{\delta\phi_B}$$

and compute

$$[\Xi, \Psi]_{SN} = \mathsf{d}\Xi \lrcorner \Psi - \mathsf{d}\Psi \lrcorner \Xi.$$

We have

$$\mathsf{d}\Xi = \int \theta^{(J)} \xi_A^{\langle J\rangle\prime} (\delta\phi) \frac{\delta}{\delta\phi_A} = \int \theta^{(J)} \frac{\partial \xi_A^{\langle J\rangle}}{\partial \phi_C^{(L)}} \delta\phi_C^{(L)} \frac{\delta}{\delta\phi_A},$$

and

$$\mathsf{d}\Xi \lrcorner \Psi = -\int \theta^{(J+K)} \frac{\partial \xi_A^{\langle J\rangle}}{\partial \phi_B^{(L)}} D_L \psi_B^{\langle K\rangle} \frac{\delta}{\delta\phi_A}.$$

Therefore, we obtain

$$[\Xi, \Psi]_{SN} = -\int \theta^{(J+K)} \left(D_L \psi_B^{\langle K\rangle} \frac{\partial \xi_A^{\langle J\rangle}}{\partial \phi_B^{(L)}} - D_L \xi_B^{\langle K\rangle} \frac{\partial \psi_A^{\langle J\rangle}}{\partial \phi_B^{(L)}} \right) \frac{\delta}{\delta\phi_A} = -[\Xi, \Psi],$$

and the proof is completed.

Statement 5.3 (Olver's Lemma [2]) *The Schouten-Nijenhuis bracket for two bivectors can be expressed in the form*

$$[\Lambda, \Psi]_{SN} = -\frac{1}{2}\int \ \xi \wedge \hat{I}'(\hat{K}\xi) \wedge \xi - \frac{1}{2}\int \ \xi \wedge \hat{K}'(\hat{I}\xi) \wedge \xi, \tag{13}$$

where the two differential operators $\hat{I}$, $\hat{K}$ are the coefficients of the bivectors in their canonical form.

Proof. Let us consider the Schouten-Nijenhuis bracket for the two bivectors and without loss of generality take them in the canonical form

$$\Lambda = \frac{1}{2}\int \theta^{(L)} \xi_A \wedge I_{AB}^{\langle L\rangle N} D_N \xi_B,$$

$$\Psi = \frac{1}{2}\int \theta^{(M)} \xi_C \wedge K_{CD}^{\langle M\rangle P} D_P \xi_D,$$

where $\xi_A = \delta/\delta\phi_A$ and operators $\hat{I}$, $\hat{K}$ are antisymmetric. Then we have

$$\mathsf{d}\Lambda = \frac{1}{2}\int \theta^{(L)} \frac{\partial I_{AB}^{\langle L\rangle N}}{\partial \phi_E^{(J)}} \delta\phi_E^{(J)} \xi_A \wedge D_N \xi_B$$

and

$$\mathsf{d}\Lambda \lrcorner \Psi = \frac{1}{4}\int \theta^{(L+M)} \frac{\partial I_{AB}^{\langle L\rangle N}}{\partial \phi_C^{(J)}} D_J \left(K_{CD}^{\langle M\rangle P} D_P \xi_D \right) \wedge \xi_A \wedge D_N \xi_B -$$

$$-\frac{1}{4}\int \theta^{(L+M)} D_P\left(\frac{\partial I_{AB}^{\langle L\rangle N}}{\partial \phi_D^{(J)}}\xi_A\wedge D_N\xi_B\right)\wedge D_J\left(\xi_C K_{CD}^{\langle M\rangle P}\right).$$

Now let us make integration by parts in the second term

$$\mathsf{d}\Lambda \,\lrcorner\, \Psi = -\frac{1}{4}\int \theta^{(L+M)}\xi_A\wedge\left(I_{AB}^{\langle L\rangle N}\right)'\left(\hat{K}^{\langle M\rangle}\xi\right)\wedge D_N\xi_B -$$

$$-\frac{1}{4}\int \theta^{(L+M+Q)}(-1)^{|P|}\binom{P}{Q}\frac{\partial I_{AB}^{\langle L\rangle N}}{\partial \phi_D^{(J)}}\xi_A\wedge D_N\xi_B\wedge D_{J+P-Q}\left(\xi_C K_{CD}^{\langle M\rangle P}\right).$$

At last we change the order of multipliers under wedge product in the second term, make a replacement $M\to M-Q$ and organize the whole expression in the form

$$\mathsf{d}\Lambda \,\lrcorner\, \Psi = -\frac{1}{4}\int \theta^{(L+M)}\xi_A\wedge\left(I_{AB}^{\langle L\rangle N}\right)'_C\left(\hat{K}_{CD}^{\langle M\rangle}\xi_D+\right.$$

$$\left.+(-1)^{|P|}\binom{P}{Q}\binom{P-Q}{R}D_{P-Q-R}K_{CD}^{\langle M-Q\rangle P}D_R\xi_C\right)\wedge D_N\xi_B.$$

Having in mind the definition of adjoint operator (9) we can represent the final result of the calculation as follows,

$$[\Lambda,\Psi]_{SN} = -\frac{1}{2}\int \theta^{(L+M)}\xi\wedge\left((\hat{I}^{\langle L\rangle})'(\hat{K}^{\langle M\rangle}\xi)+(\hat{K}^{\langle M\rangle})'(\hat{I}^{\langle L\rangle}\xi)\right)\wedge\xi,$$

thus supporting in this extended formulation the method, proposed in Ref. [2] for testing the Jacobi identity (see Section 7). For the general procedure of testing Hamiltonian properties see also [26].

6. Poisson brackets and Hamiltonian vector fields

Let us call the bivector

$$\Psi = \frac{1}{2}\int \frac{\delta}{\delta\phi_A}\wedge \hat{I}_{AB}\frac{\delta}{\delta\phi_B},$$

formed with the help of the graded antisymmetric differential operator

$$\hat{I}_{AB} = \theta^{(L)} I_{AB}^{\langle L\rangle N} D_N,$$

the *Poisson bivector* if

$$[\Psi,\Psi]_{SN} = 0.$$

The operator $\hat{I}_{AB}$ is then called the *Hamiltonian operator*. We call the value of the Poisson bivector on the differentials of two functionals F and G

$$\{F,G\} = \Psi(\mathsf{d}F,\mathsf{d}G) = \mathsf{d}G\,\lrcorner\,\mathsf{d}F\,\lrcorner\,\Psi$$

the *Poisson bracket* of these functionals.

The explicit form of the Poisson bracket can easily be obtained. It depends on the explicit form of the functional differential, which can be changed by the formal partial integration. Of course, all the possible forms are equivalent. Taking the extreme cases we get an expression through Fréchet derivatives

$$\{F, G\} = \int \theta^{(J)} \mathrm{Tr}\left(f'_A \hat{I}^{\langle J \rangle}_{AB} g'_B\right) \tag{14}$$

or through higher Eulerian operators (2)

$$\{F, G\} = \int \theta^{(J)} D_{P+Q} \left(E^P_A(f) \hat{I}^{\langle J \rangle}_{AB} E^Q_B(g)\right). \tag{15}$$

Theorem 6.1 *The Poisson bracket defined above satisfies the standard requirements of the bilinearity, antisymmetry and closeness on the space of local functionals* A, *i.e.,* Definition 2.3 *of* [6].

Proof. 1) From the previous formulas (14), (15) it is clear that $\{F, G\}$ is a local functional, 2) antisymmetry of $\{F, G\}$ is evident and 3) equivalence of the Jacobi identity to the Poisson bivector property will be proved in Section 7.

The result of interior product of the differential of a local functional H on the Poisson bivector (up to the sign) will be called the *Hamiltonian vector field* (or the *Hamiltonian 1-vector*)

$$\hat{I}\mathsf{d}H = -\mathsf{d}H \rfloor \Psi$$

corresponding to the Hamiltonian H. Evidently, the standard relations take place

$$\{F, H\} = \mathsf{d}F(\hat{I}\mathsf{d}H) = (\hat{I}\mathsf{d}H)F.$$

Theorem 6.2 *The Hamiltonian vector field corresponding to the Poisson bracket of the functionals F and H coincides up to the sign with commutator of the Hamiltonian vector fields corresponding to these functionals.*

Proof. Consider a value of the commutator of Hamiltonian vector fields $\hat{I}\mathsf{d}F$ and $\hat{I}\mathsf{d}H$ on the arbitrary functional G

$$[\hat{I}\mathsf{d}F, \hat{I}\mathsf{d}H]G = \hat{I}\mathsf{d}F(\hat{I}\mathsf{d}H(G)) - \hat{I}\mathsf{d}H(\hat{I}\mathsf{d}F(G)) =$$

$$= \hat{I}\mathsf{d}F(\{G, H\}) - \hat{I}\mathsf{d}H(\{G, F\}) = \{\{G, H\}, F\} - \{\{G, F\}, H\} =$$
$$= -\{G, \{F, H\}\} = -\hat{I}\mathsf{d}\{F, H\}(G),$$

where we have used the Jacobi identity and antisymmetry of Poisson bracket. Due to the arbitrariness of G the proof is completed.

Example 6.3
Let us consider the first structure

$$\{u(x), u(y)\} = \frac{1}{2}(D_x - D_y)\delta(x, y)$$

of the Korteweg-de Vries equation (Example 7.6 of Ref. [2])

$$u_t = u_{xxx} + uu_x.$$

Construct the adjoint graded operator to θD according to Eq.(9)

$$(\theta D)^* = -\theta D - D\theta$$

and the antisymmetric operator is

$$\hat{I} = \frac{1}{2}\Big(\theta D - (\theta D)^*\Big) = \theta D + \frac{1}{2} D\theta.$$

The Poisson bivector has a form

$$\Psi = \frac{1}{2}\int \theta\Big(\frac{\delta}{\delta u} \wedge D\frac{\delta}{\delta u}\Big).$$

The differential of a local functional H (for simplicity it is written in canonical

$$H = \int \theta h$$

form) is equal to

$$\mathsf{d}H = \int \theta h'(\delta u) = \int \theta^{(k)}(-1)^k E^k(h)\delta u,$$

where the Fréchet derivative or higher Eulerian operators can be used. Therefore, the Hamiltonian vector field generated by H is

$$\hat{I}\mathsf{d}H = -\mathsf{d}H \lrcorner \Psi = -\frac{1}{2}\int \theta \left(h'(D\frac{\delta}{\delta u}) - Dh'(\frac{\delta}{\delta u}) \right),$$

or

$$-\frac{1}{2}\int \theta^{(k)}(-1)^k \left(E^k(h)D - DE^k(h) \right) \frac{\delta}{\delta u},$$

or also

$$-\frac{1}{2}\int \theta^{(k)}(-1)^k D_i \left(E^k(h)D - DE^k(h) \right) \frac{\partial}{\partial u^{(i)}}.$$

The value of this vector field on another functional F coincides with the Poisson bracket

$$-\mathsf{d}F \lrcorner \mathsf{d}H \lrcorner \Psi = \{F, H\} = \frac{1}{2}\int \theta^{(k+l)}(-1)^{k+l} \left(E^k(f)DE^l(h) - E^k(h)DE^l(f) \right).$$

7. Proof of Jacobi identity

In this section we will prove that the Jacobi identity for the Poisson bracket is fulfilled if and only if the Schouten–Nijenhuis bracket of the corresponding Poisson bivector with itself is equal to zero. This should complete the proof of Theorem 6.1.

Let us use one of the possible forms of the Poisson brackets given in Appendix of [6]

$$\{F,G\}=\frac{1}{2}\int\theta^{(J)}\mathrm{Tr}\left(f'(\hat{I}^{\langle J\rangle}g')-g'(\hat{I}^{\langle J\rangle}f')\right),$$

where the differential operator $\hat{I}$ is not supposed to be antisymmetric for the easier comparison of this proof with that given in [6]. We remind that in less condensed notations

$$\mathrm{Tr}\left(f'(\hat{I}g')\right)=\binom{J}{M}\binom{K}{L}D_L\frac{\partial f}{\partial\phi_A^{(J)}}D_{J+K-L-M}I_{AB}^{N}D_{N+M}\frac{\partial g}{\partial\phi_B^{(K)}}$$

(in Appendix of [6] the indices M and L in the binomial coefficients of the same formula are unfortunately given in the opposite order).

We will estimate the bracket

$$\{\{F,G\},H\}=\frac{1}{2}\int\theta^{(J)}\mathrm{Tr}\left(\{f,g\}'(\hat{I}^{\langle J\rangle}h')-h'(\hat{I}^{\langle J\rangle}\{f,g\}')\right),$$

where $\{f,g\}$ denotes the integrand of $\{F,G\}$. Since Fréchet derivative is a derivation we have

$$\{f,g\}'=\frac{1}{2}\theta^{(K)}\mathrm{Tr}\Big(f''(\hat{I}^{\langle K\rangle}g',\cdot)+f'\hat{I}'^{\langle K\rangle}(\cdot)g'+g''(f'\hat{I}^{\langle K\rangle},\cdot)-(f\leftrightarrow g)\Big)$$

and

$$\mathrm{Tr}\left(\{f,g\}'\hat{I}h'\right)=\frac{1}{2}\left(f''(\hat{I}g',\hat{I}h')+f'\hat{I}'(\hat{I}h')g'+g''(f'\hat{I},\hat{I}h')-(f\leftrightarrow g)\right).$$

Let us explain that f'' denotes the second Fréchet derivative, i.e., the symmetric bilinear operator arising in calculation of the second variation of the local functional F (in the canonical form):

$$f''(\xi,\eta)=\frac{\partial^2 f}{\partial\phi_A^{(J)}\partial\phi_B^{(K)}}D_J\xi_A D_K\eta_B.$$

When we put into entries of f'' operators under the trace sign it should be understood that these operators act on everything except their own coefficients, for example,

$$\mathrm{Tr}\left(f''(\hat{I}g',\hat{I}h')\right)=\binom{L}{P}\binom{L-P}{Q}\binom{M}{S}\binom{M-S}{T}\times$$

$$\times D_{L+M-P-Q-S-T}\frac{\partial^2 f}{\partial\phi_A^{(J)}\partial\phi_B^{(K)}}D_{J+T}\left(D_P\hat{I}_{AC}\frac{\partial g}{\partial\phi_C^{(L)}}\right)D_{K+Q}\left(D_S\hat{I}_{BD}\frac{\partial h}{\partial\phi_D^{(M)}}\right)$$

and the expression remains symmetric under permutation of its entries

$$\mathrm{Tr}\left(f''(\hat{I}g',\hat{I}h')\right)=\mathrm{Tr}\left(f''(\hat{I}h',\hat{I}g')\right).$$

When the operator $\hat{I}$ stands to the right from the operator of Fréchet derivative f' as in expression

$$\mathrm{Tr}\left(g''(\hat{I}h', f'\hat{I})\right),$$

it acts on everything except f'. At last, for Fréchet derivative of the operator we have

$$\hat{I}'(\hat{I}h') = \frac{\partial I^K_{AB}}{\partial \phi^{(J)}_C} D_J \left(I^L_{CD} D_L \frac{\partial h}{\partial \phi^{(M)}_D} D_M \right) D_K.$$

Making similar calculations we get

$$\mathrm{Tr}\left(h'\hat{I}\{f,g\}'\right) = \frac{1}{2}\mathrm{Tr}\left(f''(h'\hat{I},\hat{I}g') + f'\hat{I}'(h'\hat{I})g' + g''(f'\hat{I},h'\hat{I}) - (f \leftrightarrow g)\right)$$

and therefore

$$\{\{F,G\},H\} = \frac{1}{4}\int \theta^{(J+K)}\mathrm{Tr}\Big(f''(\hat{I}^{\langle J\rangle}g', \hat{I}^{\langle K\rangle}h') - f''(h'\hat{I}^{\langle J\rangle}, \hat{I}^{\langle K\rangle}g') -$$

$$-f''(\hat{I}^{\langle J\rangle}h', g'\hat{I}^{\langle K\rangle}) + f''(g'\hat{I}^{\langle J\rangle}, h'\hat{I}^{\langle K\rangle}) + f'\hat{I}'^{\langle J\rangle}(\hat{I}^{\langle K\rangle}h' - h'\hat{I}^{\langle K\rangle})g' - (f \leftrightarrow g)\Big).$$

Just the first four terms, apart from the fifth containing Fréchet derivative of the operator $\hat{I}$, were present in our proof for nonultralocal case given in [6] (only terms with zero grading were allowed for $\hat{I}$ there). After cyclic permutation of F, G, H all terms with the symmetric operator of the second Fréchet derivative are mutually cancelled and

$$\{\{F,G\},H\} + \mathrm{c.p.} = \frac{1}{4}\int \theta^{(J+K)}\mathrm{Tr}\Big(f'\hat{I}'^{\langle J\rangle}(\hat{I}^{\langle K\rangle}h' - h'\hat{I}^{\langle K\rangle})g' -$$

$$-g'\hat{I}'^{\langle J\rangle}(\hat{I}^{\langle K\rangle}h' - h'\hat{I}^{\langle K\rangle})f' + c.p.\Big),$$

where cyclic permutations of F, G, and H are abbreviated to *c.p.*. When operator $\hat{I}$ is given in explicitly antisymmetric form all the four terms are equal. Taking into account Olver's Lemma (13) we get

$$\{\{F,G\},H\} + c.p. = -[\hat{I},\hat{I}]_{SN}(\mathsf{d}F, \mathsf{d}G, \mathsf{d}H),$$

so finishing the proof.

8. Examples of nonultralocal operators

The second structure of the Korteweg-de Vries equation may serve as a counter-example to the hypothesis [25] that all operators which are Hamiltonian with respect to the standard Poisson brackets should also be Hamiltonian in the new brackets.

Example 8.1

Let us start with the standard expression (Example 7.6 of Ref. [2])

$$\{u(x), u(y)\} = \left(\frac{d^3}{dx^3} + \frac{2}{3}u\frac{d}{dx} + \frac{1}{3}\frac{du}{dx}\right)\delta(x,y)$$

and construct the adjoint operator to

$$\hat{K} = \theta\left(D_3 + \frac{2}{3}uD + \frac{1}{3}Du\right),$$

which is

$$\hat{K}^* = -\theta\left(D_3 + \frac{2}{3}uD + \frac{1}{3}Du\right) - D\theta\left(3D_2 + \frac{2}{3}u\right) - 3D_2\theta D - D_3\theta.$$

Then the antisymmetric operator

$$\hat{I} = \frac{1}{2}(\hat{K} - \hat{K}^*) = \theta\left(D_3 + \frac{2}{3}uD + \frac{1}{3}Du\right) + D\theta\left(\frac{3}{2}D_2 + \frac{1}{3}u\right) + \frac{3}{2}D_2\theta D + \frac{1}{2}D_3\theta$$

can be used for forming the bivector

$$\Psi = \frac{1}{2}\int \xi \wedge \hat{I}\xi,$$

where $\delta/\delta u = \xi$. This bivector has a form

$$\Psi = \frac{1}{2}\int\left(\theta\xi \wedge D_3\xi + \frac{3}{2}D\theta\xi \wedge D_2\xi + (\frac{3}{2}D_2\theta + \frac{2}{3}\theta u)\xi \wedge D\xi\right).$$

Then evaluating the Schouten-Nijenhuis bracket for the bivector with the help of Statement 5.3

$$[\Psi, \Psi]_{SN} = \int\left(\frac{2}{3}\theta\xi \wedge D_3\xi \wedge D\xi + D\theta\xi \wedge D_2\xi \wedge D\xi\right)$$

and integrating the first term by parts we get

$$[\Psi, \Psi]_{SN} = \frac{1}{3}\int \theta D(\xi \wedge D\xi \wedge D_2\xi).$$

Therefore, instead of the Jacobi identity we have

$$\{\{F, G\}, H\} + c.p. = -\frac{1}{3}\int_\Omega D_{i+j+k+1}\left(E^i(f)DE^j(g)D_2E^k(h) + c.p.\right)dx.$$

So, the second structure of KdV equation can be Hamiltonian only under special boundary conditions.

Example 8.2

Now let consider another example which is also nonultralocal, but the operator remains to be Hamiltonian in the new brackets independently of boundary conditions. The Euler equations for the flow of ideal fluid can be written [2] in Hamiltonian form as follows (Example 7.10 of Ref. [2])

$$\frac{\partial\omega}{\partial t}=\mathcal{D}\frac{\delta H}{\delta\omega},$$

where

$$H=\int\frac{1}{2}|\mathbf{u}|^2d^2x,\qquad \omega=\nabla\times\mathbf{u}.$$

Let us limit our consideration by the 2-dimensional case when ω has only one component ω and

$$\mathcal{D}=\omega_xD_y-\omega_yD_x,$$

where $\omega_i=D_i\omega$, $i=(x,y)$. We can construct the antisymmetric operator

$$\hat{I}=\frac{1}{2}\Big(\theta\mathcal{D}-(\theta\mathcal{D})^*\Big)=\theta(\omega_xD_y-\omega_yD_x)+\frac{1}{2}(D_y\theta\omega_x-D_x\theta\omega_y),$$

and then the bivector

$$\Psi=\frac{1}{2}\int\xi\wedge\hat{I}\xi=\frac{1}{2}\int\theta(\omega_x\xi\wedge\xi_y-\omega_y\xi\wedge\xi_x),$$

where $\xi=\delta/\delta\omega$. Statement 5.3 gives us

$$[\Psi,\Psi]_{SN}=\int\Big(\theta\left(\omega_x(\xi\wedge\xi_{xy}\wedge\xi_y-\xi\wedge\xi_{yy}\wedge\xi_x)+\omega_y(\xi\wedge\xi_{xy}\wedge\xi_x-\xi\wedge\xi_{xx}\wedge\xi_y)\right)+$$

$$+\left(D_y\theta\omega_x-D_x\theta\omega_y\right)\xi\wedge\xi_x\wedge\xi_y\Big)$$

and after integration by parts the expression can be reduced to zero.

9. Conclusion

We have shown that there is an extension of the standard formal variational calculus which incorporates divergences without any specification of boundary conditions. It should be important to understand relations of this formalism to the constructions of the variational bicomplex [27]. It seems also rather interesting to study if some physically relevant algebras can be realized with the help of the new Poisson brackets as algebras of local functionals. One such example is considered in [28].

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