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CONTACT TRANSFORMATIONS AND LOCAL REDUCIBILITY OF ODE TO THE FORM $Y''' = 0$

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Abstract. Contact group orbits of the ODE $y''' = 0$ in corresponding jet bundles are investigated. Explicit relations for the right-hand side of an arbitrary 3 -order ODE necessary and sufficient for the existence of a contact transformation reducing this equation locally to the form $y''' = 0$ are obtained.

1. Introduction

In [1], Chern obtained conditions for an ODE to be locally reducible by a contact transformation to the form $y''' = 0$. His approach is based on the Cartan's exterior form method.

In this paper we use another approach to this problem. It is as follows.

An arbitrary contact transformation transforms the equation $y''' = 0$ to the form

$$y''' = u^3(x,y,y')\cdot y''^3 + u^2(x,y,y')\cdot y''^2 + u^1(x,y,y')\cdot y'' + u^0(x,y,y') \tag{1}$$

We identify every equation $\mathcal{Y}$ of form (1) with the section $S_{\mathcal{Y}}(p) = (\,u^3(p),$ $u^2(p), u^1(p), u^0(p)\,)$ of the product bundle $\pi : \mathbf{R}^4 \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$. An arbitrary contact transformation transforms an arbitrary equation (1) to an equation of that form. Therefore, the action of the pseudogroup Γ of all contact transformation on the bundle π is defined. Differential prolongations of this action define the action of Γ on the jet bundles $J^k\pi$ of π, $k = 0, 1, 2, \ldots, \infty$. As a result, every $J^k\pi$ may be represented as an union of nonintersecting orbits. Denote by $\operatorname{Im} j_k S_{\mathcal{Y}}$ the image of k-jet prolongation of $S_{\mathcal{Y}}$. We call the orbit of the subset $\operatorname{Im} j_k S_{\mathcal{Y}} \subset J^k\pi$ *the k-orbit of $\mathcal{Y}$*.

Here we prove that $\mathcal{Y}$ is locally reducible by a contact transformation to the form $y''' = 0$ iff $\operatorname{Im} j_2 S_{\mathcal{Y}}$ is a subset of the 2-orbit of the equation $y''' = 0$. We also obtain explicit equations defining this orbit as a submanifold in $J^2\pi$.

In [2], we used this approach to obtain the well known conditions (see [3], [4]) for a second-order ODE to be locally reducible by a point transformation to a linear form.

Key words and phrases: Ordinary differential equation, jet bundle, contact transformation, equivalence problem, linearizability.

In this paper, all complicated calculations are esecuted with the help of the computer algebraic system REDUCE 3.6. Corresponding programs are collected in the appendix to this paper.

Below, everything is supposed to be smooth.

2. Preliminaries

Let $\mathbf{R}^n$ denotes the n – dimensional arithmetical space, $[f]_x^k$ denotes the k–jet of a map f at the point x. By definition, put $A_{x_1 \dots x_k} = \dfrac{\partial^k A}{\partial x_1 \dots \partial x_k}$. Below, for any multi-index $\sigma = (\, i_1, i_2, \dots, i_r\,)$ we put $|\sigma| = i_1 + i_2 + \dots + i_r$ and $i_s \geq 0,\ s = 1, 2, \dots, r$.

2.1.

Let $\lambda : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a product bundle. Denote by $[S]_p^k$ the k-jet of the section S of λ at $p \in \mathbf{R}^n$, $k = 0, 1, 2, \dots, \infty$. The bundle of all these k-jets is denoted. by $\lambda_k : J^k\lambda \to \mathbf{R}^n$.

Let $x^1, \dots, x^n$ be standard coordinates on the base $\mathbf{R}^n$ of λ and let $u^1, \dots, u^m$ be standard coordinates on the fiber $\mathbf{R}^m$ of λ. Then x^i; u^j_σ denote standard coordinates of $J^k\lambda$, where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$,
$\sigma = (\, i_1, i_2, \dots, i_n\,)$ and $0 \leq |\sigma| \leq k$.

The natural projection $\lambda_{k_1,k_2} : J^{k_1}\lambda \longrightarrow J^{k_2}\lambda,\ k_2 \leq k_1 \leq \infty$ projects each k_1-jet to its k_2-jet.

Every section S of λ defines the section j_kS of λ_k by the formula $j_kS(p) = [S]_p^k$, $p \in \mathbf{R}^n$.

Let $L_S^{(k)} = \operatorname{Im} j_kS$. Any k-jet $\theta_k = [S]_p^k$ can be identified with the tangent space to the submanifold $L_S^{(k-1)}$ at the point $\theta_{k-1} = \lambda_{k,k-1}(\theta_k)$. This space is denoted by $\mathcal{K}_\theta$.

We treat any submanifold $\mathcal{E} \subset J^k\lambda$ as a k-th order PDE system with n independent and m dependent variables. Then a solution of $\mathcal{E}$ is a section S of λ with $L_S^{(k)} \subset \mathcal{E}$.

Let $\mathcal{E} \subset J^k\lambda$ be a PDE system. The first prolongation of $\mathcal{E}$ is the submanifold $\mathcal{E}^{(1)} \subset J^{k+1}\lambda$ defined by

$$\mathcal{E}^{(1)} = \{\ \theta_{k+1} \in J^{k+1}\lambda \ \Big|\ \mathcal{K}_{\theta_{k+1}} \subset T_{\theta_k}(\mathcal{E}),\ \theta_{k-1} = \lambda_{k+1,k}(\theta_{k+1})\ \}.$$

The r-th prolongation $\mathcal{E}^{(r)}$, $r \geq 2$ is defined by induction on r $\mathcal{E}^{(r)} = (\,\mathcal{E}^{(r-1)}\,)^{(1)}$. If $\mathcal{E}$ is defined by the equations $F^\alpha(x^i, u^j_\sigma) = 0$, $\alpha = 1, 2, \dots, r$, then $\mathcal{E}^{(r)}$ is defined by the equations $D_\sigma F^\alpha = 0$, $\alpha = 1, 2, \dots, r$, $0 \leq |\sigma| \leq r$, where $D_\sigma = D_{(i_1, i_2, \dots, i_n)} = D^{i_1}_{x^1} \circ \dots \circ D^{i_n}_{x^n}$,

$$D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{|\sigma|=0}^{\infty} u^i_{\sigma+1_j} \frac{\partial}{\partial u^i_\sigma}, \quad j = 1, 2, \dots, n$$

is the total derivative operator with respect to x^j, $\sigma + 1_j =$
$(\, i_1, \dots, i_{j-1}, i_j, i_{j+1}, \dots, i_n\,) + 1_j = (\, i_1, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_n\,)$.

We use the following fact from the theory of completely integrable PDE systems. Let $\theta_k \in \mathcal{E}$ and $\theta_{k-1} = \lambda_{k,k-1}(\theta_k)$. The tangent space to $\mathcal{E} \cap \lambda^{-1}_{k,k-1}(\theta_{k-1})$ is called the symbol of $\mathcal{E}$ at θ_k and is denoted by $\operatorname{Smbl}_{\theta_k}\mathcal{E}$.

Theorem 2.1 ([5]) *Let $\mathcal{E} \subset J^k\lambda$ be a PDE system. Assume that*
1) $\lambda_{k+1,k}\Big|_{\mathcal{E}^{(1)}} \colon \mathcal{E}^{(1)} \to \mathcal{E}$ is surjective,
2) $Smbl_{\theta_k}\mathcal{E} = \{0\}$ for all $\theta_k \in \mathcal{E}$.
Then for any $\theta_k \in \mathcal{E}$ there is a solution f of $\mathcal{E}$ with $[f]^k_p = \theta_k,\ p = \lambda_k(\theta_k)$.

2.2.

Let $\lambda : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ be a product bundle, let x be the coordinate on the base of λ and let y be the coordinate on the fiber of λ. We denote the standard coordinates on $J^1\lambda$ by x, y, z.

We recall (see, for example, [6]) that a contact transformation is a most general smooth invertible transformation of variables that preserves the structure of an ODE: it is a diffeomorphism of $J^1\lambda$ defined (in the coordinates x, y, z) by the formula

$$\begin{array}{rcl} X &=& X(x,y,z) \\ Y &=& Y(x,y,z) \\ Z &=& (Y_x + zY_y)/(X_x + zX_y), \end{array} \tag{2}$$

where functions $X(x,y,z)$ and $Y(x,y,z)$ are connected by the relation

$$X_z(Y_x + zY_y) - Y_z(X_x + zX_y) = 0 \tag{3}$$

Below, we denote the space $J^1\lambda$ by $\mathbf{B}$. The point $(0,0,0) \in \mathbf{B}$ is denoted by$\mathbf{0}$.

Lemma 2.2 *The Lie pseudogroup of all contact transformations of* $\mathbf{B}$ *is transitive.*

Proof. Let $p = (x_0, y_0, z_0)$ be an arbitrary point of $\mathbf{B}$. Then the contact transformation

$$\left\{ \begin{array}{rcl} X &=& x - x_0 \\ Y &=& z_0 \cdot (x - x_0) + y - y_0 \\ Z &=& z - z_0 \end{array} \right. \tag{4}$$

takes p to $\mathbf{0}$. ∎

A vector field on $\mathbf{B}$ is a *contact vector field* if its flow consists of contact transformations. It is easily shown that a contact vector field ξ can be represented (in coordinates x, y, z) by the formula

$$\xi = \xi_\varphi = -\varphi_z\frac{\partial}{\partial x} + (\varphi - z\varphi_z)\frac{\partial}{\partial y} + (\varphi_x + z\varphi_y)\frac{\partial}{\partial z}, \tag{5}$$

where the function $\varphi = \varphi(x,y,z)$ is called the *generating function* of ξ. Obviously, the generating function defines the contact vector field completely.

2.3.

Let $\mathcal{Y}_0$ be an arbitrary equation of form (1)

$$Y''' = U^3(X,Y,Z) \cdot Y''^3 + U^2(X,Y,Z) \cdot Y''2 + U^1(X,Y,Z) \cdot Y'' + U^0(X,Y,Z) \tag{6}$$

Acting on $\mathcal{Y}_0$ by an arbitrary contact transformation (2), we obtain a new equation $\mathcal{Y}$ of form (1)

$$y''' = u^3(x,y,z)\cdot y''^3 + u^2(x,y,z)\cdot y''^2 + u^1(x,y,z)\cdot y'' + u^0(x,y,z)$$

Coefficients of the latter equation are expressed in terms of coefficients of the former equation and derivatives of the functions $X(x,y,z), Y(x,y,z)$ defining the contact transformation by the formulas

$$\begin{aligned}
u^3 &= F\cdot[\ X_{zz}Z_z - Z_{zz}X_z \\
&\quad +U^3Z_z^3 + U^2Z_z^2X_z + U^1Z_zX_z^2 + U^2X_z^3\ ],\\
u^2 &= F\cdot[\ X_{zz}(Z_x+zZ_y) - Z_{zz}(X_x+zX_y),\\
&\quad +2(\,(X_{xz}+zX_{yz})Z_z - (Z_{xz}+zZ_{yz})X_z\,)\\
&\quad +X_yZ_z - Z_yX_z\\
&\quad +3U^3(Z_x+zZ_y)Z_z^2\\
&\quad +U^2(\,Z_x^2(X_x+zX_y) + 2Z_z(Z_x+zZ_y)X_z\,)\\
&\quad +U^1(\,X_x^2(Z_x+zZ_y) + 2X_z(X_x+zX_y)Z_z\,)\\
&\quad +3U^0(X_x+zX_y)X_z^2\ ],\\
u^1 &= F\cdot[\ (X_{xx}+2zX_{xy}+(z)^2X_{yy})Z_z,\\
&\quad +(Z_{xx}+2zZ_{xy}+(z)^2Z_{yy})X_z\\
&\quad +2(\,(X_{xz}+zX_{yz})(Z_x+zZ_y)\\
&\quad -(Z_{xz}+zZ_{yz})(X_x+zX_y)\,)\\
&\quad +(Z_x+zZ_y)X_y - Z_y(X_x+zX_y)\\
&\quad +3U^3(Z_x+zZ_y)^2Z_z\\
&\quad +U^2(\,X_z(Z_x+zZ_y)^2 + 2Z_z(Z_x+zZ_y)(X_x+zX_y)\,)\\
&\quad +U^1(\,X_z(Z_x+zZ_y)(X_x+zX_y) + 2Z_z(X_x+zX_y)^2\,)\\
&\quad +3U^0(X_x+zX_y)^2X_z\ ],\\
u^0 &= F\cdot[\ (X_{xx}+2zX_{xy}+(z)^2X_{yy})(Z_x+zZ_y)\\
&\quad -(Z_{xx}+2zZ_{xy}+(z)^2Z_{yy})(X_x+zX_y)\\
&\quad +U^3(Z_x+zZ_y)^3\\
&\quad +U^2(Z_x+zZ_y)^2(X_x+zX_y)\\
&\quad +U^1(Z_x+zZ_y)(X_x+zX_y)^2\\
&\quad +U^0(X_x+zX_y)^3\ ],
\end{aligned} \tag{7}$$

where $F = \dfrac{X_x+zX_y}{X_xY_y - Y_xX_y}$. These formulas were obtained with the help of REDUCE-program 6.1.

2.4.

Let us consider relations (7), where $u^3(x,y,z),\dots,u^0(x,y,z)$ are coefficients of some equation $\mathcal{Y}$ of form (1). Then relations (7) may be considered as a PDE system with respect to unknown functions $X(x,y,z), Y(x,y,z)$. Obviously, this system is compatible whenever $\mathcal{Y}_0$

is locally reducible by a contact transformation to $\mathcal{Y}$. We add equation (3) to this PDE system and denote the resulting system by $\tilde{\mathcal{E}}(\mathcal{Y}_0, \mathcal{Y})$.

Let $\tau : \mathbf{R}^2 \times \mathbf{B} \longrightarrow \mathbf{B}$ be a product bundle; then $\tilde{\mathcal{E}}(\mathcal{Y}_0, \mathcal{Y}) \subset J^3\tau$. We say that $\theta_3 \in J^3\tau$ is a *contact transformation's jet* iff there is a contact transformation (2) such that θ_3 is the 3-jet of the section $(x, y, z) \mapsto (\, X(x, y, z), Y(x, y, z)\,)$ of τ.

Let $\mathcal{E}(\mathcal{Y}_0, \mathcal{Y}) \subset \tilde{\mathcal{E}}(\mathcal{Y}_0, \mathcal{Y})$ be the subset consisting of all contact transformation's jets.

For $\mathcal{Y}_0 = \{\, y''' = 0 \,\}$ we write $\mathcal{E}(\mathcal{Y})$ instead of $\mathcal{E}(\mathcal{Y}_0, \mathcal{Y})$.

We say that contact transformation (2) is satisfies the PDE system $\mathcal{E}(\mathcal{Y}_0, \mathcal{Y})$ iff the pair of functions $(\, X(x, y, z), Y(x, y, z)\,)$ is a solution of this system.

Thus the equations $\mathcal{Y}_0$ and $\mathcal{Y}$ are locally equivalent iff there is a contact transformation satisfying $\mathcal{E}(\mathcal{Y}_0, \mathcal{Y})$.

The main goal of this paper is to obtain the compatibility conditions for the PDE system $\mathcal{E}(\mathcal{Y})$.

2.5.

Let $\pi : \mathbf{R}^4 \times \mathbf{B} \longrightarrow \mathbf{B}$ be a product bundle. We denote the standard coordinates of the fiber $\mathbf{R}^4$ of π by u^3, u^2, u^1, u^0.

Let $\mathcal{Y}$ be an arbitrary equation of form (1) and let its coefficients u^3, u^2, u^1, u^0 be defined on the domain $V \subset \mathbf{B}$. We identify $\mathcal{Y}$ with the section $S_{\mathcal{Y}}$ of π defined by the formula

$$S_{\mathcal{Y}}(p) = (\; u^3(p),\; u^2(p),\; u^1(p),\; u^0(p)\;),\;\; p \in V$$

We denote by S_0 the trivial section corresponding to the equation $y''' = 0$.

2.6.

Let f be an arbitrary contact transformation on the base $\mathbf{B}$ of π. Formulas (2), (3), (7) define the lifting of f to the (local) diffeomorphism $f^{(0)}$ of $J^0\pi$.

The k-th order differential prolongation of these relations defines the lifting of $f^{(0)}$ to the (local) diffeomorphism $f^{(k)}$ of the jet bundle $J^k\pi, k = 1, 2, \ldots \infty$ (see, for example, [6]).

Obviously, the lifting of contact transformations generates the lifting of any contact vector fields ξ_φ to the vector field $\xi_\varphi^{(k)}$ in $J^k\pi \quad k = 0, 1, 2, \ldots, \infty$.

It is well known (see, for example, [6]) that $\xi_\varphi^{(\infty)}$ is defined by the formula

$$\xi_\varphi^{(\infty)} = -\varphi_z \cdot D_x + (\varphi - z\varphi_z) \cdot D_y + (\varphi_x + z\varphi_y) \cdot D_z + \sum_{|\sigma|=0}^{\infty} D_\sigma(\psi^i(\varphi)) \frac{\partial}{\partial u^i_\sigma},$$

where $\psi^3(\varphi), \ldots, \psi^0(\varphi)$ are components of the "velocity of deformation"

$$\frac{d}{dt}(\, f_t^{(0)} \circ S \circ f_t\,)\Big|_{t=0}, \tag{8}$$

of an arbitrary section S transformed by the flow $f_t^{(0)}$ of $\xi_\varphi^{(0)}$. The explicit form of the formula (8) is as follows

$$\begin{pmatrix} \psi^3(\varphi) \\ \psi^2(\varphi) \\ \psi^1(\varphi) \\ \psi^0(\varphi) \end{pmatrix} = \begin{pmatrix} u^3_x\varphi_z - u^3_y(\varphi - z\varphi_z) - u^3_z(\varphi_x + z\varphi_y) \\ -u^3(3\varphi_{xz} + 3z\varphi_{yz} + 2\varphi_y) + u^2\varphi_{zz} \\ +\varphi_{zzz} \\ u^2_x\varphi_z - u^2_y(\varphi - z\varphi_z) - u^2_z(\varphi_x + z\varphi_y) \\ -3u^3(\varphi_{xx} + 2z\varphi_{xy} + z^2\varphi_{yy}) \\ -u^2(\varphi_{xz} + z\varphi_{yz} + \varphi_y) + 2u^1\varphi_{zz} \\ +3(\varphi_{xzz} + z\varphi_{yzz} + \varphi_{yz}) \\ u^1_x\varphi_z - u^1_y(\varphi - z\varphi_z) - u^1_z(\varphi_x + z\varphi_y) \\ -2u^2(\varphi_{xx} + 2z\varphi_{xy} + z^2\varphi_{yy}) \\ +u^1(\varphi_{xz} + z\varphi_{yz}) + 3u^0\varphi_{zz} \\ +3(\varphi_{xy} + z\varphi_{yy}) + 3(\varphi_{xxz} + 2z\varphi_{xyz} + z^2\varphi_{yyz}) \\ u^0_x\varphi_z - u^0_y(\varphi - z\varphi_z) - u^0_z(\varphi_x + z\varphi_y) \\ -u^1(\varphi_{xx} + 2z\varphi_{xy} + z^2\varphi_{yy}) \\ +u^0(3\varphi_{xz} + 3z\varphi_{yz} + \varphi_y) \\ +\varphi_{xxx} + 3z\varphi_{xxy} + 3z^2\varphi_{xyy} + z^3\varphi_{yyy} \end{pmatrix} \tag{9}$$

It can be obtained with the help of REDUCE-program 6.2.

If $\xi_\varphi(p) = 0$, then the restriction of $\xi^{(k)}_\varphi$ on the fiber $\pi_k^{-1}(p)$ is defined by the formula

$$\xi^{(k)}_\varphi \Big|_{\pi_k^{-1}(p)} = \sum_{|\sigma|=0}^{k} D_\sigma(\psi^i(\varphi))(p)\frac{\partial}{\partial u^i_\sigma}. \tag{10}$$

2.7.

Let Γ be the Lie pseudogroup of all contact transformation of $\mathbf{B}$. Then $\Gamma^{(k)} = \{\ f^{(k)} \ \big|\ f \in \Gamma\ \}$ is a Lie pseudogroup acting on $J^k\pi, \quad k = 0, 1, 2, \ldots\infty$. With respect to this action, $J^k\pi$ splits into a union of nonintersecting orbits. We recall that the subset

$$\mathrm{Orb}\,(\theta_k) = \{\ f^{(k)}(\theta_k) \in J^k\pi \ \big|\ f^{(k)} \in \Gamma^{(k)}\ \} \ \subset\ J^k\pi$$

is named the orbit of the point $\theta_k \in J^k\pi$ with respect to the action of $\Gamma^{(k)}$ on $J^k\pi$.

Let $\mathcal{Y}$ be an arbitrary equation of form (1) and let its coefficients be defined on the domain $V \subset \mathbf{B}$. The k-orbit of $\mathcal{Y}$ is the submanifold $\mathrm{Orb}_k(\mathcal{Y}) \subset J^k\pi$ defined by the formula

$$\mathrm{Orb}_k(\mathcal{Y}) = \bigcup_{p\in V} \mathrm{Orb}\,(\ [S_\mathcal{Y}]^k_p\ ).$$

We denote by Orb_k the k-orbit of the equation $y''' = 0$.

3. Orbits of the equation $\mathbf{y''' = 0}$

In this section we calculate $\mathrm{Codim}\,\mathrm{Orb}_k, \quad k = 0, 1, 2, \ldots$

3.1.

Lemma 3.1 $Orb_k = Orb(\ [S_0]^k_{\mathbf{0}}\ ),\quad k = 0, 1, 2, \ldots \infty.$

Proof. Let $p = (x_0, y_0, z_0)$ be an arbitrary point of $\mathbf{B}$. Contact transformation (7) takes p to $\mathbf{0}$. It follows from relations (28) that this transformation transforms the equation $y''' = 0$ onto itself. Therefore, $f^{(k)}$ takes $[S_0]^k_{\mathbf{0}}$ to $[S_0]^k_p$. ∎

It follows from the transitivity of Γ that

$$\dim \mathrm{Orb}\,(\theta_k) = \dim(\ \mathrm{Orb}\,(\theta_k) \cap \pi_k^{-1}(p)\ ) + \dim \mathbf{B},\quad \forall p \in \mathbf{B},\ \forall \theta_k \in J^k\pi$$

Let p be an arbitrary point of $\mathbf{B}$ and let θ_k be an arbitrary point of $\pi_k^{-1}(p)$. Let us consider the Lie group

$$G^k(p) = \{\ [f]^{k+2}_p \ \Big|\ f \in \Gamma,\quad f(p) = p\ \}.$$

This group acts on the fiber $\pi_k^{-1}(p)$ of the bundle π_k by the formula

$$(\ [f]^{k+2}_p \times \theta_k\ ) \longmapsto f^{(k)}(\theta_k). \tag{11}$$

Obviously, $\mathrm{Orb}\,(\theta_k) \cap \pi_k^{-1}(p)$ is an orbit of this action.

Let $H(\theta_k) \subset G^k(p)$ be the isotropy group of the point $\theta_k \in J^k\pi$. We denote by $H^k(p)$ the isotropy group of the point $[S_0]^k_p$.

It's well known that $\mathrm{Orb}\,(\theta_k) \cap \pi_k^{-1}(p)$ is diffeomorphic to the quotient group $G^k(p)/H(\theta_k)$.

Let $\mathcal{G}^k(p)$, $\mathcal{H}(\theta_k)$ and $\mathcal{H}^k(p)$ be the Lie algebras of the groups $G^k(p)$, $H(\theta_k)$ and $H^k(p)$ respectively. Then

$$\dim(\ \mathrm{Orb}\,(\theta_k) \cap \pi_k^{-1}(p)\ ) = \dim \mathcal{G}^k(p) - \dim \mathcal{H}(\theta_k).$$

3.2.

Then

$$\mathcal{G}^k(p) = \{\ [\xi_\varphi]^{2+k}_p \ \Big|\ \xi_\varphi \text{ is a contact vector field and } \xi_\varphi(p) = 0\ \}$$

It follows from (5) that the condition $\xi_\varphi(p) = 0$ is equivalent to the relations

$$\varphi(p) = 0, \quad \varphi_x(p) = -z\varphi_y(p), \quad \varphi_z(p) = 0. \tag{12}$$

Therefore, an element $[\xi_\varphi]^{2+k}_p \in \mathcal{G}^k(p)$ is defined by the following collection of derivatives of φ at a point p

$$\begin{array}{cccc} & \varphi_{x^2}, & & \varphi_{x^{3+k}}, \\ \varphi_y, & \varphi_{xy},\ \varphi_{xz}, & \cdots & \varphi_{x^{2+k}y},\ \varphi_{x^{2+k}z}, \\ & \varphi_{y^2},\ \varphi_{yz},\ \varphi_{z^2}, & & \cdot\quad\cdot\quad\cdot \\ & & & \varphi_{y^{3+k}},\ \varphi_{y^{2+k}z},\ \cdots\ \varphi_{yz^{2+k}},\ \varphi_{z^{3+k}} \end{array} \tag{13}$$

It is easy to prove the following statement.

Lemma 3.2 *The space of collections (13) defining $0 \in \mathcal{G}^k(p)$ is one-dimensional and is generated by the following elements*

$$
\begin{matrix} 0, \end{matrix} \qquad
\begin{matrix} 0, \\ 0,\ 0, \\ 0,\ 0,\ 0, \end{matrix} \qquad \cdots \qquad
\begin{matrix} 0, \\ 0,\ 0, \\ \cdot\ \cdot\ \cdot \\ 0,\ 0,\ 0, \end{matrix} \qquad
\begin{matrix} (-z)^{3+k}, \\ (-z)^{2+k},\ 0, \\ \cdot\ \ \cdot\ \ \cdot \\ (-z)^2,\ 0,\ \cdots\ 0, \\ (-z)^1,\ 0,\ 0,\ \cdots\ 0, \\ (-z)^0,\ 0,\ 0,\ 0,\ \cdots\ 0. \end{matrix}
$$

Proof. From (5) it follows that $[\xi_\varphi]_p^{2+k} = 0$ iff

$$D_\sigma(\varphi_z)\Big|_p = 0, \quad D_\sigma(\varphi - z\varphi_z)\Big|_p = 0, \quad D_\sigma(\varphi_x - z\varphi_y)\Big|_p = 0.$$

for every $\sigma = (i_1, i_2, i_3)$ with $0 \le |\sigma| \le 2+k$. Therefore, $\varphi_{x^{i_1}y^{i_2}z^{i_3}} = 0$ for $i_3 \ge 1$ and $0 \le i_1 + i_2 + i_3 \le 3+k$, $\varphi_{x^{i_1}y^{i_2}z^{i_3}} = 0$ for $0 \le i_1+i_2+i_3 \le 2+k$, and $\varphi_{x^{i_1}y^{i_2}} = (-z)^{i_1}\varphi_{y^{i_2}}$ for $i_1 + i_2 = 3+k$. ■

Now it follows from (13) and lemma 3.2 that

$$\dim \mathcal{G}^k(p) = 6 + \sum_{i=0}^{k} \frac{(i+4)(i+5)}{2}, \quad k = 0, 1, 2, \ldots. \tag{14}$$

3.3.

The action (11) of the group $G^k(p)$ on $\pi_k^{-1}(p)$ generates the representation of its Lie algebra $\mathcal{G}^k(p)$ by vector fields on $\pi_k^{-1}(p)$. It follows from (10) that this representation is defined in coordinates by the formula

$$[\xi_\varphi]_p^k \longmapsto \sum_{|\sigma|=0}^{k} (\ D_\sigma\psi^i(\varphi)\ )(p) \cdot \frac{\partial}{\partial u^i_\sigma},$$

where $\psi^3(\varphi), \ldots, \psi^0(\varphi)$ are components of the matrix (9).

Therefore, a collection (13) defines an element of $\mathcal{H}(\theta_k)$ iff it is a solution of the following system of homogeneous linear algebraic equations

$$D_\sigma(\psi^i(\varphi))\,|_{\theta_k} = 0, \quad i = 3, 2, 1, 0; \quad |\sigma| = 0, 1, 2, \ldots, k \tag{15}$$

At the point $[S_0]^k_{\mathbf{0}}$, this system has the form

$$D_\sigma(\ \varphi_{zzz}\ )\Big|_{\mathbf{0}} = 0, \tag{16}$$

$$D_\sigma(\ \varphi_{xzz} + z\varphi_{yzz} + \varphi_{yz}\ )\Big|_{\mathbf{0}} = 0, \tag{17}$$

$$D_\sigma(\ \varphi_{xxz} + 2z\varphi_{xyz} + z^2\varphi_{yyz} + \varphi_{xy} + z\varphi_{yy}\ )\Big|_{\mathbf{0}} = 0, \tag{18}$$

$$D_\sigma(\ \varphi_{xxx} + 3z\varphi_{xxy} + 3z^2\varphi_{xyy} + z^3\varphi_{yyy}\ )\Big|_{\mathbf{0}} = 0, \tag{19}$$

$$0 \le |\sigma| \le k$$

Now, it is easy to calculate

$$\begin{array}{rcl} \dim \mathcal{H}^0_p & = & 12 \\ \dim \mathcal{H}^1_p & = & 15 \end{array} \quad \forall p \in \mathbf{B} \tag{20}$$

Lemma 3.3 *For $2 \leq k \leq \infty$, a solution of system (16) – (19) can be represented in the form*

$$\Big(\varphi_y, \quad \begin{array}{c} \varphi_{x^2}, \\ \varphi_{xy}, \varphi_{xz}, \\ \varphi_{y^2}, \varphi_{yz}, \varphi_{z^2}, \end{array} \quad \begin{array}{c} *, \\ *, *, \\ *, *, *, \\ *, *, *, *, \end{array} \quad \ldots \quad \begin{array}{c} *, \\ *, *, \\ \cdot \quad \cdot \quad \cdot \\ *, *, \cdots, *, *, \end{array} \quad \begin{array}{c} *, \\ \cdot\cdot\cdot \\ *, \cdots, *, \\ \varphi_{y^{1+k}}, *, \cdots, *, \end{array}$$

$$\begin{array}{c} *, \\ \cdot\cdot\cdot \\ *, \cdots, *, \\ \varphi_{xy^{1+k}}, *, \cdots, *, \\ \varphi_{y^{2+k}}, \varphi_{y^{1+k}z}, *, \cdots, *, \end{array} \quad \begin{array}{c} *, \\ \cdot\cdot\cdot \\ *, \cdots, *, \\ \varphi_{x^2y^{1+k}}, *, \cdots, *, \\ \varphi_{xy^{2+k}}, \varphi_{xy^{1+k}z}, *, \cdots, *, \\ \varphi_{y^{3+k}}, \varphi_{y^{2+k}z}, \varphi_{y^{1+k}z^2}, *, \cdots, * \end{array}\Big),$$

where unknowns of the form $\varphi_{x^{i_1}y^{i_2}z^{i_3}}$ are free, while unknowns denoted by $*$ are dependent ones (they are linear functions of free unknowns).

Proof. By S_k denote system (16) - (19).

For $k = 2$ the lemma's statement is proved by the direct calculation. (It can be done with the help of REDUCE-program (6.4).)

Let r be a natural number and $r \geq 2$. Suppose that the lemma's statement is proved for all $2 \leq k \leq r$. Let us prove it for $k = r + 1$.

We assume that an unknown is dependent in S_{r+1} if it is dependent in S_r in correspondence to the lemma's statement.

Step 1. Consider the unknowns φ_σ of the highest order (that is, unknowns of order $r+4$). It follows from the system S_{r+1} that the unknowns

$$\begin{array}{c} \varphi_{x^2y^{r+2}}, \\ \varphi_{xy^{r+3}}, \ \varphi_{xy^{r+2}z}, \\ \varphi_{y^{r+4}}, \ \varphi_{y^{r+3}z}, \ \varphi_{y^{r+2}z^2} \end{array}$$

are free and all other unknowns of order $r + 4$ are expressed in terms of unknowns of lower orders.

Step 2. Consider the following unknowns of order $r + 3$

$$\begin{array}{c} \varphi_{x^2y^{r+1}}, \\ \varphi_{xy^{r+2}}, \ \varphi_{xy^{r+1}z}, \\ \varphi_{y^{r+3}}, \ \varphi_{y^{r+2}z}, \ \varphi_{y^{r+1}z^2}. \end{array}$$

They are free in S_r. We can choose the unknowns $\varphi_{x^2y^{r+1}}, \varphi_{xy^{r+1}z}, \varphi_{y^{r+1}z^2}$ to be dependent in S_{r+1}. Indeed, the equations (18) with $\sigma = (1, r, 0)$ and (19) with $\sigma = (0, r, 1)$ of S_{r+1} are as follows:

$$\varphi_{x^3y^rz} + \varphi_{x^2y^{r+1}} = 0,$$
$$\varphi_{x^3y^rz} + 3\varphi_{x^2y^{r+1}} = 0.$$

It follows from these equations that $\varphi_{x^2y^{r+1}} = 0$.

Equations (17) with $\sigma = (1, r, 0)$ and (18) with $\sigma = (0, r, 1)$ of S_{r+1} are as follows

$$\varphi_{x^2y^rz} + \varphi_{xy^{r+1}z} = 0,$$
$$\varphi_{x^2y^rz^2} + 2\varphi_{xy^{r+1}z} + \varphi_{xy^{r+1}z} + \varphi_{y^{r+2}} = 0.$$

It follows from these equations that $\varphi_{xy^{r+1}z} = -\dfrac{1}{2}\varphi_{y^{r+2}}$.

Equation (17) with $\sigma = (0, r, 1)$ of S_{r+1} is

$$\varphi_{y^{r+1}z^2} + \varphi_{y^{r+1}z^2} = 0,$$

Therefore, $\varphi_{y^{r+1}z^2} = 0$.

We can choose the unknowns

$$\varphi_{xy^{r+2}},$$
$$\varphi_{y^{r+3}},\ \varphi_{y^{r+2}z},$$

to be free. Indeed, $\varphi_{y^{r+3}}$ does not enter the equations of S_{r+1}. The unknowns $\varphi_{xy^{r+2}}$ enters only equation (18) with $\sigma = (0, r+1, 0)$

$$\varphi_{x^2y^{r+1}z} + \varphi_{xy^{r+2}} = 0$$

and $\varphi_{x^2y^{r+1}z}$ enters this equation only.

The unknown $\varphi_{y^{r+2}z}$ enters only the equation (17) with $\sigma = (0, r+1, 0)$

$$\varphi_{xy^{r+1}z^2} + \varphi_{y^{r+2}z} = 0$$

and $\varphi_{xy^{r+1}z^2}$ enters this equation only.

Step 3. Consider the following unknowns of order $r+2$

$$\varphi_{xy^{r+1}},$$
$$\varphi_{y^{r+2}},\ \varphi_{y^{r+1}z}.$$

They are free in S_r. The unknowns $\varphi_{xy^{r+1}}, \varphi_{y^{r+1}z}$ are dependent in S_{r+1}. Indeed, the equations (18) with $\sigma = (0, r, 0)$, (19) with $\sigma = (0, r-1, 2)$ and (18) with $\sigma = (1, r-1, 1)$ from S_{r+1} are as follows

$$\varphi_{x^2y^rz} + \varphi_{xy^{r+1}} = 0,$$
$$\varphi_{x^3y^{r-1}z^2} + 6\varphi_{x^2y^rz} + 6\varphi_{xy^{r+1}} = 0,$$
$$\varphi_{x^3y^{r-1}z^2} + 2\varphi_{x^2y^rz} + 4\varphi_{x^2y^rz} + \varphi_{xy^{r+1}} = 0.$$

It follows from these equations that $\varphi_{xy^{r+1}} = 0$.

Equations (17) with $\sigma = (0, r, 0)$ and (18) with $\sigma = (0, r-1, 2)$ of S_{r+1} are as follows

$$\varphi_{xy^r z^2} + \varphi_{y^{r+1}z} = 0,$$
$$4\varphi_{xy^r z^2} + 2\varphi_{y^{r+1}z} + \varphi_{xy^r z^2} + 2\varphi y^{r+1} z = 0.$$

It follows from these equations that $\varphi_{y^{r+1}z} = 0$.

The unknown $\varphi_{y^{r+2}}$ enters only (18) with $\sigma = (0, r, 1)$

$$\varphi_{x^2 y^r z^2} + 3\varphi_{xy^{r+1}z} + \varphi_{y^{r+2}} = 0$$

and $\varphi_{x^2 y^r z}$ yet enters the equation (17) with $\sigma = (1, r, 0)$ only. Therefore, we can choose it to be free in S_{r+1}.

Step 4. The unknown $\varphi_{y^{r+1}}$ is dependent one in S_{r+1}. Indeed, the equations (18) with $\sigma = (0, r-1, 1)$, (19) with $\sigma = (0, r-2, 3)$ and (17) with $\sigma = (1, r-1, 0)$ of S_{r+1} are as follows

$$\varphi_{x^2 y^{r-1} z^2} + 3\varphi_{xy^r z} + \varphi_{y^{r+1}} = 0,$$
$$3\varphi_{x^2 y^{r-1} z^2} + 3\varphi_{xy^r z} + 2\varphi_{y^{r+1}} = 0,$$
$$\varphi_{x^2 y^{r-1} z^2} + \varphi_{xy^r z} = 0.$$

It follows from these equations that $\varphi_{y^{r+1}} = 0$.

Step 5. At last we can choose the unknowns

$$\begin{array}{ccc} & \varphi_{x^2}, & \\ \varphi_y, & \varphi_{xy},\ \varphi_{xz}, & \\ & \varphi_{y^2},\ \varphi_{yz},\ \varphi_{z^2} & \end{array}$$

to be free. Indeed, it is obvious for unknowns φ_y, φ_{x^2}, φ_{xz}, φ_{z^2} because they does not enter the equations of S_{r+1}.

The unknown φ_{xy} enters only the equation (18) with $\sigma = (0, 0, 0)$

$$\varphi_{x^2 z} + \varphi_{xy} = 0$$

and $\varphi_{x^2 z}$ enters only this equation too.

The unknown φ_{yz} enters only the equation (17) with $\sigma = (0, 0, 0)$

$$\varphi_{xz^2} + \varphi_{yz} = 0$$

and φ_{xz^2} enters only this equation too.

The unknown φ_{y^2} enters only the equation (18) with $\sigma = (0, 0, 1)$

$$\varphi_{x^2 z^2} + 3\varphi_{xyz} + \varphi_{y^2} = 0$$

and the unknowns $\varphi_{x^2 z^2}$, φ_{xyz} yet enters the equation (17) with $\sigma = (1, 0, 0)$

$$\varphi_{x^2 z^2} + \varphi_{xyz} = 0$$

of S_{r+1} only. ∎

Corollary 3.4 *For any $p \in \mathbf{B}$*

$$\begin{aligned} &\dim \mathcal{H}^k(p) = 16, \quad 2 \le k < \infty \\ &\dim \mathcal{H}^\infty(p) = 7. \end{aligned} \tag{21}$$

Let

$$\rho_{l,m} : \mathcal{G}^l(p) \longrightarrow \mathcal{G}^m(p), \quad \infty \ge l \ge m$$

be the natural projection taking a l-jet to its m-jet. Let us consider the algebra $\rho_{\infty,0}(\,\mathcal{H}^\infty(p)\,)$ and its subalgebra $\mathcal{H}^{\infty,2}(p)$ defined by the formula

$$\mathcal{H}^{\infty,2}(p) = \{\ [\xi_\varphi]_p^2 \in \rho_{\infty,2}(\,\mathcal{H}^\infty(p)\,)\ \Big|\ [\xi_\varphi]_p^1 = 0\ \}. \tag{22}$$

Corollary 3.5 $\qquad \mathcal{H}^{\infty,2}(p) = \{0\}.$

Proof. For $\mathbf{0} \in \mathbf{B}$, the statement immediately follows from lemma 3.3. Let $p \in \mathbf{B}$ and let f be a contact transformation takng p to $\mathbf{0}$. It is easy to check that $\mathcal{H}^{\infty,2}(p) = f_*^{(\infty)}\mathcal{H}^{\infty,2}(\mathbf{0}) = \{0\}$. ■

3.4.

Now we can calculate the codimension of Orb_k, $k = 0,1,2,\ldots$ We have $\mathrm{Codim\,Orb}_k = \dim \pi_k^{-1}(p) - (\,\dim \mathcal{G}^k(p) - \dim \mathcal{H}^k(p)\,)$. It's clear that

$$\dim \pi_k^{-1}(p) = 4\sum_{i=0}^{k} \frac{(i+1)(i+2)}{2}, \quad k = 0,1,2,\ldots. \tag{23}$$

Taking into account formulas (23),(14), (20), and (21), we obtain

$$\begin{aligned} &\mathrm{Codim\,Orb}_k = 0, \quad k = 0,1, \\ &\mathrm{Codim\,Orb}_k = \sum_{i=0}^{k} \frac{4(i+1)(i+2) - (i+4)(i+5)}{2} + 10, \quad k \ge 2. \end{aligned} \tag{24}$$

It follows from (24) that the submanifold $\mathrm{Orb}_2 \subset J^2\pi$ is defined by four equations.

4. Explicit relations

In this section, we obtain equations defining the submanifold $\mathrm{Orb}_2 \subset J^2\pi$.

4.1.

Here we find the equations defining the submanifold $\mathrm{Orb}_2 \cap \pi_2^{-1}(\mathbf{0}) \subset \pi_2^{-1}(\mathbf{0})$.

Let us consider an isotropy algebra $\mathcal{H}(\theta_2)$ of an arbitrary point $\theta_2 \in \pi_2^{-1}(\mathbf{0})$. Each element $[\xi_\varphi]_{\mathbf{0}}^2 \in \mathcal{H}(\theta_2)$ is defined by the collection of the derivatives (13, with $k = 2$) which is a solution of system of homogeneous linear algebraic equations (15, with $k = 2$)

$$D_\sigma(\psi^i(\varphi))\Big|_{\theta_2} = 0, \quad i = 3,2,1,0, \quad |\sigma| = 0,1,2. \tag{25}$$

Obviously, the number of equation of this system is equal to 40; it follows from (13) that the number of unknowns in this system is equal to 53. Let us consider the matrix of this system. Its elements are polynomials of u^i_σ, $i = 3, 2, 1, 0$, $|\sigma| = 0, 1, 2$. System (25) can be reduced by elementary transformations to the equivalent system with coefficient matrix $\mathrm{Mat}\,(\theta_2)$ of the form

$$\mathrm{Mat}\,(\theta_2) = \begin{pmatrix} A \\ B \end{pmatrix} \tag{26}$$

where A is a 36×53-matrix with $\mathrm{Rank}\, A = 36$ and the matrix B is the 4×53-matrix

$$\begin{pmatrix} 0 & \cdots & 0 & F^3 & 0 & F^3 & F^2 \\ 0 & \cdots & 0 & 2F^2 & F^3 & F^2 & 2F^1 \\ 0 & \cdots & 0 & 0 & F^1 & F^0 & 0 \\ 0 & \cdots & 0 & F^1 & 2F^2 & F^1 & F^0 \end{pmatrix}$$

where

$$\begin{aligned}
\tilde F^3 &= 27u^3_{xz} - 9u^2_{zz} + 18u^3_x u^2 + 81u^3_y + 18u^3_z u^1 \\
&\quad -18u^2_x u^3 - 18u^2_z u^2 + 36u^1_z u^3 - 54(u^3)^2 u^0 \\
&\quad +18u^3 u^2 u^1 - 4(u^2)^3, \\
\tilde F^2 &= -9u^3_{xx} - 3u^2_{xz} + 6u^1_{zz} - 18u^3_x u^1 \\
&\quad -18u^3_z u^0 + 6u^2_x u^2 - 24u^2_y + 6u^2_z u^1 \\
&\quad -36u^0_z u^3 + 18u^3 u^2 u^0 - 12u^3 (u^1)^2 + 2(u^2)^2 u^1, \\
\tilde F^1 &= 6u^2_{xx} - 3u^1_{xz} - 9u^0_{zz} + 36u^3_x u^0 - 6u^1_x u^2 \\
&\quad +21u^1_y - 6u^1_z u^1 + 18u^0_x u^3 + 18u^0_z u^2 \\
&\quad +18u^3 u^1 u^0 - 12(u^2)^2 u^0 + 2u^2 (u^1)^2, \\
\tilde F^0 &= -9u^1_{xx} + 27u^0_{xz} - 36u^2_x u^0 + 18u^1_x u^1 \\
&\quad +18u^1_z u^0 - 18u^0_x u^2 - 54u^0_y - 18u^0_z u^1 \\
&\quad -54u^3 (u^0)^2 + 18u^2 u^1 u^0 - 4(u^1)^3.
\end{aligned} \tag{27}$$

The matrix (26) was obtained with the help of REDUCE-program 6.3.

Thus we have $\mathrm{Codim\,Orb}\,(\theta_2) = \dim \pi_k^{-1}(\mathbf{0}) - (\ \dim \mathcal{G}^k(p) - \dim \mathcal{H}^k(p)\ )$. It folows from lemma 3.2 that $\dim \mathcal{H}^k(p) =$(the number of unknowns of the system (25)) $-\mathrm{Rank\,Mat}\,(\theta_2) - 1 = 52 - \mathrm{Rank\,Mat}\,(\theta_2)$. Now, taking into account the formulas (23) and (14), we obtain $\mathrm{Codim\,Orb}\,(\theta_2) = 40 - \mathrm{Rank\,Mat}\,(\theta_2)$

Hence we have that $\mathrm{Codim\,Orb}\,(\theta_2) = 4$ iff $\tilde F^i(\theta_2) = 0,\ i = 3, 2, 1, 0$.

Theorem 4.1 *$Orb_2 \cap \pi_2^{-1}(\mathbf{0}) = \{\ \theta_2 \in \pi_2^{-1}(\mathbf{0}) \ \big|\ \tilde F^i(\theta_2) = 0,\ i = 3, 2, 1, 0\ \}$.*

Proof. Let $M \subset \pi_2^{-1}(\mathbf{0})$ be the submanifold defined by the equations $\tilde F^i(\theta_2) = 0$, $i = 3, 2, 1, 0$. It can be easily checked that M is a path-connected manifold. Obviously, for every $\theta_2 \in M$ the submanifold $\mathrm{Orb}\,(\theta_2) \cap \pi_2^{-1}(\mathbf{0})$ is an open subset of M. Therefore $\mathrm{Orb}\,_2 \cap \pi_2^{-1}(\mathbf{0}) = M$ ∎

4.2.

Let $p = (x_0, y_0, z_0)$ be an arbitrary point of $\mathbf{B}$. Let us consider the contact transformation $\hat{p}$ defined by the equations

$$\left\{ \begin{array}{lcl} X &=& x + x_0 \\ Y &=& z_0 \cdot x + y + y_0 \\ Z &=& z + z_0 \end{array} \right. \tag{28}$$

This transformation takes $\mathbf{0}$ to p. The lifting $\hat{p}^{(2)}$ takes every point $\theta_2 \in \pi_2^{-1}(p)$ to the point $\hat{p}^{(2)}(\theta_2) \in \pi_2^{-1}(\mathbf{0})$. Therefore submanifold $\mathrm{Orb}_2 \cap \pi^{-1}(p)$ is defined by the equations $\tilde{F}^i \circ \hat{p}^{(2)} = 0, \quad i = 3,2,1,0$. Let functions F^i, $i = 3,2,1,0$, be defined by the formula

$$F^i(\theta_2) = \tilde{F}^i \circ \widehat{\pi_2(\theta_2)}^{(2)}(\theta_2), \quad i = 3,2,1,0, \quad \theta_2 \in J^2\pi.$$

Then we obtain the following theorem.

Theorem 4.2 *$Orb_2 = \{ \theta_2 \in J^2\pi \mid F^i(\theta_2) = 0, \ i = 3,2,1,0 \}$.*

Now we deduce the explicit form of the functions F^i. Suppose the points $\theta_2 \in \pi_2^{-1}(p)$ and $\hat{p}^{(2)}(\theta_2) \in \pi_2^{-1}(\mathbf{0})$ in standard coordinates on $J^2\pi$ have the following form.

$$\begin{array}{rcl} \theta_2 &=& (X, Y, Z; U^3, \ldots, U^0; \ldots; U^3_{XX}, \ldots, U^0_{ZZ}) \\ \hat{p}^{(2)}(\theta_2) &=& (0,0,0; u^3, \ldots, u^0; \ldots; u^3_{xx}, \ldots, u^0_{zz}). \end{array}$$

Then it follows from (7) and (28) that

$$\begin{array}{rcl} u^i &=& U^i, \\ u^i_x &=& U^i_X + ZU^i_Y, \\ u^i_y &=& U^i_Y, \\ u^i_z &=& U^i_Z, \\ u^i_{xx} &=& U^i_{XX} + 2ZU^i_{XY} + Z^2U^i_{YY}, \\ u^i_{xy} &=& U^i_{XY} + ZU^i_{YY}, \\ u^i_{xz} &=& U^i_{XZ} + ZU^i_{YZ}, \\ u^i_{yy} &=& U^i_{YY}, \\ u^i_{yz} &=& U^i_{YZ}, \\ u^i_{zz} &=& U^i_{ZZ}, \qquad\qquad i = 3,2,1,0. \end{array}$$

Substituting these expressions into (27), we get

$$\begin{aligned}
F^3 &= 27u^3_{xz} + 27zu^3_{yz} - 9u^2_{zz} + 18u^3_x u^2 + 18zu^3_y u^2 \\
&\quad +81u^3_y + 18u^3_z u^1 - 18u^2_x u^3 - 18zu^2_y u^3 - 18u^2_z u^2 \\
&\quad +36u^1_z u^3 - 54(u^3)^2 u^0 + 18u^3 u^2 u^1 - 4(u^2)^3, \\
F^2 &= -9u^3_{xx} - 18zu^3_{xy} - 3u^2_{xz} - 3zu^2_{yz} + 6u^1_{zz} \\
&\quad -18u^3_x u^1 - 18zu^3_y u^1 - 18u^3_z u^0 + 6u^2_x u^2 + 6zu^2_y u^2 \\
&\quad -24u^2_y + 6u^2_z u^1 - 36u^0_z u^3 + 18u^3 u^2 u^0 - 12u^3(u^1)^2 \\
&\quad +2(u^2)^2 u^1 - 9z^2 u^3_{yy}, \\
F^1 &= 6u^2_{xx} + 12zu^2_{xy} - 3u^1_{xz} - 3zu^1_{yz} - 9u^0_{zz} \\
&\quad +36u^3_x u^0 + 36zu^3_y u^0 - 6u^1_x u^2 - 6zu^1_y u^2 + 21u^1_y \\
&\quad -6u^1_z u^1 + 18u^0_x u^3 + 18zu^0_y u^3 + 18u^0_z u^2 + 18u^3 u^1 u^0 \\
&\quad -12(u^2)^2 u^0 + 2u^2(u^1)^2 + 6z^2 u^2_{yy}, \\
F^0 &= -9u^1_{xx} - 18zu^1_{xy} + 27u^0_{xz} + 27zu^0_{yz} - 36u^2_x u^0 \\
&\quad -36zu^2_y u^0 + 18u^1_x u^1 + 18zu^1_y u^1 + 18u^1_z u^0 - 18u^0_x u^2 \\
&\quad -18zu^0_y u^2 - 54u^0_y - 18u^0_z u^1 - 54u^3(u^0)^2 + 18u^2 u^1 u^0 \\
&\quad -4(u^1)^3 - 9z^2 u^1_{yy}.
\end{aligned} \tag{29}$$

4.3.

Let $\mathrm{Orb}\,_2^{(k)}$ be the k-th order differential prolongation of $\mathrm{Orb}\,_2^{(k)}$ (see subsection 2.1).

$$\mathrm{Orb}\,_2^{(k)} = \{\ \theta_{2+k} \in J^{2+k}\pi \ \Big|\ D_\sigma F^i(\theta_{2+k}) = 0, \quad |\sigma| \le k, \quad i = 3,2,1,0\ \}.$$

Theorem 4.3 *$Orb\,_2^{(k)} = Orb\,_{2+k}, \quad k = 0,1,2,\dots,\infty.$*

Proof. Notice that

$$\mathrm{Orb}\,_{2+k} \subset \mathrm{Orb}\,_2^{(k)}, \quad k = 0,1,2,\dots,\infty.$$

Indeed, if $\theta_{2+k} \in \mathrm{Orb}\,_{2+k}$, then there is a contact transformation f such that $\theta_{2+k} = f^{(2+k)}([S_0]^{2+k}_{\mathbf{0}})$. This transformation transforms $\mathcal{Y}_0$ to the equation $\mathcal{Y}$ with $\theta_{2+k} = [S_\mathcal{Y}]^{2+k}_p$. Therefore, $\mathrm{Im}\,(j_2 S_\mathcal{Y}) \subset \mathrm{Orb}\,_2$. It follows that $\mathrm{Im}\,(j_{2+k} S_\mathcal{Y}) \subset \mathrm{Orb}\,_2^{(k)}, \quad k = 0,1,2,\dots,\infty$. Hence $\theta_{2+k} \in \mathrm{Orb}\,_2^{(k)}$.

Suppose that the equality $\mathrm{Orb}\,_{2+k} = \mathrm{Orb}\,_2^{(k)}$ is proved for $k = 1,2,\dots,r$. Let us prove it for $k = r+1$.

Lemma 4.4 *If $Orb\,_{2+r} = Orb\,_2^{(r)}$ and $\pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap Orb\,_{3+r} = \pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap Orb\,_2^{r+1}$ then $Orb\,_{3+r} = Orb\,_2^{(r+1)}$.*

Proof. Let $\theta_{3+r} \in \mathrm{Orb}\,_2^{(r+1)}, \quad \theta_{2+r} = \pi_{3+r,2+r}(\theta_{3+r})$, and let f be a contact transformation such that $f^{(2+r)}(\theta_{2+r}) = [S_0]^{2+r}_{\mathbf{0}}$. We have $\mathcal{K}_{\theta_{3+r}} \subset T_{\theta_{2+r}}(\mathrm{Orb}\,_2^{(r)}) = T_{\theta_{2+r}}(\mathrm{Orb}\,_{2+r})$. Therefore, $\mathcal{K}_{f^{3+r}(\theta_{3+r})} = f^{(2+r)}_*(\mathcal{K}_{\theta_{3+r}}) \subset f^{(2+r)}_*(T_{\theta_{2+r}}(\mathrm{Orb}\,_{2+r})) = T_{[S_0]^{2+r}_{\mathbf{0}}}(\mathrm{Orb}\,_2(r)))$. It follows that

$f^{3+r}(\theta_{3+r}) \in \mathrm{Orb}\,_2^{(r+1)}$. Obviously $f^{3+r}(\theta_{3+r}) \in \pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap \mathrm{Orb}\,_2^{(r+1)}$ $= \pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap \mathrm{Orb}\,_{3+r}$. Then it follows $\theta_{3+r} \in \mathrm{Orb}\,_{3+r}$. This concludes the proof of the lemma.

Obviously, $\pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap \mathrm{Orb}\,_{3+r} \subset \pi^{-1}_{3+r,2+r}([S_0]^{2+r}_{\mathbf{0}}) \cap \mathrm{Orb}\,_2^{r+1}$. Therefore, to complete the proof of the theorem it is sufficient to show that the dimensions of these manifolds are the same. That is,
$\dim \mathrm{Smbl}\,_{[S_0]^{3+r}_{\mathbf{0}}}(\mathrm{Orb}\,_{3+r}) = \dim \mathrm{Smbl}\,_{[S_0]^{3+r}_{\mathbf{0}}}(\mathrm{Orb}\,_2^{(1+r)})$.
The following lemma implies this equality.

Lemma 4.5

$$\mathit{Codim}\,\mathit{Smbl}\,_{[S_0]^{2+k}_{\mathbf{0}}}(\mathit{Orb}\,_{2+k}) = \mathit{Codim}\,\mathit{Smbl}\,_{[S_0]^{2+k}_{\mathbf{0}}}(\mathit{Orb}\,_2^{(k)}),\ k = 0, 1, 2, \ldots$$

Proof. Notice that $\mathrm{Codim}\,\mathrm{Smbl}\,_{[S_0]^2_{\mathbf{0}}}(\mathrm{Orb}\,_2) = 4$. For $k \geq 1$, it follows from (24) that

$$\begin{aligned} \mathrm{Codim}\,\mathrm{Smbl}\,_{[S_0]^{2+k}_{\mathbf{0}}}(\mathrm{Orb}\,_{2+k}) &= 4\frac{(k+3)(k+4)}{2} - \frac{(k+6)(k+7)}{2} \\ &= \frac{3}{2}(k^2 + 5k + 2). \end{aligned}$$

From theorem 4.2 we obtain that $\mathrm{Smbl}\,_{[S_0]^2_{\mathbf{0}}}(\mathrm{Orb}\,_2)$ is defined by the system of linear algbraic equations

$$\left\{ \begin{aligned} 3u^3_{x^2} + u^2_{xz} - 2u^1_{z^2} &= 0 \\ 2u^2_{x^2} - u^1_{xz} - 3u^0_{z^2} &= 0 \\ -u^1_{x^2} + 3u^0_{xz} &= 0 \\ 3u^3_{xz} - u^2_{z^2} &= 0. \end{aligned} \right. \qquad S_2$$

Hence $\mathrm{Smbl}\,_{[S_0]^{2+k}_{\mathbf{0}}}(\mathrm{Orb}\,_2^{(k)})$ is defined by the k-th order prolongation of this system, $k = 1, 2, \ldots$. We denote this prolongation by $S_2^{(k)}$. It is obvious that equations of $S_2^{(k)}$ generated by the first three equations of S_2 are linear independent.

Let us consider the equation in $S_2^{(k)}$ generated by the last equation of S_2.

$$D^a_x D^b_y D^c_z(\ 3u^3_{xz} - u^2_{z^2}\ ) = 0, \quad a + b + c = k. \tag{30}$$

Let it be a linear combination of equations of $S_2^{(k)}$ generated by the first three equations of S_2. Then we have

$$D^a_x D^b_y D^c_z(\ 3u^3_{xz} - u^2_{z^2}\ ) = D^{a-1}_x D^b_y D^{c+1}_z(\ 3u^3_{x^2} + u^2_{xz} - 2u^1_{z^2}\ ) + \cdots,$$

where the dots stand for a linear combination of equations generated by the second and the third equation of S_2 and $a \geq 1$. Transfering $D^{a-1}_x D^b_y D^{c+1}_z(\ 3u^3_{x^2} + u^2_{xz} - 2u^1_{z^2}\ )$ into the left-hand side of this relation, we obtain the following relation

$$-2u^2_{x^a y^b z^{c+2}} + 2u^1_{x^{a-1} y^b z^{c+3}} = -D^{a-2}_x D^b_y D^{c+2}_z(\ 2u^2_{x^2} - u^1_{xz} - 3u^0_{z^2}\ ) + \cdots,$$

where the dots stand for a linear combination of equations generated by the third equation of S_2 and $a \geq 2$. Transfering $-D_x^{a-2} D_y^b D_z^{c+2}(\ 2u^2_{x^2} - u^1_{xz} - 3u^0_{z^2}\ )$ into the left-hand side of this relation, we obtain the following

$$u^1_{x^{a-1}y^bz^{c+3}} + 3u^0_{x^{a-2}y^bz^{c+4}} = -D_x^{a-3} D_y^b D_z^{c+3}(\ -u^1_{x^2} + u^0_{xz}\ ) + \cdots,$$

where $a \geq 3$.

Thus, equations of form (30) are a linear combinations of equations from $S_2^{(k)}$ generated by the first three equations of S_2 if and only if $a \geq 3$.

Obviously, the number of these equations in $S_2^{(k)}$ is $\dfrac{(k-2)(k-1)}{2}$. Therefore, the rank of $S_2^{(k)}$ is equal to

$$4\frac{(k+1)(k+2)}{2} - \frac{(k-2)(k-1)}{2} = \frac{3}{2}(k^2 + 5k + 2).$$

This completes the lemma's proof. ■

5. Integrability conditions

Theorem 5.1 *Let $\mathcal{Y}$ be an ODE of form t (1). Let p be a point in the domain U of $S_\mathcal{Y}$. Then $\mathcal{Y}$ is reducible by a contact transformation to the form $y''' = 0$ in a neighborhood of p iff there exists a neighborhood $V \subset U$ of p such that $Im(j_2 S_\mathcal{Y}\big|_V) \subset Orb_2$.*

Proof. The necessity statement is obvious. Let us prove the sufficiency. We can assume that $\mathrm{Im}\,(j_2 S_\mathcal{Y}) \subset \mathrm{Orb}_2$. Let us consider PDE system $\mathcal{E} = \tau_{\infty,3}(\ \mathcal{E}(\mathcal{Y})^{(\infty)}\ )$. Obviously, $\mathcal{E} \subset \mathcal{E}(\mathcal{Y})$ and solution sets for these systems are the same. It follows from the theorems 4.2, 4.3 that $\mathrm{Im}\, j_\infty S_\mathcal{Y} \subset \mathrm{Orb}_\infty$. Therefore, $\mathcal{E}(\mathcal{Y})^{(\infty)} \neq \emptyset$. It follows that $\mathcal{E} \neq \emptyset$. Now we prove that $\mathcal{E}$ has solutions. To this end we prove that $\mathcal{E}$ is a completely integrable PDE system (see theorem 2.1).

First, let us prove that $\tau_{4,3} : \mathcal{E}^{(1)} \to \mathcal{E}$ is a surjection. Let $\theta_3 \in \mathcal{E}$; then there is $\theta_\infty \in \mathcal{E}(\mathcal{Y})^{(\infty)}$ such that $\tau_{\infty,3}(\theta_\infty) = \theta_3$. Let $\tau_{\infty,4}(\theta_\infty) = \theta_4$. Taking into account $\mathcal{K}_{\theta_\infty} \subset T_{\theta_\infty}(\mathcal{E}(\mathcal{Y})^{(\infty)})$ and $\mathcal{E} = \tau_{\infty,3}(\ \mathcal{E}(\mathcal{Y})^{(\infty)}\ )$, we obtain $\mathcal{K}_{\theta_4} = (\tau_{\infty,3})_*(\mathcal{K}_{\theta_\infty}) \subset T_{\theta_3}(\mathcal{E})$. It follows that $\theta_4 \in \mathcal{E}^{(1)}$. Thus the projection $\tau_{4,3} : \mathcal{E}^{(1)} \longrightarrow \mathcal{E}$ is surjection.

Second, let us prove that $\mathrm{Smbl}_{\theta_3}\mathcal{E} = \{0\}$ for any $\theta_3 \in \mathcal{E}$. Let $\theta_3 \in \mathcal{E}$. By definition $\mathrm{Smbl}\,\mathcal{E} = T_{\theta_3}(\ \tau_{3,2}^{-1}(\theta_2) \cap \mathcal{E}\ )$, where $\theta_2 = \tau_{3,2}(\theta_3)$. Obviously,

$$\tau_{3,2}^{-1}(\theta_2) \cap \mathcal{E} = \tau_{\infty,3}(\ \tau_{\infty,2}^{-1}(\theta_2) \cap \mathcal{E}(\mathcal{Y})^{(\infty)}\ )$$

and

$$\mathrm{Smbl}_{\theta_3}\mathcal{E} = (\tau_{\infty,3})_* T_{\theta_\infty}(\ \tau_{\infty,2}^{-1}(\theta_2) \cap \mathcal{E}(\mathcal{Y})^{(\infty)}\ ).$$

Let $p = \tau_3(\theta_3)$. We define the subgroup $H^{\infty,2}(p)$ of the isotropy group $H^\infty(p)$ of the point $[S_0]_p^\infty \in J^\infty\pi$ by the formula

$$H^{\infty,2}(p) = \{\ [f]_p^\infty \in H^\infty(p)\ \Big|\ [f]_p^1 = [\mathrm{Id}]_p^1\ \},$$

where Id is the identity diffeomorphism. Let $\theta_\infty, \bar\theta_\infty \in \tau^{-1}_{\infty,2}(\theta_2) \cap \mathcal{E}(\mathcal{Y})^{(\infty)}$; then $\theta^{-1}_\infty \cdot \bar\theta_\infty \in H^{\infty,2}(p)$. Consequently,

$$\tau^{-1}_{\infty,2}(\theta_2) \cap \mathcal{E}(\mathcal{Y})^{(\infty)} = \theta_\infty \cdot H^{\infty,2}(p).$$

Obviously, the Lie algebra of $H^{\infty,2}(p)$ is $\mathcal{H}^{\infty,2}(p)$ defined by (22). Then

$$T_{\theta_\infty}(\tau^{-1}_{\infty,2}(\theta_2) \cap \mathcal{E}(\mathcal{Y})^{(\infty)}) = (\theta_\infty)_* \mathcal{H}^{\infty,2}(p).$$

It follows from corollary 3.5 that $\mathcal{H}^{\infty,2}(p) = \{\, 0 \,\}$. ∎

Taking into account theorem 4.2, we deduce the following

Corollary 5.2 *Let $\mathcal{Y}$ be an ODE of form (1) and let p be a point in the domain of the coefficients $u^3(x,y,y'), u^2(x,y,y'), u^1(x,y,y'), u^0(x,y,y')$ of $\mathcal{Y}$. Then $\mathcal{Y}$ is reducible to the form $y''' = 0$ by a contact transformation in a neighborhood of p iff the collection*

$$(\ u^3(x,y,z), u^2(x,y,z), u^1(x,y,z), u^0(x,y,z)\ )$$

is a solution of the PDE system

$$F^3 = 0, \quad F^2 = 0, \quad F^1 = 0, \quad F^0 = 0$$

in a neighborhood of p.

6. Appendix

This section contains REDUCE programs that were used in calculations in this paper.

In texts of these programs we denote

$\dfrac{\partial^{i1+i2+i3} f}{(\partial x)^{i1}(\partial y)^{i2}(\partial z)^{i3}}$ by $f\overbrace{x\ldots x}^{i1}\overbrace{y\ldots y}^{i2}\overbrace{z\ldots z}^{i3}$ or by $f(i1,i2,i3)$

and we also denote

$\dfrac{\partial^{i1+i2+i3} u^n}{(\partial x)^{i1}(\partial y)^{i2}(\partial z)^{i3}}$ by $un\overbrace{x\ldots x}^{i1}\overbrace{y\ldots y}^{i2}\overbrace{z\ldots z}^{i3}$ or by $u(n,i1,i2,i3)$,

where $n = 3,2,1,0$.

6.1.

The following program calculates the law of coefficient's transformation for 3-order ODEs of form (1) with respect to contact transformations. Here, an initial ODE has the form

$$p3 = W(3) * p2 * *3 + W(2) * p2 * *2 + W(1) * p2 + W(0);$$

a contact transformation has the form

$$(\ x, y, z\ ) \longmapsto (\ q(x,y,z), p(x,y,z), r(x,y,z)\ ),$$

where $r = (df(p,x) + z * df(p,y))/(df(q,x) + z * df(q,y))$ *and* $df(p,z)(df(q,x) + z * df(q,y)) - df(q,z) * (df(p,x) + z * df(p,y)) = 0.$

The program expresses p3 *and* p2 *in terms of* $y3$, $y2$ *and* z *and substitutes these expressions in the initial ODE. As a result an ODE of the form*

$$y3 = H * [u(3) * y2 * *3 + u(2) * y2 * *2 + u(1) * y2 + u0]$$

is obtained. Next, the program calculates H. *At the end it returns the coefficients of the resulting ODE expressed in terms of* $W(3), W(2), W(1), W(0)$ *and derivatives of the functions* q, p, r.

Program 6.1 .

```
OFF NAT;

DEPEND q,x,y,z; DEPEND p,x,y,z; DEPEND r,x,y,z;
OPERATOR DX,W,u,n;

FOR ALL A LET
    DX(A)=df(A,x)+df(A,y)*z+df(A,z)*y2+df(A,y2)*y3+df(A,y3)*y4;
LET
    df(p,z)= df(q,z)*(df(p,x)+z*df(p,y))/(df(q,x)+z*df(q,y)),
    p2 = DX(r)/DX(q),
    p3 = DX(p2)/DX(q),
    n(3)=6, n(2)=2, n(1)=1, n(0)=1;        % Here n(i)=i!.

G := p3 - W(3)*p2**3 - W(2)*p2**2 - W(1)*p2 - W(0)$
H:= DF(G,y3)$
yy:= G/H - y3$      % It's a preliminary form of right-hand side of
                    % the transformed ODE is
H :=DEN(yy)$
G:=H*yy$
Clear yy$
H:=SUB(r=(z*df(p,y)+df(p,x))/(z*df(q,y)+df(q,x)),H)$
H:=1/H$
Clear r$

OUT result$
    % y3:=H*yy;      % It's a final form of the transformed ODE.
    FACTOR W(3),W(2),W(1),W(0)$
    FOR i:=3 STEP -1 UNTIL 0 DO
        WRITE u(i),"=H*[ ",SUB(y2=0,DF(G,y2,i))/n(i)," ],"$
    WRITE "where H=",H$
    CLEAR G,H$
SHUT result$

END;
```

6.2.

The following program calculates the velocity of deformation of jet-sections of π with respect to contact transformations.

Let ξ_f be an arbitrary contact vector field in **B***; let $f_s(x,y,z) = (\,q(s,x,y,z), p(s,x,y,z), r(s,x,y,z)\,)$ be its flow and let $S = (\,W(3), W(2), W(1), W(0)\,)$ be an arbitrary section of the bundle π. Then the deformation velocity of S is defined by formula (8) $\frac{d}{ds}(\,f_s^{(0)} \circ S \circ f_s\,)\Big|_{s=0}$. The program substitutes the transformed section $f_s^{(0)} \circ S \circ f_s = (\ u(3), u(2), u(1), u(0)\ )$ into the procedure $\frac{d}{ds}\Big|_{s=0}$. This procedure is denoted by DS in the program. The transformed section $(\ u(3), u(2), u(1), u(0)\ )$ is obtained with the help of program 6.1.*

Program 6.2 .

```
OFF NAT;

OPERATOR q,p,r,W,u;
DEPEND q, s,x,y,z; DEPEND p, s,x,y,z; DEPEND r, s,x,y,z;
DEPEND W(3),s; DEPEND W(2),s; DEPEND W(1),s; DEPEND W(0),s;

PROCEDURE DS(AA);
BEGIN
   SCALAR A;
   A:= df(AA,s);

   A:=
   sub(df(q,x)=1, sub(df(q,y)=0, sub(df(q,z)=0,
   sub(df(q,x,2)=0, sub(df(q,x,y)=0, sub(df(q,x,z)=0,
   sub(df(q,y,2)=0, sub(df(q,y,z)=0, sub(df(q,z,2)=0,
   sub(df(p,x)=0, sub(df(p,y)=1,
   sub(df(r,x)=0, sub(df(r,y)=0, sub(df(r,z)=1,
   sub(df(r,x,2)=0, sub(df(r,x,y)=0, sub(df(r,x,z)=0,
   sub(df(r,y,2)=0, sub(df(r,y,z)=0, sub(df(r,z,2)=0,
   A))))))))))))))))))));

   A:=
   sub(df(q,s,x)=-fxz, sub(df(q,s,y)=-fyz, sub(df(q,s,z)=-fzz,
   sub(df(q,s,x,2)=-fxxz, sub(df(q,s,x,y)=-fxyz, sub(df(q,s,x,z)=-fxzz,
   sub(df(q,s,y,2)=-fyyz, sub(df(q,s,y,z)=-fyzz, sub(df(q,s,z,2)=-fzzz,
   sub(df(p,s,x)=fx-z*fxz, sub(df(p,s,y)=fy-z*fyz,
   sub(df(r,s,x)=fxx + z*fxy, sub(df(r,s,y)=fxy + z*fyy,
   sub(df(r,s,z)=fxz + z*fyz + fy,
   sub(df(r,s,x,2)=fxxx + z*fxxy, sub(df(r,s,x,y)=fxxy + z*fxyy,
   sub(df(r,s,x,z)=fxxz + z*fxyz + fxy, sub(df(r,s,y,2)=fxyy + z*fyyy,
   sub(df(r,s,y,z)=fxyz + z*fyyz + fyy,
   sub(df(r,s,z,2)=fxzz + z*fyzz + 2*fyz, A))))))))))))))))))));
```

```
    A:= sub(W(3)=u3,
    sub(df(W(3),s)= u3x*fz - u3y*(f-z*fz) - u3z*(fx+z*fy),A));

    A:= sub(W(2)=u2,
    sub(df(W(2),s)= u2x*fz - u2y*(f-z*fz) - u2z*(fx+z*fy),A));

    A:= sub(W(1)=u1,
    sub(df(W(1),s)= u1x*fz - u1y*(f-z*fz) - u1z*(fx+z*fy),A));

    A:= sub(W(0)=u0,
    sub(df(W(0),s)= u0x*fz - u0y*(f-z*fz) - u0z*(fx+z*fy),A));

    RETURN A;
END;
```

% Values of functions u^3, u^2, u^1, u^0 (7) can be obtained with the help of
% program 6.1.

```
g(3):=u(3)$
g(2):=u(2)$
g(1):=u(1)$
g(0):=u(0)$

ORDER u3x, u3y, u3z, u2x, u2y, u2z, u1x, u1y, u1z, u0x, u0y, u0z, u3, u2, u1, u0;

OUT result$
    DS(g(3));
    DS(g(2));
    DS(g(1));
    DS(g(0));
SHUT result$

END$
```

6.3.

The following program performs the following tasks:
- it calculates the linear algebraic system (15) with $\theta_k \in \pi_k^{-1}(\mathbf{0})$,
- it reduces this system by elementary transformations to an equivalent system with coefficient matrix of the form

$$\begin{pmatrix} A \\ B \end{pmatrix},$$

where A is a step form matrix of maximal rank such that the first element of every step is a nonzero integer, components of B are zeros or polynomials with of nonzero degree.

Program 6.3 .

```
OFF NAT;

LET k=2;       % k=0,1,... is order of the jet bundle.
```

```
OPERATOR f,u,g;
DEPEND f(i1,i2,i3), X,Y,Z;
DEPEND u(n,i1,i2,i3), n,X,Y,Z;

FOR ALL n,i1,i2,i3 LET
   DF(f(i1,i2,i3),X)=f(i1+1,i2,i3),
   DF(f(i1,i2,i3),Y)=f(i1,i2+1,i3),
   DF(f(i1,i2,i3),Z)=f(i1,i2,i3+1),

   DF(u(n,i1,i2,i3),X)=u(n,i1+1,i2,i3),
   DF(u(n,i1,i2,i3),Y)=u(n,i1,i2+1,i3),
   DF(u(n,i1,i2,i3),Z)=u(n,i1,i2,i3+1);
```

% Functions $\psi^3(\varphi), \psi^2(\varphi), \psi^1(\varphi), \psi^0(\varphi)$ are the left-hand sides of
% equations of system (15) with $k = 0$. They can be obtained with the help
% of program 6.2.

g(0):=$\psi^3(\varphi)$
g(1):=$\psi^2(\varphi)$
g(2):=$\psi^1(\varphi)$
g(3):=$\psi^0(\varphi)$

```
% The following procedure creates the array "arrders" of all Nth
% derivatives of the function FN=FN(x,y,z)
PROCEDURE DERS(N,FN);
begin
   scalar j3,der,num,j;
   num:=(N+1)*(N+2)/2-1;   % num+1 is the number of all N-th deriv-s.
   array arrders(num);
   j:=-1;
   for j1:=N step (-1) until 0 do
   << j3:=-1;
      for j2:=N-j1 step (-1) until 0 do
      << j3:=j3+1;
         j:=j+1;
         arrders(j):=df(FN,x,j1,y,j2,z,j3)
      >>
   >>;
end;

% The following procedure creates the array "arrders1" of all N-th
% derivatives of the function FN=FN(x,y,z) with
% Z=0, f(0,0,0)=0, f(1,0,0)=0, f(0,0,1)=0.
PROCEDURE DERS1(N,FN);
begin
   scalar j3,der,num,j;
   num:=(N+1)*(N+2)/2-1;   % num+1 is the number of all N-th deriv-s.
   array arrders1(num);
```

```
    j:=-1;
    for j1:=N step (-1) until 0 do
    << j3:=-1;
        for j2:=N-j1 step (-1) until 0 do
        << j3:=j3+1;
            j:=j+1;
            arrders1(j):=SUB(f(0,0,0)=0,SUB(f(1,0,0)=0,SUB(f(0,0,1)=0,
            SUB(z=0,df(FN,x,j1,y,j2,z,j3) ) )));
        >>
    >>;
end;
```

% The following procedure returns a number of all derivatives of order
% $\leq$ N for the function f(0,0,0)

```
PROCEDURE NUMVARS(N);
begin
    scalar s;
    s:=0;
    for m:=0:N do s:=s+(m+1)*(m+2)/2;
    return s;
end;
```

% The following procedure creates the array "arrvars" of all derivatives
% of order $\leq$ N for the function f(0,0,0) except the derivatives
% f(0,0,0), f(1,0,0) and f(0,0,1).

```
PROCEDURE VARS(N);
begin
    scalar num,j1,j2;
    num:=NUMVARS(N)-3-1;
    array arrvars(num);
    arrvars(0):=f(0,1,0);
    for m:=2:N do
    << j1:=NUMVARS(m-1)-3;
        j2:=NUMVARS(m)-3-1;
        DERS(m,f(0,0,0));
        for j:=j1:j2 do arrvars(j):=arrders(j-j1)
    >>;
    clear arrders;
end;
```

% The following procedure creates an array "arrpsys". This array consists
% of left-hand sides of equations of system (15) with
% $\theta_k \in \pi_k^{-1}(\mathbf{0})$ and $k = N$.

```
PROCEDURE PSYS(N);
begin
    scalar num,j1,j2,r,rr;
```

```
    num:=4*NUMVARS(N)-1;
    array arrpsys(num);
    for m:=0:N do
    ≪ r:=(m+1)*(m+2)/2;
        rr:=4*NUMVARS(m-1);
        for j:=0:3 do
        ≪ DERS1(m,g(j));
            j1:=rr+j*r;
            j2:=j1+r-1;
            for s:=j1:j2 do arrpsys(s):=arrders1(s-j1)
        ≫
    ≫;
    clear arrders1;
end;

VARS(k+3)$
Lngth:=NUMVARS(k+3)-3-1$   % It's the length of "arrvars"
PSYS(k)$
Len:=4*NUMVARS(k)-1$   % It's the length of "arrpsys"
q:=0$

OUT result$

for j:=0:Lngth do
≪ Vr:=arrvars(j)$ WRITE Vr;
    s:=-1$
    REPEAT
    ≪ s:=s+1$
        Eqv:=arrpsys(s)$
        Cff:=DF(Eqv,Vr)
    ≫
    UNTIL ( ( NUMBERP(Cff) AND (Cff NEQ 0) ) OR (s=Len) )$
    IF ( NUMBERP(Cff) AND (Cff NEQ 0) ) THEN
    ≪ arrpsys(s):=0$
        q:=q+1$
        WRITE " E",q,": ",Eqv,"=0";
        FOR r:=0:Len DO
        ≪ Cff1:=DF(arrpsys(r),Vr)$
            IF (Cff1 NEQ 0) THEN arrpsys(r):=Cff*arrpsys(r)-Cff1*Eqv
        ≫$
        CLEAR Eqv,Cff,Cff1$
    ≫
≫$
CLEAR arrvars,Lngth$
WRITE "Conditions";
FOR w:=0:Len DO
```

```
    IF (arrpsys(w) NEQ 0) THEN
    ≪ q:=q+1$
        WRITE " E",q,": ",arrpsys(w),"=0";
        arrpsys(w):=0$
    ≫$
CLEAR q,Len, arrpsys$
SHUT result$
END$
```

6.4.

If we add to the program 6.3 the following substitution command

FOR ALL n,i1,i2,i3 LET u(n,i1,i2,i3)=0;

then the resulting program reduces the system of homogeneous linear algebraic equations (16) – (19) to an equivalent system with coefficient matrix of the step form.

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