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SECONDARY OPERATIONAL CALCULUS

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Abstract. Outlines of operational calculus in differential algebras are given. By operational calculus we mean techniques reducing solution of linear differential equations with constant coefficients to solution of appropriate algebraic systems of linear equations. It is shown that in the classical differential algebra such techniques cannot be realized, because of the lack of nontrivial exponentials. At the same time, the differential Lie algebra of vertical derivations, possesses a full spectrum of exponentials, and the corresponding operational calculus (secondary operational calculus) is developed.

In the first part of the paper (Sections 1,2) I recall the basic elements of differential algebras and propose a techniques reducing solution of linear differential equations with constant coefficients to solution of appropriate algebraic systems of linear equations. In the second part (Section 3) I use the above techniques to derive the operational calculus over the differential Lie algebra of vertical derivations in the classical differential algebra. I don't know any essential applications of presented results, but I am sure that being so nice and natural they must be of use somewhere, while the general scheme may encourage one to search for other fruitful realizations.

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1. Differential algebras

In this section I recall the basic elements of differential algebras. The only rather new element is the notion of *multiplicator*. I give its definition and list the main properties, including connections with derivations. The proofs are straightforward and easy.

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1.1. Notation. Let $\mathcal{A}$ be a topological algebra over a field $\mathbb{F}$ of characteristic zero.

We denote by
$$\operatorname{cen}\mathcal{A} = \{f \in \mathcal{A} : fg = gf,\ \forall g \in \mathcal{A}\}$$
the *center*, and by
$$\operatorname{ann}\mathcal{A} = \{f \in \mathcal{A} : fg = gf = 0,\ \forall g \in \mathcal{A}\}$$
the *annihilator* of $\mathcal{A}$.

We denote by $\mathfrak{L}(\mathcal{A})$ the linear space of all continuous linear mappings on $\mathcal{A}$. The *composition rule* $F, G \mapsto F \circ G$ defines in $\mathfrak{L}(\mathcal{A})$ the structure of a unital associative algebra $\mathfrak{L}_\circ(\mathcal{A})$, while the *commutator rule* $F, G \mapsto [F, G] = F \circ G - G \circ F$ defines in $\mathfrak{L}(\mathcal{A})$ the structure of a Lie algebra $\mathfrak{L}_{[\,,]}$.

The (left) *action* $\alpha : \mathcal{A} \to \mathfrak{L}(\mathcal{A})$ is defined by the rule $f \mapsto \alpha(f) = \widehat{f}$, $\widehat{f}(g) = fg$, $f, g \in \mathcal{A}$.

1.2. Multiplicators. A mapping $M \in \mathfrak{L}(\mathcal{A})$ is called a *multiplicator* (see [1,2]) if $M(fg) = M(f)g = fM(g)$ for any $f, g \in \mathcal{A}$. We denote by $\mathfrak{M}(\mathcal{A})$ the set of all multiplicators on $\mathcal{A}$.

One can check the following properties of $\mathfrak{M}(\mathcal{A})$.

(1) *$\mathfrak{M}(\mathcal{A})$ is a unital subalgebra of $\mathfrak{L}_\circ(\mathcal{A})$.*
(2) *The kernel $\ker M = \{f \in \mathcal{A} : M(f) = 0\}$ and the image $\operatorname{im} M = M(\mathcal{A})$ are two-sided ideals of $\mathcal{A}$ for any $M \in \mathfrak{M}(\mathcal{A})$.*
(3) *$M : \operatorname{ann}\mathcal{A} \to \operatorname{ann}\mathcal{A}$ and $M : \operatorname{cen}\mathcal{A} \to \operatorname{cen}\mathcal{A}$ for any $M \in \mathfrak{M}(\mathcal{A})$.*
(4) *$[M, N](f) \in \operatorname{ann}\mathcal{A}$ for any $M, N \in \mathfrak{M}(\mathcal{A})$ and $f \in \mathcal{A}$. In particular, $\mathfrak{M}(\mathcal{A})$ is commutative if $\operatorname{ann}\mathcal{A} = 0$.*

Let $\mathcal{A}$ be an associative algebra.

(5) *$\alpha(f) \in \mathfrak{M}(\mathcal{A})$ for any $f \in \operatorname{cen}\mathcal{A}$, i.e., $\alpha : \operatorname{cen}\mathcal{A} \to \mathfrak{M}(\mathcal{A})$.*
(6) *The intersection $\ker\alpha \cap \operatorname{cen}\mathcal{A} = \{f \in \operatorname{cen}\mathcal{A} : \alpha(f) = 0\} = \operatorname{ann}\mathcal{A}$. In particular, the mapping $\alpha : \operatorname{cen}\mathcal{A} \to \mathfrak{M}(\mathcal{A})$ is injective if $\operatorname{ann}\mathcal{A} = 0$.*
(7) *If there exists an approximation of unity $\{\delta_n\} \subset \mathcal{A}$ ($\delta_n f \to f$ and $f\delta_n \to f$, $n \to \infty$, for any $f \in \mathcal{A}$), then $\operatorname{ann}\mathcal{A} = 0$ and the image $\alpha(\operatorname{cen}\mathcal{A})$ is dense in $\mathfrak{M}(\mathcal{A})$.*

Let $\mathcal{A}$ be a unital associative algebra.

(8) *$\alpha : \operatorname{cen}\mathcal{A} \simeq \mathfrak{M}(\mathcal{A})$, the inverse α^{-1} acts by the rule $M \mapsto \alpha^{-1}(M) = M(e)$, $M \in \mathfrak{M}(\mathcal{A})$, where e is the unit element of $\mathcal{A}$. Hence, we may identify the algebras $\mathfrak{M}(\mathcal{A})$ and $\operatorname{cen}\mathcal{A}$.*

1.3. Derivations. A mapping $Z \in \mathfrak{L}(\mathcal{A})$ is called a *derivation* or *differentiation* if the *Leibniz rule* $Z(fg) = Z(f)g + fZ(g)$ is valid for any $f, g \in \mathcal{A}$ (see [2–5], for example). Denote by $\mathfrak{D}(\mathcal{A})$ the set of all derivations on $\mathcal{A}$. Clearly, $\mathfrak{D}(\mathcal{A})$ is a linear subspace of $\mathfrak{L}(\mathcal{A})$. One can check the following properties of $\mathfrak{D}(\mathcal{A})$.

(1) *$\mathfrak{D}(\mathcal{A})$ is a left $\mathfrak{M}(\mathcal{A})$-module with $MZ = M \circ Z$, $M \in \mathfrak{M}(\mathcal{A})$, $Z \in \mathfrak{D}(\mathcal{A})$*
(2) *$\mathfrak{D}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{L}_{[\,,]}$.*
(3) *$Z : \operatorname{ann}\mathcal{A} \to \operatorname{ann}\mathcal{A}$ and $Z : \operatorname{cen}\mathcal{A} \to \operatorname{cen}\mathcal{A}$ for any $Z \in \mathfrak{D}(\mathcal{A})$.*
(4) *The commutator $[Z, M] \in \mathfrak{M}(\mathcal{A})$ for any $Z \in \mathfrak{D}(\mathcal{A})$ and $M \in \mathfrak{M}(\mathcal{A})$.*

In particular, for every $Z \in \mathfrak{D}(\mathcal{A})$ the rule $\widehat{Z}(M) = [Z, M]$ defines a mapping $\widehat{Z} : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{A})$. Moreover, by the Jacobi identity,

(5) *$\widehat{Z}(M \circ N) = \widehat{Z}(M) \circ N + M \circ \widehat{Z}(N)$ for any $Z \in \mathfrak{D}(\mathcal{A})$ and $M, N \in \mathfrak{M}(\mathcal{A})$.*

Hence, the rule $Z \mapsto \beta(Z) = \widehat{Z}$ defines the action $\beta : \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathfrak{M}(\mathcal{A}))$.

(6) *The action $\beta : \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathfrak{M}(\mathcal{A}))$ is a morphism of Lie algebras.*

(7) *The Leibniz rule $Z(M(f)) = \widehat{Z}(M)(f) + M(Z(f))$ is valid for any $Z \in \mathfrak{D}(\mathcal{A})$, $M \in \mathfrak{M}(\mathcal{A})$ and $f \in \mathcal{A}$.*

Let $\mathcal{A}$ be an associative algebra, and $Z \in \mathfrak{D}(\mathcal{A})$.

(8) *$\widehat{Z}(\alpha(f)) = \alpha(Z(f))$ for any $f \in \operatorname{cen} \mathcal{A}$, i.e., the action of Z on $\mathfrak{M}(\mathcal{A})$ is compatible with the action α on $\operatorname{cen} \mathcal{A}$.*

1.4. Constants. For a subset $\mathfrak{Z} \subset \mathfrak{D}(\mathcal{A})$ we denote by

$$\operatorname{con}_{\mathfrak{Z}} \mathcal{A} = \{f \in \mathcal{A} : Z(f) = 0, \ \forall Z \in \mathfrak{Z}\}$$

the subalgebra of all $\mathfrak{Z}$-*constants* of the algebra $\mathcal{A}$. In particular, $\operatorname{con} \mathcal{A} = \operatorname{con}_{\mathfrak{D}(\mathcal{A})} \mathcal{A}$ is the subalgebra of *absolute constants* or simply *constants* of $\mathcal{A}$. Clearly, the field $\mathbb{F} \subset \operatorname{con}_{\mathfrak{Z}} \mathcal{A}$ for any $\mathfrak{Z} \subset \mathfrak{D}(\mathcal{A})$.

1.5. Differential algebras. A *differential algebra* is a pair $(\mathcal{A}, \mathfrak{C}(\mathcal{A}))$, where $\mathcal{A}$ is an algebra and $\mathfrak{C}(\mathcal{A})$ is a subalgebra of the Lie algebra $\mathfrak{D}(\mathcal{A})$ (see [3–5] for general theory).

We call a differential algebra $(\mathcal{A}, \mathfrak{C}(\mathcal{A}))$ *standard* if([1])

(1) $\mathfrak{C}(\mathcal{A})$ is a free finite-dimensional left $\mathfrak{M}(\mathcal{A})$-submodule of $\mathfrak{D}(\mathcal{A})$ with a pair-wise commuting basis $D_1, \ldots, D_m \in \mathfrak{D}(\mathcal{A})$ ($[D_\mu, D_\nu] = 0$, $\mu, \nu \in \overline{1, m}$), so

$$\mathfrak{C}(\mathcal{A}) \equiv [D] = \left\{ Z = \sum_{\mu=1}^{m} F^\mu D_\mu : F^\mu \in \mathfrak{M}(\mathcal{A}), \ \mu \in \overline{1, m} \right\}.$$

We call a standard differential algebra $(\mathcal{A}, [D])$ *regular* if

(2) $\mathfrak{D}(\mathcal{A}) = [D] \oplus \mathfrak{V}(\mathcal{A})$, where $\mathfrak{V}(\mathcal{A})$ is a left $\mathfrak{M}(\mathcal{A})$-submodule and a subalgebra of the Lie algebra $\mathfrak{D}(\mathcal{A})$;

(3) $[D_\mu, V] \in \mathfrak{V}(\mathcal{A})$ for all $\mu \in \overline{1, m}$ and $V \in \mathfrak{V}(\mathcal{A})$.

The number m is called the *differential dimension* of $(\mathcal{A}, [D])$, derivations $Z \in [D]$ are called *Cartan*, derivations $V \in \mathfrak{V}(\mathcal{A})$ are called *vertical*, $\mathfrak{C}(\mathcal{A})$-constants are called *differential constants* or $\mathfrak{C}$-*constants*.

2. Operational calculus

In this section I present a general scheme permitting, in principle, to reduce solution of linear differential equations with constant coefficients in frames of given differential

([1])Here and below we use the notation $\overline{k, l} \equiv \{k, k+1, \ldots, l\}$.

algebra to solution of appropriate algebraic systems of linear equations. This general scheme motivates constructions in the next section and may lead, I believe, to future realizations.

Let $(\mathcal{A},[D])$ be a standard differential algebra, and denote by

$$\mathbb{M} \equiv \operatorname{con}_{[\widehat{D}]} \mathfrak{M}(\mathcal{A}) = \{M \in \mathfrak{M}(\mathcal{A}) : \widehat{D}_\mu(M) = [D_\mu, M] = 0,\ \mu \in \overline{1,m}\}$$

the unital subalgebra of *differential constants* of $\mathfrak{M}(\mathcal{A})$.

2.1. Exponentials. We define (differential) exponentials as common eigenvectors of the derivations $D_1,\dots,D_m$. Namely, an element $f \in \mathcal{A}$ is called a ζ-*exponential*, $\zeta = (\zeta_1,\dots,\zeta_m) \in \mathbb{M}_m$ (2), if

$$D_\mu(f) = \zeta_\mu f, \quad \mu \in \overline{1,m}.$$

We denote by $\zeta\text{-}\exp\mathcal{A}$ the linear space of all ζ-exponentials in $\mathcal{A}$, and we denote by $\exp\mathcal{A} = \sum_{\zeta\in\mathbb{M}_m} \zeta\text{-}\exp\mathcal{A}$ the linear space of all finite sums of exponentials in $\mathcal{A}$.

One can check the *characteristic property* of exponentials:

(1) *$fg \in (\zeta+\eta)\text{-}\exp\mathcal{A}$ for any $f \in \zeta\text{-}\exp\mathcal{A}$ and $g \in \eta\text{-}\exp\mathcal{A}$.*

In particular, $\exp\mathcal{A}$ is a subalgebra of $\mathcal{A}$. Moreover, if the subalgebra $\mathbb{M}$ is commutative, then $\exp\mathcal{A}$ is an $\mathbb{M}$-algebra.

2.2. $\mathfrak{C}$-differential operators. Let $\mathbb{M}[\zeta]$ be the ring of all polynomials in $\zeta = (\zeta_1,\dots,\zeta_m)$ over $\mathbb{M}$. A *$\mathfrak{C}$-differential operator with constant coefficients on* $(\mathcal{A},[D])$ is a mapping $P(D) : \mathcal{A} \to \mathcal{A}$, where $P(z) \in \mathbb{M}[z]$. Denote by $\mathbb{M}[D]$ the unital associative commutative ring of all such operators, so, by definition, $\mathbb{M}[\zeta] \simeq \mathbb{M}[D]$. Clearly,

$$P(D)(f) = P(\zeta)(f) \quad \text{for any} \quad P(D) \in \mathbb{M}[D] \quad \text{and} \quad f \in \zeta\text{-}\exp\mathcal{A}.$$

2.3. Operational calculus. An *operational calculus in the differential algebra* $(\mathcal{A},[D])$ is a techniques, reducing the solution of $\mathfrak{C}$-differential equations

$$P(D)(u) = f, \quad f,u \in \mathcal{A}, \quad f \text{ given}, \quad u \text{ to be found},$$

(or systems of such equations) to the solution of appropriate algebraic systems of linear equations. It is clear that differential exponentials should play the crucial role in any such techniques.

2.4. Secondary calculus. Let $(\mathcal{A},[D])$ be a regular differential algebra. In this case, every Cartan derivation $D_\mu \in [D]$, $\mu \in \overline{1,m}$, defines the derivation $\widehat{D}_\mu \in \mathfrak{D}(\mathfrak{V}(\mathcal{A}))$ by the rule $\widehat{D}_\mu(V) = [D_\mu, V]$, $V \in \mathfrak{V}(\mathcal{A})$. Moreover, $[\widehat{D}_\mu, \widehat{D}_\nu] = 0$, $\mu,\nu \in \overline{1,m}$. Hence, *the differential Lie algebra of vertical derivations $(\mathfrak{V}(\mathcal{A}),[\widehat{D}])$ is defined, where*

$$[\widehat{D}] = \left\{ Z = \sum_{\mu=1}^m F^\mu \widehat{D}_\mu : F^\mu \in \mathfrak{M}(\mathfrak{V}(\mathcal{A})),\ \mu \in \overline{1,m} \right\}$$

One can try to develop an operational calculus in this secondary differential algebra. Moreover, it may happen that the original algebra $(\mathcal{A},[D])$ has not enough differential exponents to construct any sensible operational calculus in it, while the secondary algebra $(\mathfrak{V}(\mathcal{A}),[\widehat{D}])$ admits one. In the next section we present an example of such situation.

(2) We distinguish upper and lower indices, hence, $\mathbb{M}_m = \mathbb{M} \times \cdots \times \mathbb{M}$ (m times).

3. Secondary operational calculus in the classical differential algebra

The results of this section can be treated from two points of view. On one side, they present an example of situation, where the original differential algebra has a poor spectral analysis, while derived algebra admits a rich one, with an efficient operational calculus over it. On the other side, being so neat and natural they surely should find a place in general theory and applications.

3.1. Notation. We use the following multiindex notation:

$i = (i_\mu;\ \mu \in \overline{1,m}) \in \mathbb{I}$ are multiindices, $i_\mu \in \{0, 1, \dots\}$, $\mathbb{I} = \{0, 1, \dots\}_m$(3);
$|i| = i_1 + \dots + i_m$ for $i \in \mathbb{I}$;
$i + j = (i_1 + j_1, \dots, i_m + j_m)$ for $i, j \in \mathbb{I}$;
$(\mu) = (0, \dots, 1, \dots, 0) \in \mathbb{I}$ for $\mu \in \overline{1,m}$ (1 stands at μ-th place);
in particular, $i + (\mu) = (i_1, \dots, i_\mu + 1, \dots, i_m)$ for $i \in \mathbb{I}$ and $\mu \in \overline{1,m}$;

3.2. Classical differential algebra. Let

$x = (x^\mu;\ \mu \in \overline{1,m}) \in \mathbb{R}^m$ be independent variables,
$u = (u^\alpha;\ \alpha \in \overline{1,a}) \in \mathbb{R}^a$ be dependent variables,
$\mathbf{u} = (u_i^\alpha;\ \alpha \in \overline{1,a},\ i \in \mathbb{I}) \in \mathbb{R}^a_{\mathbb{I}}$(4) be differential variables, i.e.,

$$u_i^\alpha \longleftrightarrow \frac{\partial^{|i|} u^\alpha}{\partial x^i} = \frac{\partial^{i_1 + \dots + i_m} u^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^m)^{i_m}}, \quad (x, \mathbf{u}) \in M = \mathbb{R}^m \times \mathbb{R}^a_{\mathbb{I}}.$$

Let us denote by $\mathcal{A} = \mathcal{C}^\infty(M)$ the algebra of all complex-valued smooth functions $f = f(x, \mathbf{u}) = f(x^\mu, u_i^\alpha)$ on M, depending on a finite number of differential variables u_i^α, $|i| \le p$, where $p = p(f)$ is the *differential order of a function* f (see [3–7], for more detail). Clearly, $\mathcal{A}$ is a unital associative commutative algebra, $\mathfrak{M}(\mathcal{A}) = \mathcal{A}$, while

$$\mathfrak{D}(\mathcal{A}) = \left\{ X = \sum_\mu \chi^\mu \partial_{x^\mu} + \sum_{\alpha, i} \eta_i^\alpha \partial_{u_i^\alpha};\ \chi^\mu, \eta_i^\alpha \in \mathcal{A} \right\}$$

is the Lie algebra of all derivations of $\mathcal{A}$ (or vector fields on M, in geometrical language). Further, denote by

$$D_\mu = \partial_{x^\mu} + \sum_{\alpha, i} u^\alpha_{i+(\mu)} \partial_{u_i^\alpha} \in \mathcal{A}, \quad \mu \in \overline{1,m},$$

the total derivatives (see [3–7], for example). One can check that $[D_\mu, D_\nu] = 0$, for any $\mu, \nu \in \overline{1,m}$, hence, a differential algebra $(\mathcal{A}, [D])$ is defined. Moreover, $\mathfrak{D}(\mathcal{A}) = [D] \oplus \mathfrak{V}(\mathcal{A})$, where

$$\mathfrak{V}(\mathcal{A}) = \left\{ V = \sum_{\alpha, i} \phi_i^\alpha \partial_{u_i^\alpha};\ \phi_i^\alpha \in \mathcal{A} \right\}$$

is the Lie algebra of all vertical derivations of $\mathcal{A}$. It is easy to check that the assumptions (1) – (3) of Section 1.5 are valid, i.e., $(\mathcal{A}, [D])$ *is a regular differential algebra.*

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(3) $\{0, 1, \dots\}_m = \{0, 1, \dots\} \times \dots \times \{0, 1, \dots\}$ (m times), see the footnote (2).
(4) see the footnote (2).

One can check that $\operatorname{con}_{[D]} \mathcal{A} = \mathbb{C}$ (in fact, due to precisely this property the derivations D_μ are called *total*). Further, *the equations*

$$D_\mu(f) = \zeta_\mu f, \quad \mu \in \overline{1,m}, \quad f(x,\mathbf{u}) \in \mathcal{A}, \quad \zeta = (\zeta_\mu) \in \mathbb{C},$$

have only trivial solutions $f = f(x) = c\exp\left(\sum_\mu \zeta_\mu x^\mu\right)$, $c \in \mathbb{C}$, *so there is no hope to develop a meaningful operational calculus in* $\mathcal{A}$.

3.3. Differential algebra of vertical derivations. The algebra $(\mathcal{A}, [D])$ is regular (see Section 3.2). Hence, in accordance with Section 2.4, the differential Lie algebra $(\mathfrak{V}(\mathcal{A}), [\widehat{D}])$ of all vertical derivations of $\mathcal{A}$ is defined. One can check that

$$\mathfrak{M}(\mathfrak{V}(\mathcal{A})) = \Big\{ M \in \mathfrak{L}(\mathfrak{V}(\mathcal{A})) : M([V,W]) = [M(V), W],\ \forall V, W \in \mathfrak{V}(\mathcal{A}) \Big\} = \mathbb{C},$$

hence,

$$[\widehat{D}] = \Big\{ Z = \sum_\mu c^\mu \widehat{D}_\mu \ ;\ c^\mu \in \mathbb{C} \Big\}.$$

3.4. Exponentials. *Let* $\zeta = (\zeta_\mu) \in \mathbb{C}_m$(5). *A vertical derivation* $V \in \mathfrak{V}(\mathcal{A})$ *is a* ζ-*exponential, i.e.,* $[D_\mu, V] = \zeta_\mu V$, $\mu \in \overline{1,m}$, *iff there exists* $\varphi = (\varphi^\alpha) \in \mathcal{A}^a$ *such that*

$$V = \sum_{\alpha,i} ((D-\zeta)_i \varphi^\alpha) \partial_{u_i^\alpha} \equiv E(\zeta, \varphi),$$

where $(D-\zeta)_i = (D_1 - \zeta_1)^{i_1} \ldots (D_m - \zeta_m)^{i_m}$.

The proof is done by direct calculations.

In particular,

$$E(0,\varphi) = \sum_{\alpha,i} (D_i \varphi^\alpha) \partial_{u_i^\alpha} \equiv E_\varphi, \quad D_i = D_1^{i_1} \ldots D_m^{i_m},$$

is the *evolution derivation* (see [5,6], for example). The characteristic property takes the form

$$[E(\zeta,\varphi), E(\xi,\psi)] = E(\zeta+\xi, \chi), \quad \zeta, \xi \in \mathbb{C}_m, \quad \varphi, \psi \in \mathcal{A}^a,$$

where $\chi^\alpha = E(\zeta,\varphi)(\psi^\alpha) - E(\xi,\psi)(\varphi^\alpha)$.

3.5. Involution in $\mathcal{A}^a_{\mathbb{I}}$. Consider the linear space

$$\mathcal{A}^a_{\mathbb{I}} = \{\varphi_i^\alpha \in \mathcal{A} : \alpha \in \overline{1,a},\ i \in \mathbb{I}\}$$

and define the linear mapping $I : \mathcal{A}^a_{\mathbb{I}} \to \mathcal{A}^a_{\mathbb{I}}$ by the rule $\boldsymbol{\varphi} \mapsto \boldsymbol{\psi} = I(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = (\varphi_i^\alpha), \boldsymbol{\psi} = (\psi_i^\alpha) \in \mathcal{A}^a_{\mathbb{I}}$, and

$$\psi_i^\alpha = \sum_{p+q=i} (-1)^{|q|} \tbinom{i}{p} D_p \varphi_q^\alpha, \quad \tbinom{i}{p} = \frac{i!}{p!q!},$$

here $i, p, q \in \mathbb{I}$, $i! = i_1! \ldots i_m!$, and so on.

(5)see the footnote (2).

Proposition. *The linear mapping I is an involution, i.e., $I^2 = 1 \equiv \mathrm{id}_{\mathcal{A}^a_{\mathbb{I}}}$.*

Proof. Indeed, let $\boldsymbol{\varphi} = (\varphi^\alpha_i) \in \mathcal{A}^a_{\mathbb{I}}$, $\boldsymbol{\psi} = I(\boldsymbol{\varphi})$, then

$$\begin{aligned} I^2(\boldsymbol{\varphi})^\alpha_i &= I(\boldsymbol{\psi})^\alpha_i = \sum_{p+q=i} (-1)^{|q|}\tbinom{i}{p} D_p \psi^\alpha_q \\ &= \sum_{p+q=i} (-1)^{|q|}\tbinom{i}{p} D_p \Big[\sum_{k+l=q} (-1)^{|l|}\tbinom{q}{k} D_k \varphi^\alpha_l \Big] \\ &= \sum_{p+k+l=i} (-1)^{|k+l|+|l|}\tbinom{i}{p}\tbinom{k+l}{k} D_{p+k}\varphi^\alpha_l = \sum_{p+k+l=i} (-1)^{|k|}\frac{i!}{p!k!l!} D_{p+k}\varphi^\alpha_l \\ &= \sum_{q+l=i} \tbinom{i}{l} \Big[\sum_{p+k=q} (-1)^{|k|}\tbinom{q}{k} \Big] D_q \varphi^\alpha_l = \varphi^\alpha_i, \quad \alpha \in \overline{1,a}, \quad i \in \mathbb{I}, \end{aligned}$$

because

$$\sum_{p+k=q} (-1)^{|k|}\tbinom{q}{k} = (1-1)^q = \delta^q_0 = \begin{cases} 1, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

3.6. Taylor mapping. Define the linear mapping $T : \mathcal{C}^\infty(\mathbb{C}^m) \to \mathbb{C}^{\mathbb{I}}$ by the rule $f \mapsto T(f)$, where $f \in \mathcal{C}^\infty(\mathbb{C}^m)$, $T(f) = (T(f)^i) \in \mathbb{C}^{\mathbb{I}}$,

$$T(f)^i = \frac{1}{i!} f^{(i)}(0) = \frac{1}{i!} \frac{\partial^{|i|} f(0)}{(\partial\zeta_1)^{i_1} \dots (\partial\zeta_m)^{i_m}}.$$

One can check the following properties of T:

(1) $T(\zeta_k)^i = \delta^i_k$, $\quad i, k \in \mathbb{I}$, $\quad \zeta_k = \zeta_1^{k_1} \dots \zeta_m^{k_m}$;
(2) $T(fg) = T(f) * T(g)$, i.e., $T(fg)^i = \sum_{p+q=i} T(f)^p T(g)^q$, $i \in \mathbb{I}$.

The component-wise generalization of T is defined by the rule

$$T : \mathcal{C}^\infty(\mathbb{C}_m) \otimes \mathfrak{V}(\mathcal{A}) \xrightarrow{T \otimes \mathrm{id}_{\mathfrak{V}(\mathcal{A})}} \mathbb{C}^{\mathbb{I}} \otimes \mathfrak{V}(\mathcal{A}) = \mathfrak{V}(\mathcal{A})^{\mathbb{I}}.$$

Proposition. *Let $E(\zeta,\varphi)$ be a ζ-exponential* (see Section 3.4), *then the Taylor image $T(E(\zeta,\varphi)) = E(\varphi) \equiv (E(\varphi)^i)$, where* (cf. [1])

$$E(\varphi)^i = (-1)^{|i|} \sum_{\alpha,k} \tbinom{i+k}{k} (D_k \varphi^\alpha) \partial_{u^\alpha_{i+k}}, \quad i \in \mathbb{I}.$$

Proof. Indeed,

$$\begin{aligned} T(E(\zeta,\varphi))^i &= T\Big[\sum_{\alpha,p} ((D-\zeta)_p \varphi^\alpha)\partial_{u^\alpha_p} \Big]^i = T\Big[\sum_{\alpha,p} \sum_{k+l=p} (-1)^{|l|}\tbinom{p}{k} \zeta_l (D_k\varphi^\alpha)\partial_{u^\alpha_p} \Big]^i \\ &= \sum_{\alpha,k,l} (-1)^{|l|}\tbinom{k+l}{k} T(\zeta_l)^i (D_k\varphi^\alpha)\partial_{u^\alpha_{k+l}} = \sum_{\alpha,k} (-1)^{|i|}\tbinom{k+i}{k} (D_k\varphi^\alpha)\partial_{u^\alpha_{k+i}}. \end{aligned}$$

3.7. Exponential representation. Every vertical derivation of $\mathcal{A}$ can be expanded in ζ-exponentials in the following sense.

Theorem. *Let* $V = \sum_{\alpha,i} \varphi^\alpha_i \partial_{u^\alpha_i} \in \mathfrak{V}(\mathcal{A})$, *then* $V = \sum_i E(\psi_i)^i$, *where*

$$\boldsymbol{\psi} = (\psi^\alpha_i) = I(\boldsymbol{\varphi}), \quad \boldsymbol{\varphi} = (\varphi^\alpha_i) = I(\boldsymbol{\psi}), \quad \psi_i = (\psi^\alpha_i;\ \alpha \in \overline{1,a}) \in \mathcal{A}^a.$$

Proof. Indeed, by Section 3.5,

$$\begin{aligned}\sum_i E(\psi_i)^i &= \sum_i (-1)^{|i|} \sum_{\alpha,k} \tbinom{i+k}{k}(D_k\psi^\alpha_i)\partial_{u^\alpha_{i+k}} \\ &= \sum_{\alpha,p}\Big[\sum_{i+k=p}(-1)^{|i|}\tbinom{p}{k}(D_k\psi^\alpha_i)\Big]\partial_{u^\alpha_p} = \sum_{\alpha,p} I(\boldsymbol{\psi})^\alpha_p\partial_{u^\alpha_p} = \sum_{\alpha,p}\varphi^\alpha_p\partial_{u^\alpha_p} = V.\end{aligned}$$

Thus, we have the isomorphisms

(1) *the coordinate representation* $K : \mathfrak{V}(\mathcal{A}) \to \mathcal{A}^a_{\mathbb{I}}$, with $V = \sum_{\alpha,i}\varphi^\alpha_i\partial_{u^\alpha_i} \mapsto \boldsymbol{\varphi} = (\varphi^\alpha_i)$;
(2) *the exponential representation* $E : \mathfrak{V}(\mathcal{A}) \to \mathcal{A}^a_{\mathbb{I}}$, with $V = \sum_i E(\psi_i)^i \mapsto \boldsymbol{\psi} = (\psi^\alpha_i)$;
(3) *the involution* $I : \mathcal{A}^a_{\mathbb{I}} \to \mathcal{A}^a_{\mathbb{I}}$, defined in Section 3.5;

and the commutative diagram

$$\begin{array}{ccc} \mathfrak{V}(\mathcal{A}) & = & \mathfrak{V}(\mathcal{A}) \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle E} \\ \mathcal{A}^a_{\mathbb{I}} & \xrightarrow{I} & \mathcal{A}^a_{\mathbb{I}} \end{array}$$

3.8. Difference operators in $\mathcal{A}^a_{\mathbb{I}}$. Every polynomial $P(\zeta) = \sum_i c^i\zeta_i \in \mathbb{C}[\zeta]$, $c^i \in \mathbb{C}$, $\zeta_i = \zeta_1^{i_1}\ldots\zeta_m^{i_m}$, $|i| \le n$, $n = n(P)$ is a degree of P, defines a linear mapping (*difference operator*) $\widetilde{P} : \mathcal{A}^a_{\mathbb{I}} \to \mathcal{A}^a_{\mathbb{I}}$ by the rule:

$$\widetilde{P}(\boldsymbol{\varphi}) = \boldsymbol{\psi}, \quad \boldsymbol{\varphi} = (\varphi^\alpha_i), \quad \boldsymbol{\psi} = (\psi^\alpha_i) \in \mathcal{A}^a_{\mathbb{I}}, \quad \psi^\alpha_i = \sum_k c^k\varphi^\alpha_{i+k}.$$

One can check that

$\widetilde{PQ} = \widetilde{P}\circ\widetilde{Q}$ for any $P, Q \in \mathbb{C}[\zeta]$.

Thus, the unital associative commutative ring $\mathfrak{S}(\mathcal{A}^a_{\mathbb{I}})$ of all difference operators in $\mathcal{A}^a_{\mathbb{I}}$ is defined. By definition, there is a natural isomorphism $\mathbb{C}[\zeta] \simeq \mathfrak{S}(\mathcal{A}^a_{\mathbb{I}})$.

3.9. Operational transform. Every polynomial $P(\zeta) = \sum_i c^i\zeta_i \in \mathbb{C}[\zeta]$ determines a $\widehat{D}$-differential operator $P(\widehat{D}) : \mathfrak{V}(\mathcal{A}) \to \mathfrak{V}(\mathcal{A})$ by the rule (see Sections 2.2 and 2.4):

$$P(\widehat{D}) = \sum_i c^i\widehat{D}_i, \quad \widehat{D}_i = \widehat{D}_1^{i_1}\ldots\widehat{D}_m^{i_m},$$

remind that $\widehat{D}_\mu(V) = [D_\mu, V]$, so, for example, $\widehat{D}^2_\mu(V) = [D_\mu,[D_\mu,V]]$, $V \in \mathfrak{V}(\mathcal{A})$.

Proposition. *Let $V = \sum_i E(\varphi_i)^i \in \mathfrak{V}(\mathcal{A})$, then $P(\widehat{D})(V) = \sum_i E(\psi_i)^i$, where $\boldsymbol{\psi} = \widetilde{P}(\boldsymbol{\varphi})$.*

Proof. Indeed, let $\varphi = (\varphi^\alpha) \in \mathcal{A}^a$, $i \in \mathbb{I}$, then for any $\mu \in \overline{1,m}$ we shall have

$$\begin{aligned}
\widehat{D}_\mu(E(\varphi)^i) &= [D_\mu, E(\varphi)^i] = (-1)^{|i|} \sum_{\alpha,k} \tbinom{i+k}{k} [D_\mu, (D_k\varphi^\alpha)\partial_{u^\alpha_{i+k}}] \\
&= (-1)^{|i|} \sum_{\alpha,k} \tbinom{i+k}{k} [(D_{k+(\mu)}\varphi^\alpha)\partial_{u^\alpha_{i+k}} - (D_k\varphi^\alpha)\partial_{u^\alpha_{i+k-(\mu)}}] \\
&= (-1)^{|i|} \sum_{\alpha,k} \left[\tbinom{i+k-(\mu)}{k-(\mu)} - \tbinom{i+k}{k}\right] (D_k\varphi^\alpha)\partial_{u^\alpha_{i+k-(\mu)}} \\
&= (-1)^{i-(\mu)} \sum_{\alpha,k} \tbinom{i-(\mu)+k}{k} (D_k\varphi^\alpha)\partial_{u^\alpha_{i-(\mu)+k}} = E(\varphi)^{i-(\mu)}.
\end{aligned}$$

By induction, $\widehat{D}_k(E(\varphi)^i) = E(\varphi)^{i-k}$ for any $k \in \mathbb{I}$. Hence,

$$\begin{aligned}
P(\widehat{D})\big(\textstyle\sum_i E(\varphi_i)^i\big) &= \sum_k c^k \sum_i \widehat{D}_k(E(\varphi_i)^i) = \sum_k c^k \sum_i E(\varphi_i)^{i-k} \\
&= \sum_k c^k \sum_j E(\varphi_{k+j})^j = \sum_j E\big(\textstyle\sum_k c^k \varphi_{j+k}\big)^j \\
&= \sum_j E(\widetilde{P}(\varphi)_j)^j.
\end{aligned}$$

Remark. Clearly, $\widehat{D}_\mu \circ T = T \circ \widehat{D}_\mu$, $\mu \in \overline{1,m}$. Using this fact, we could present the first part of the proof as follows,

$$\begin{aligned}
\widehat{D}_\mu(E(\varphi)^i) &= \widehat{D}_\mu(T(E(\zeta,\varphi))^i) = T(\widehat{D}_\mu(E(\zeta,\varphi)))^i = T(\zeta_\mu E(\zeta,\varphi))^i \\
&= \sum_{p+q=i} \delta^p_{(\mu)} E(\varphi)^q = E(\varphi)^{i-(\mu)}.
\end{aligned}$$

Corollary. *The operational transform*

$$\mathbb{C}[\widehat{D}] \ni P(\widehat{D}) \longleftrightarrow E \circ P(\widehat{D}) \circ E^{-1} = \widetilde{P} \in \mathfrak{S}(\mathcal{A}^a_{\mathbb{I}})$$

is defined.

3.10. Operational calculus. Let an equation

$$(*) \qquad P(\widehat{D})(V) = F, \quad F \in \mathfrak{V}(\mathcal{A}) \text{ is known}, \quad V \in \mathfrak{V}(\mathcal{A}) \text{ is to find},$$

be given. Consider the algebraic system of the *difference equations*

$$(**) \qquad \widetilde{P}(\boldsymbol{\varphi}) = \boldsymbol{\psi}, \quad \boldsymbol{\psi} \in \mathcal{A}^a_{\mathbb{I}} \text{ is known}, \quad \boldsymbol{\varphi} \in \mathcal{A}^a_{\mathbb{I}} \text{ is to find}.$$

The above results lead to the following

Theorem. *A vertical derivation $V = \sum_i E(\varphi_i)^i$ is a solution of $(*)$, where $F = \sum_i E(\psi_i)^i$, iff $\boldsymbol{\varphi}$ is a solution of $(**)$.*

The generalization of this techniques to the systems of $\widehat{D}$-differential equations with constant coefficients is immediate.

Remark. The systems of difference equations $(**)$ arise in the formal theory of overdetermined systems of the linear PDE with constant coefficients (see [8], for example).

3.11. Examples. In many cases the difference system $(**)$ can be solved explicitly.

1. The equations

$$\widehat{D}_\mu(V) - \zeta_\mu V = 0, \quad \mu \in \overline{1,m},$$

with some fixed $\zeta \in \mathbb{C}_m$ have the difference system

$$\varphi^\alpha_{i+(\mu)} - \zeta_\mu \varphi^\alpha_i = 0, \quad \alpha \in \overline{1,a}, \quad i \in \mathbb{I}.$$

The latter has the immediate solution $\varphi^\alpha_i = \zeta_i \varphi^\alpha$, $\varphi = (\varphi^\alpha) \in \mathcal{A}^a$ is arbitrary, and we recover $V = E(\zeta, \varphi)$.

2. Let $m = a = 1$, for simplicity. The equation

$$\widehat{D}(V) - \zeta V = F, \quad \zeta \in \mathbb{C}, \quad F \in \mathfrak{V}(\mathcal{A}),$$

has the difference system

$$\varphi_{i+1} - \zeta\varphi_i = \psi_i, \quad \text{or} \quad \varphi_{i+1} = \zeta\varphi_i + \psi_i, \quad i = 0, 1, \dots.$$

We find that $\varphi_0 = \varphi \in \mathcal{A}$ is arbitrary, $\varphi_i = \zeta^i\varphi + \sum_{k=0}^{i-1} \zeta^k \psi_{i-k-1}$, $i = 1, 2, \dots$, so

$$V = E(\zeta, \varphi) + \sum_{k=0}^{\infty} \zeta^k \sum_{i>k} E(\psi_{i-k-1})^i.$$

In particular, if $\zeta = 0$, then $V = \sum_{i=0}^{\infty} E(\psi_{i-1})^i$ is a $\widehat{D}$-*primitive* of F, $\psi_{-1} = \varphi \in \mathcal{A}$ is arbitrary.

3. Again, let $m = a = 1$. The equation

$$\widehat{D}^2(V) - \zeta V = 0, \quad \zeta \in \mathbb{C},$$

has the difference system

$$\varphi_{i+2} - \zeta\varphi_i = 0, \quad i = 0, 1, \dots,$$

with the general solution $\varphi_{2k} = \zeta^k\varphi$, $\varphi_{2k+1} = \zeta^k\psi$, $k = 0, 1, \dots$, where $\varphi, \psi \in \mathcal{A}$ are arbitrary. Hence,

$$V = C(\sqrt{\zeta}, \varphi) + S(\sqrt{\zeta}, \psi) = E(\sqrt{\zeta}, \varphi_+) + E(-\sqrt{\zeta}, \varphi_-),$$

where

$$C(\lambda, \varphi) = \frac{1}{2}[E(\lambda, \varphi) + E(-\lambda, \varphi)], \quad S(\lambda, \psi) = \frac{1}{2}[E(\lambda, \psi) - E(-\lambda, \psi)],$$
$$\varphi_+ = \frac{1}{2}(\varphi + \psi), \quad \varphi_- = \frac{1}{2}(\varphi - \psi).$$

4. Let $m = 2$, $a = 1$, $i = (k,l)$, $k,l \in \mathbb{I} = \{0,1,\dots\}$. A vertical derivation V is called *$\widehat{D}$-harmonic* if it is a solution of the *$\widehat{D}$-Laplace equation*

$$\widehat{\Delta}(V) \equiv (\widehat{D}_1^2 + \widehat{D}_2^2)(V) = 0.$$

The corresponding difference system

$$\varphi_{k+2,l} + \varphi_{k,l+2} = 0, \quad k,l \in \mathbb{I},$$

has the general solution

$$\varphi_{2k,l} = (-1)^k \varphi_{l+2k}, \quad \varphi_{2k+1,l} = (-1)^k \psi_{l+2k}, \quad k,l \in \mathbb{I},$$

where $\varphi_l, \psi_l \in \mathcal{A}$ are arbitrary. Thus, a vertical derivation V is $\widehat{D}$-harmonic iff it has a representation

$$V = \sum_{k,l} (-1)^k \big[E(\varphi_l)^{2k,l} + E(\psi_l)^{2k+1,l}\big],$$

where

$$E(\varphi)^{kl} = (-1)^{k+l} \sum_{p,q} \tbinom{k+p}{p}\tbinom{l+q}{q}(D_1^p D_2^q \varphi)\partial_{u_{k+p,l+q}}.$$

5. Let $m = a = 2$, $i = (k,l)$, $k,l \in \mathbb{I} = \{0,1,\dots\}$. The *$\widehat{D}$-Cauchy–Riemann system*

$$\begin{cases} \widehat{D}_1(U) = \widehat{D}_2(V), \\ \widehat{D}_2(U) = -\widehat{D}_1(V), \end{cases} \quad \text{where} \quad \begin{cases} U = \sum_i E(\varphi_i)^i, \\ V = \sum_i E(\psi_i)^i, \end{cases}$$

has the difference system

$$\varphi^\alpha_{k+1,l} = \psi^\alpha_{k,l+1}, \quad \varphi^\alpha_{k,l+1} = -\psi^\alpha_{k+1,l}, \quad \alpha \in \overline{1,2}, \quad k,l \in \mathbb{I}.$$

The latter has the compatibility conditions

$$\varphi^\alpha_{k+2,l} + \varphi^\alpha_{k,l+2} = 0, \quad \psi^\alpha_{k+2,l} + \psi^\alpha_{k,l+2} = 0, \quad \alpha \in \overline{1,2}, \quad k,l \in \mathbb{I}.$$

By the previous example,

$$\begin{aligned} &\varphi^\alpha_{2k,l} = (-1)^k f^\alpha_{l+2k}, \quad \varphi^\alpha_{2k+1,l} = (-1)^k g^\alpha_{l+2k}, \quad \psi^\alpha_{2k,l} = (-1)^k h^\alpha_{l+2k}, \\ &\psi^\alpha_{2k+1,l} = (-1)^k r^\alpha_{l+2k}, \quad \alpha \in \overline{1,2}, \quad k,l \in \mathbb{I}. \end{aligned}$$

The difference system is satisfied iff

$$g^\alpha_k = h^\alpha_{k+1}, \quad r^\alpha_k = -f^\alpha_{k+1}, \quad \alpha \in \overline{1,2}, \quad k \in \mathbb{I}.$$

Hence, solutions of the $\widehat{D}$-Cauchy–Riemann system are pairs (U,V) of vertical derivations having representations

$$U = \sum_{k,l} E(\varphi_{kl})^{kl}, \quad V = \sum_{k,l} E(\psi_{kl})^{kl},$$

where

$$\begin{aligned} \varphi^\alpha_{2k,l} &= (-1)^k f^\alpha_{l+2k}, \quad \varphi^\alpha_{2k+1,l} = (-1)^k h^\alpha_{l+2k+1}, \\ \psi^\alpha_{2k,l} &= (-1)^k h^\alpha_{l+2k}, \quad \psi^\alpha_{2k+1,l} = (-1)^{k+1} f^\alpha_{l+2k+1}, \end{aligned}$$

with arbitrary $f^\alpha_k, h^\alpha_k \in \mathcal{A}$, $\alpha \in \overline{1,2}$, $k,l \in \mathbb{I}$.

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