Special Geometric Structures in String Theory
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THE MATHEMATICAL FORMULATION OF THE M5–BRANE

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Abstract. I discuss three issues in M5–brane theory. These issues have in common that they are much better understood in the context of strings and Dirichlet branes, or D–branes. D–branes and the M5–brane are related in the sense that the M5–brane can be considered as the strong coupling limit of the D4–brane. In this limit an extra worldvolume direction opens up. The first issue I discuss is how the non-Abelian Born–Infeld vector of the D–branes should generalize to non-Abelian two–form gauge fields living on the worldvolume of the M5–brane. Next, I discuss how the noncommutative geometry of D-branes generalizes to a non-commutative loop space geometry on the M5–brane. The third and last issue I discuss is how the noncommutative open strings recently discovered in the context of D–branes generalize to a noncommutative open membrane.

1. Introduction

String theory is considered to be the most promising candidate for solving the problem of the notorious infinities of quantum gravity (for a textbook, see [1, 2, 3, 4, 5]. The basic assumption underlying string theory is that the particles in nature are not pointlike but, instead, are described by the vibrational modes of little strings whose basic length scale is the Planck length. String theory predicts a finite number of massless particles, among which is the graviton as the carrier of the gravitational force, plus an infinite number of very heavy particles with a mass of the order of the Planck mass. The idea is that this infinite number of particles, massless and massive ones, controls the problemetic infinities of quantum gravity. To introduce fermions in the spectrum, like leptons and quarks, one must consider a supersymmetric string, or superstring, theory. In the low-energy limit, or at large distance scales, the superstring theory can be approximated by a supergravity theory. To avoid the occurrence of anomalies during the quantization procedure one usually starts with a superstring theory in D=10 spacetime dimensions. In a second step one rolls up six of the spatial dimensions in order to obtain four–dimensional physics. This rolling up of dimensions is called “compactification”.

Having gone from pointlike particles to string-like objects, i.e. objects with a one–dimensional extension or "1–branes", it is natural to consider the further generalization to membranes or "2–branes". Indeed in 1987 we proposed the D=11 supermembrane and conjectured that its low-energy limit was given by D=11 supergravity [6]. The interesting thing about D=11 supergravity is that it is the highest–dimensional supergravity theory one can construct[1]. It was soon discovered that there were problems with the spectrum of the D=11 supermembrane when considered to be at the same footing as a fundamental D=10 superstring. For instance, the spectrum did not contain the massless fields of D=11 supergravity [7, 8].

In the mid-nineties there was a revival of interest in membranes and its higher-dimensional partners, the "p–branes", due to the progress made in understanding some of the nonperturbative features of string theory. In particular, it was discovered that, for large values of the string coupling constant, the spectrum of string theory contains p-branes for various values of p. In other words string theory really is a theory of strings *and* branes. Another discovery was that the D=10 superstring and D=11 supermembrane are related in the sense that the strong coupling linit of D=10 supergravity leads to the opening up of an extra dimension and D=11 supergravity [9]. It is unclear how to take this strong coupling limit of the complete D=10 superstring theory. It is supposed to lead to a so-called D=11 M-theory whose precise formulation is yet unknown.

We do know that the low–energy limit of M-theory is D=11 supergravity. This D=11 supergravity theory has not only membrane, or M2–brane, solutions [10] but also 5–brane, or M5–brane, solutions [11]. The M5–brane is the magnetic version of the M2–brane[2]. The M5–brane is much less understood than the M2–brane. In the same way as the M2–brane is the strong coupling limit of the fundamental string, or F1–brane, the M5–brane is the strong coupling limit of the D4–brane. It is the purpose of this lecture to gain insight into the mathematical formulation of the M5–brane by making use of what we already know about the D–branes, in particular the D4–brane. Any progress in our understanding of the M5–brane is likely to lead to further insights into M–theory.

[1]We assume here a Minkowski signature of spacetime. The problem with going beyond D = 11 is that one must then introduce massless particles of spin higher than two and it is not known how to couple such high–spin particles to gravity.

[2]In general the magnetic version of a p–brane is a (D-p-4)–brane. Note that only in D=4 the magnetic version of a 0–brane (electron) is again a 0–brane (magnetic monopole).

2. D–branes

A particularly interesting class of branes in string theory are the D–branes [12]. They differ from the ordinary branes, or p–branes, in the sense that open strings can end on the D-brane. In fact, the D–branes are naturally described as open superstrings with the boundary condition that the endpoints of the open superstring is constrained to move on the worldvolume of the D-brane. These are called Dirichlet boundary conditions. The dynamics of ordinary p-branes are described by a Nambu-Goto action containg the (embedding) scalars of the p-brane, i.e. for each transverse direction there is a scalar. The extra structure of the D–brane (endpoints of open strings moving on its worldvolume) results in the existence of not only (embedding) scalars but also a so-called Born-Infeld (BI) vector that couples to the endpoints of the open string. The scalars and vectors together form a so-called Dirac-Born-Infeld (DBI) action. For zero scalars this action equals the Nambu-Goto, or Dirac, action while for zero vector the action equals the Born-Infeld action [13].

We now discuss three characteristic properties of D-branes. In the next Section we will discuss these three properties in the context of the M5-brane of M–theory.

(A) Yang–Mills. It has been shown that if one considers N coinciding D–branes the N Born–Infeld vectors are not the only massless vector fields. There are additional massless vectors arsing from strings that are stretched between the different D-branes. This leads to the following symmetry enhancement [14]:

$$\underbrace{U(1)\times\cdots\times U(1)}_{N\ \text{times}} \longrightarrow U(N)\,. \tag{1}$$

In other words the U(1) Maxwell theory of a single D–brane gets extended to a U(N) Yang–Mills theory of N coinciding D–branes. In general this Yang-Mills theory is coupled to the (9–p) embedding scalars living on the worldvolume. It is not clear how to describe these scalars in the case of N coinciding D–branes. A special case is the D9–brane for which there are no transverse directions and hence no embedding scalars. To avoid the complication of how to describe the scalars it is often convenient to concentrate on this case.

One can view the number N as a deformation parameter of the worldvolume theory of the D–brane. There exists another deformation parameter, the basic string length ℓ_s or the Regge slope parameter $\alpha' = \ell_s^2$. The α'–corrections deform the Maxwell theory into a BI theory where α' multiplies the higher-order BI curvatures in the action. One can also consider the two deformations at the same time, i.e. consider the α'–corrections of N coinciding D–branes. The resulting worldvolume theory is called the non-Abelian Born–Infeld (NBI) theory but its complete structure is not known. For some recent progress, see e.g. [15, 16, 17]. We have sumarized the situation in the diagram below.

$$\begin{array}{ccc}
\text{Maxwell} & \overset{\text{N}}{\Rightarrow} & \text{Yang-Mills} \\
\alpha' \Downarrow & & \alpha' \Downarrow \\
\text{B.I.} & \overset{\text{N}}{\Rightarrow} & \underline{\text{N.B.I.}}\ ??
\end{array}$$

(B) Noncommutative Geometry. Recently, there has been an interest in considering D–branes in the background of a nonzero NS–NS B–field. Since this B–field occurs in the definition of the BI curvature

$$\mathcal{F} = dV + B \tag{2}$$

its effect is that $\mathcal{F} \neq 0$. This leads for instance to an effective open string metric and coupling constant:

$$\left(G_{\text{os}}^{-1}\right)_{ab} = \eta_{ab} - \left(\mathcal{F}^2\right)_{ab}\,, \tag{3}$$

$$\lambda_{\text{os}} = \sqrt{-\det\left(\eta_{ab} + \mathcal{F}_{ab}\right)}\,. \tag{4}$$

It turns out that a non-zero B–field also leads to an effective noncommutative geometry of the worldvolume of the D–brane [18, 19]. One way to see this is to consider a particular decoupling limit [20] in which the bulk closed string states decouple from the worldvolume open string states. Furthermore the massive open string states decouple and one ends up with a field theory living on the worldvolume of the D–branes. In this limit, assuming that $\mathcal{F}$ is constant, the open string action describing the open string ending on the D–brane is dominated by its Wess–Zumino (WZ) term given by

$$S_{\text{string}} \sim \int_{\text{bound.}} d\tau\ \underline{\mathcal{F}_{ab} X^a}\,\dot{X}^b\,. \tag{5}$$

The subscript "bound." indicates the boundary of the open string worldvolume, i.e. the worldline of its endpoints. The underlined terms effectively describe a standard vector coupling to a particle. It is well known, e.g. from the quantum Hall effect, that this coupling leads to a noncommutative geometry for those coordinates that couple to the vector. This is simply due to the reason that these coordinates occur with a *linear* time derivative in the action. Upon canonical quantization such terms lead to (noncommutative) Dirac brackets between the coordinates. To be more explicit let us assume that the nonzero components of $\mathcal{F}_{ab}$ are given by $\mathcal{F}_{rs}$ where r, s refer to a certain number of spatial directions:

$$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{F}_{rs} \end{pmatrix} . \tag{6}$$

The spatial coordinates X^r and X^s then become noncommuting with noncommutativity proportional to the inverse of $(\mathcal{F})_{rs}$:

$$\{X^r, X^s\}^{\mathrm{D}} = \left(\mathcal{F}^{-1}\right)^{rs} . \tag{7}$$

Here $\{.,.\}^{\mathrm{D}}$ denotes the Dirac bracket.

(C) Noncommutative Open strings. So far we only considered spacelike noncommutative coordinates. In the context of the D3–brane it is natural to also consider timelike noncommutative coordinates. In the case of the D3–brane the two cases are related to each other if one considers the S–duality in the presence of a nonzero B–field [21]. The reason of this is that under S–duality the magnetic (spatial) components of the 4 x 4 matrix $\mathcal{F}$ are converted into electric (involving time directions) components of $\mathcal{F}$. We already mentioned that in the magnetic case a decoupling limit leads to a noncommutative Yang–Mills theory on the D–brane. It has been pointed out that in the electric case the massive open string states do not decouple and hence one is ending up with an open string theory instead of a field theory [21]. This open string theory is called a noncommutative open string (NCOS). I have summarized the situation in the diagram below.

$$\begin{array}{ccc} \text{magnetic field} & \overset{\mathrm{S}}{\Leftrightarrow} & \text{electric field} \\ \Downarrow & & \Downarrow \\ \text{nonc. Y.M.} & & \text{N.C.O.S.} \end{array}$$

3. The M5–Brane

D–branes are hyperplanes with open strings ending on them. There is a 1–form gauge field on the D–brane worldvolume that couples to the 0–brane endpoints of the open string. There is a natural generalization of this to open p–branes ending on q–branes ($p < q$) with a (p+1)–form gauge field living on the worldvolume of the q–brane that couples to the (p–1)–brane boundary of the open p–brane. String theory predicts that such objects exist in D=11 with p=2 and q=5, i.e. open M2–branes ending on an M5–brane[3]. In fact, it was believed for some time that open M2–branes did not exist, see e.g. [22]. The reason for this is that the open superstring boundary conditions break the N=2, D=10 supersymmetry to a N=1, D=10 supersymmetry. Such a supersymmetry breaking is not possible in D=11 without breaking Lorentz invariance. It was only after the introduction of D–branes [12] that it was realized that a system of branes ending on branes naturally breaks Lorentz symmetry and hence an open M2–brane ending on a M5–brane could be defined [23, 24].

The M5–brane is the magnetic partner of the M2–brane in the sense that the M2–brane couples to the (electric) 3–form gauge potential of D=11 supergravity whereas the M5–brane couples to the (magnetic) dual 6–form potential. There is a 2–form potential living on the M5–brane that couples to the string boundary of the M2–brane. A nontrivial feature is that this worldvolume 2–form is selfdual. It is part of a N=2, D=6 tensor multiplet that also contains the 5 embedding scalars corrresponding to the transverse directions of the M5-brane. The tensor multiplet describes 8+8 (bosonic + fermionic) worldvolume degrees of freedom. The 5 embedding scalars X^μ $(\mu = 0, 1, \cdots , 10)$[4] occur via the worldvolume metric g_{ab} $(a = 0, 1, \cdots 5)$ and the 3 degrees of freedom described by the selfdual two form b occur via the worldvolume 3–form curvature $\mathcal{H}$:

$$g_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) : \qquad 5 \text{ d.o.f.} \tag{8}$$

$$\mathcal{H} = db + C(X) : \qquad 3 \text{ d.o.f.} \tag{9}$$

Here $g_{\mu\nu}(X)$ is the D=11 gravity field and $C(X)$ is the 3–form potential of D=11 supergravity.

[3] Branes living in a D=11 target spacetime are part of M-theory and therefore called M–branes.

[4] Note that 6 of the X^μ are gauge degrees of freedom due to the worldvolume reparametrizations.

The 3–form curvature $\mathcal{H}$ satisfies the following nonlinear selfduality condition [25]:

$$C_a{}^d\mathcal{H}_{dbc} = \sqrt{\frac{-\mathrm{det}g}{3!}}\epsilon_{abcdef}\mathcal{H}^{def}\,, \tag{10}$$

with the so–called Boillat metric given by [26]

$$C^{ab} = \frac{1}{\sqrt{1+\frac{1}{24}\mathcal{H}^2}}\left[\left(1+\frac{1}{12}\mathcal{H}^2\right)g^{ab} - \frac{1}{4}\left(\mathcal{H}^2\right)^{ab}\right]\,. \tag{11}$$

It can be shown that the worldvolume equations of motion can be formulated in terms of this Boillat metric

Having discussed the necessary M5–brane preliminaries we are now in a position to discuss in which sense the three properties of D–branes discussed in the previous Section carry over to the M5–brane.

3.1. **Non-abelian Two–form Gauge Fields.** We mentioned that the Maxwell theory of a single D–brane gets extended to a U(N) Yang–Mills theory of N coinciding D–branes. We also know that the single Maxwell 1–form of the D4–brane, in the strong coupling limit, becomes the selfdual 2–form of the M5–brane. The question is what happens in the case of N coinciding M5–branes. Naively one would expect some kind of non-Abelian 2–form gauge theory, see the diagram below. Such a theory has not been constructed yet.

$$\begin{array}{ccc}
\text{M5: 2–form} & \overset{\text{N}}{\Rightarrow} & ?? \\
\Downarrow & & \Downarrow \\
\text{Maxwell} & \overset{\text{N}}{\Rightarrow} & \text{U(N) Y.M.}
\end{array}$$

We assume here that the result can be described by a worldvolume field theory[5]. There is a theorem stating that this field theory cannot be local [27].

[5]It has been suggested that instead the worldvolume theory is described by a (nonperturbative, selfdual) string theory. If that is the case we assume that this string theory, in some limit, can be effectively described by a field theory.

One possible approach to find out what the worldvolume theory is that describes N coinciding M5–branes is to consider its reduction to five dimensions. In D=5 a Maxwell 1–form A_μ is (Poincaré) dual to an Abelian 2–form $B_{\mu\nu}$. To construct a non-Abelian 2–form gauge theory in five dimensions one should be able to extend the duality to the non-Abelian case and dualize a Yang–Mills vector A_μ^I into a Yang–Mills 2–form $B_{\mu\nu}{}^I$. This problem has already been considered in the seventies, see e.g. [28]. Sofar an explicit realization of the non–Abelian 2–form gauge theory on the M5–brane worldvolume has not been found yet. We have summarized the situation in the diagram below.

From several points of view one has come to the conclusion that whatever the worldvolume theory of N coinciding M5–branes is, it is bound to contain *nonlocal* structures. This is also natural given the fact that we are dealing with strings instead of particles moving in the worldvolume. I have tried, in collaboration with Chris Hull, to formulate such a nonlocal worldvolume theory, but sofar we did not succeed to give an explicit result that satisfies the test that, after dimensional reduction, it reproduces the (local) DBI worldvolume theory of the D4–brane.

$$\begin{array}{ccc} A_\mu & \overset{\text{duality}}{\Rightarrow} & B_{\mu\nu} \\ \Downarrow & & \Downarrow \\ A_\mu{}^I & \overset{\text{non}-\text{abelian duality}}{\Rightarrow} & B_{\mu\nu}{}^I \end{array}$$

One way to naturally incorporate nonlocal structures, is to work with a loop space. A loop space ΣM of a manifold M is defined as the space of maps from the circle S^1 to the manifold M. In fact, we will encounter exactly these loop spaces in the next Subsection where we consider the M5–brane analogue of the D–brane noncommutative geometry.

3.2. **Noncommutative Geometry.** We now wish to consider the M5–brane in the presence of a nonvanishing D=11 3–form field C and investigate what the resulting geometry of the M5–brane worldvolume is. One immediate effect of $C \neq 0$

is that the 3–form $\mathcal{H}$

$$\mathcal{H} = db + C \tag{12}$$

is nonvanishing. To investigate the resulting geometry we consider, for constant $\mathcal{H}$, a decoupling limit, discussed in [29], where the open membrane action is dominated by its Wess–Zumino term:

$$S_{\text{membrane}} \sim \int_{\text{bound.}} d\tau d\sigma \; \mathcal{H}_{abc} X^a \dot{X}^b X'^c \, . \tag{13}$$

Here "bound." indicates the string boundary of the open membrane. This string boundary lives on the M5–brane worldvolume.

For convenience we assume that the field strength $\mathcal{H}$ can be diagonalised as follows [20]:

$$\mathcal{H}_{012} = \frac{-h}{\sqrt{1+\ell_p^6 h^2}} \, , \qquad\qquad \mathcal{H}_{345} = h \, . \tag{14}$$

In the parameterisation (14) the action (13) splits into two independent Lagrangians for the two sets of coordinates X^α and X^a ($\alpha = 0, 1, 2$; $a = 3, 4, 5$):

$$S = \frac{h}{\sqrt{1+\ell_p^6 h^2}} \int_{\partial M} d\tau d\sigma \; \epsilon_{\alpha\beta\gamma} X^\alpha \dot{X}^\beta X'^\gamma \tag{15}$$

$$+ \quad h \int_{\partial M} d\tau d\sigma \; \epsilon_{abc} X^a \dot{X}^b X'^c \, . \tag{16}$$

The action (13) is invariant under worldsheet reparameterisations:

$$\delta_\xi X^\alpha = \xi^i \partial_i X^\alpha \, , \quad \delta_\eta X^a = \eta^i \partial_i X^a \, , \qquad i = 0, 1 \, . \tag{17}$$

Note that, due to the absence of a worldsheet metric, there is no need to identify the vector fields ξ and η.

The equations of motion are:

$$\epsilon_{\alpha\beta\gamma} \dot{X}^\beta X'^\gamma = 0 \, , \quad \epsilon_{abc} \dot{X}^b X'^c = 0 \, . \tag{18}$$

Assume now that the string boundary inside the M5–brane has a noncompact extension in the time direction. In that case we can impose the gauge choice $X^0 = \tau$. Substituting this into the equations of motion we obtain

$$X'^\alpha = 0 \, , \tag{19}$$

which means that the spatial extension of the string must be in the a direction. Assuming that $|\vec{X}'| \neq 0$ we obtain

$$\dot{X}^a = 0 \, , \tag{20}$$

which implies additional Dirichlet conditions in the a directions.

Let us continue by analysing the phase space dynamics of the three coordinates $X^\alpha = (X^0, X^1, X^2)$. The canonical momenta are given by

$$\Pi_\alpha(\sigma) := \frac{\delta S}{\delta \dot{X}^\alpha(\sigma)} = -\frac{h}{3}\epsilon_{\alpha\beta\gamma} X^\beta X'^\gamma \ , \tag{21}$$

indicating that there are 3 primary constraints $\phi_\alpha(\sigma)$:

$$\phi_a := \Pi_a + \frac{h}{3}\epsilon_{abc} X^b X'^c \approx 0 \ . \tag{22}$$

The non–trivial canonical Poisson brackets are:

$$\{X^a(\sigma), \Pi_b(\sigma')\} = \delta^a_b\, \delta(\sigma - \sigma') \tag{23}$$

and the nonzero Hamiltonian is given by

$$H = \int d\sigma\, \lambda^a(\sigma)\phi_a(\sigma)\,, \tag{24}$$

where $\lambda^a(\sigma)$ are three Lagrange multipliers. To proceed with the canonical analysis we study the consistency conditions

$$\dot{\phi}_a(\sigma) = \lambda^b(\sigma) M_{ba}(\sigma) \approx 0\,, \tag{25}$$

where

$$\{\phi_a(\sigma), \phi_b(\sigma')\} = M_{ab}(\sigma)\delta(\sigma - \sigma') \ , \quad M_{ab} = h\, \epsilon_{abc} X'^c \ . \tag{26}$$

Note that in the α space we can impose $X^0 = \tau$ and, via the equations of motion, $X'^\alpha = 0$. This implies that $M_{\alpha\beta} = 0$. In other words, the 3 primary constraints $\phi_\alpha(\sigma)$ are all first class. There are no second class constraints in the $0, 1, 2$ directions.

In contrast, let us now consider the canonical analysis of the three Euclidean coordinates $X^a = (X^3, X^4, X^5)$. A similar analysis as above leads to the same result except that in this case we have assumed that $|\vec{X}'| \neq 0$ and therefore

$$M_{ab} X'^b = 0\,. \tag{27}$$

The matrix M_{ab} is thus non–degenerate in the two–dimensional subspace orthogonal to $\vec{X}'$. It is convenient to introduce a projection onto this subspace as follows ($I = 1, 2$):

$$\begin{aligned}
P_I{}^a(\sigma) P_J{}^b(\sigma)\delta_{ab} &= \delta_{IJ} \ , & (28)\\
\delta^{IJ} P_I{}^a(\sigma) P_J{}^b(\sigma) &= \delta^{ab} - \frac{X'^a X'^b}{|\vec{X}'|^2} \ , & (29)\\
\epsilon^{IJ} P_I{}^a(\sigma) P_J{}^b(\sigma) &= \frac{\epsilon^{abc} X'^c}{|\vec{X}'|} \ . & (30)
\end{aligned}$$

The three constraints ϕ_a now split into the two second class constraints

$$\chi_I := P_I{}^a \phi_a \ , \tag{31}$$

with the now non–degenerate matrix

$$\{\chi_I(\sigma), \chi_J(\sigma')\} := M_{IJ}(\sigma)\delta(\sigma - \sigma') \ , \qquad M_{IJ} = P_I{}^a P_J{}^b M_{ab} \ , \tag{32}$$

and one first class constraint

$$\phi := X'^a \phi_a \equiv X'^a \Pi_a \ , \tag{33}$$

which acts as the generator of σ–reparameterisations.

Unlike for the 0,1,2 directions we have two second class constraints in the 3,4,5 directions. The presence of these two second class constraints leads to a nontrivial Dirac bracket between the X^a coordinates given by

$$\{X^a(\sigma), X^b(\sigma')\} = -\frac{1}{h}\frac{\epsilon^{abc} X'^c}{|\vec{X}'|^2}\delta(\sigma - \sigma') \,. \tag{34}$$

The conclusion is that the membrane probe sees a noncommutative geometry in the a directions of the M5–brane worldvolume.

A basic difference with the D–brane case is that we are not dealing with points X^a but with *loops* $X^a(\sigma)$. This leads us naturally to loop spaces. As a historical note, it is perhaps of interest to note that, whereas the idea of lightlike integrability applied to a superspace geometry naturally leads to the superspace constraints of Yang-Mills [30], the same idea when applied to a loop superspace geometry leads to the constraints of supergravity coupled to Yang-Mills [31]. In the latter work the definition of a loop space covariant derivative plays a central role. The gauge field part of this covariant derivative is given by the pull-back of the selfdual anti-symmetric tensor, i.e.

$$\mathcal{D}_\mu(\sigma) = \frac{\delta}{\delta X^\mu(\sigma)} + b_{\nu\mu} X'^\nu \,. \tag{35}$$

The M5–brane with nonzero C field naturally leads us to consider a noncommutative version of loop space. The notion of a noncommutative loop space extending the notion of a noncommutative geometry has not yet been defined in the mathematics literature. One of the open questions is how to define the loop space analogue of the star product. A reparametrization independent definition of the star product has been given [32]. Moreover, a path integral description of this star product exists [33]. It is not clear how to extend this to loop spaces. It should lead to a generalized star product defining the product of two loop space functionals $F[X(\sigma)]$ and $G[X(\sigma)]$:

$$(F \star G)\,[X(\sigma)] \,. \tag{36}$$

By now we have discussed two issues in M5–brane theory: the issue of non-Abelian 2–forms and the issue of noncommutative loop space. These two issues are not unrelated. It is believed that the natural framework to describe the M5–brane worldvolume theory, incorporating nonlocal structures for coinciding M5–branes, is a U(1) gauge theory on loop space. Following (35), the natural gauge field is given by the pull-back of the selfdual 2–form:

$$X'^{\nu} b_{\mu\nu} \,. \tag{37}$$

There are now two ways to deform the Abelian gauge theory: (1) a commutative to noncommutative deformation and (2) an Abelian to non-Abelian deformation. The noncommutative deformation implies that $\{X^\mu, X^\nu\} \neq 0$ while the non-Abelian deformation means that $b_{\mu\nu} \to b^I_{\mu\nu} T^I$:

$$\begin{array}{cc} \underline{\text{noncommutative}} & \underline{\text{non} - \text{Abelian}} \\[1em] \{X^\mu, X^\nu\} \neq 0 & b^I_{\mu\nu} T^I \end{array}$$

It is known that in the case of noncommutative geometry a noncommutative U(1) gauge theory is related to a non-Abelian gauge theory [20]. Based on this it is natural to suggest that a noncommutative U(1) gauge theory on loop space is related to a non-Abelian loop space gauge theory. If correct, the two issues are indeed related.

3.3. **Open Membranes.** We now come to a discussion of our third, and last, issue. We have seen that, in the case of D–branes, a magnetic B–field leads to a noncommutative Yang–Mills theory but that an electric B–field leads to a noncommutative open string (NCOS). A NCOS theory can be defined for the other D–branes as well. In the case of the D4–brane it is natural to ask what the M–theory limit of NCOS is. It has been pointed out that, for the D4–brane, the strong coupling limit of NCOS, i.e. the limit of large values of the open string coupling constant G_{os}, is described by a noncommutative open membrane (NCOM) living on the M5–brane worldvolume [34, 35]. In this picture, for large values of the basic open membrane parameter ℓ_g, the NCOM is described by a field theory of non-Abelian 2–forms on the M5–brane worldvolume. The parameter ℓ_g is the analogue of the basic string length parameter ℓ_s, see the diagram below. Here R indicates the radius of the compactifying circle.

$$
\begin{tikzcd}
\text{N.C.O.M.} \arrow[r, Rightarrow, "\ell_g"] \arrow[d, Rightarrow, "G_{os}"'] & \text{Non. Ab. 2-forms} \arrow[d, Rightarrow, "R"] \\
\text{N.C.O.S.} \arrow[r, Rightarrow, "\ell_s"] & \text{Y.M.}
\end{tikzcd}
$$

A remarkable feature of the above picture is that the above diagram is precisely the worldvolume analogue of the target space picture relation between M-theory and string theory. Here the closed string coupling constant g_s takes over the role of the open string coupling constant G_{os} and the D=11 Planck length parameter ℓ_p plays the role of the open membrane parameter ℓ_g. In short, the target space analogue of the worldvolume diagram above is given by

$$
\begin{tikzcd}
\text{M-theory} \arrow[r, Rightarrow, "\ell_p"] \arrow[d, Rightarrow, "g_s"'] & \text{D=11 SUGRA} \arrow[d, Rightarrow, "R"] \\
\text{IIA-superstring} \arrow[r, Rightarrow, "\ell_s"] & \text{IIA SUGRA}
\end{tikzcd}
$$

A natural question to ask is whether there is an effective open membrane metric, i.e. a membrane analogue of the effective open string metric G_{os} given in (3). The open membrane metric should, upon reduction, give rise to the open string metric. In view of this, it is convenient to consider the dimensional reduction of the M5–brane to the D4–brane in more detail. The 3–form curvature $\hat{\mathcal{H}}$ reduces to a 2–form curvature $\mathcal{F}$ and a 3–form curvature $\mathcal{H}$ as follows:[6]

$$
\hat{\mathcal{H}}_{aby} \equiv \mathcal{F}_{ab}\,, \qquad\qquad \hat{\mathcal{H}}_{abc} \equiv \mathcal{H}_{abc}\,. \tag{38}
$$

[6]Hatted indices and fields refer to the M5–brane.

The selfduality relation (10) of $\hat{\mathcal{H}}$ reduces to a nonlinear relation between $\mathcal{H}$ and $\mathcal{F}$ which can be used to solve for $\mathcal{H}$ in terms of $\mathcal{F}$ as follows:

$$\frac{1}{3!}\epsilon_{abcde}\mathcal{H}^{cde} = \frac{\mathcal{D}_0\mathcal{F}_{ab} + (\mathcal{F}^3)_{ab}}{\sqrt{\mathcal{D}}}\,. \tag{39}$$

The 5×5 matrix $\mathcal{D}$ is defined in terms of $\mathcal{F}$ as

$$\begin{aligned} \mathcal{D} &\equiv -\det\left(\eta_{ab} + \mathcal{F}_{ab}\right) \\ &= \mathcal{D}_0 + \underbrace{\tfrac{1}{8}(\mathrm{tr}\,\mathcal{F}^2)^2 - \tfrac{1}{4}\mathrm{tr}\,\mathcal{F}^4}_{\mathrm{rank4}}\,, \end{aligned} \tag{40}$$

with $\mathcal{D}_0$ given by

$$\mathcal{D}_0 \;=\; 1 - \frac{1}{2}\mathrm{tr}\,\mathcal{F}^2\,. \tag{41}$$

The terms in (40) underbraced with "rank 4" indicate the terms that are nonvanishing if $\mathcal{D}$ is a rank 4 matrix. They do vanish whenever $\mathcal{D}$ is of rank 2. In that case we have $\mathcal{D} = \mathcal{D}_0$. Note that, in our conventions, tr $\mathcal{F}^2 > 0$ (< 0) for electric (magnetic) $\mathcal{F}$. This leads to the following inequalities for $\mathcal{F}$:

$$0 < \mathcal{D}_0 < 1 : \text{electric}\,, \qquad \mathcal{D}_0 > 1 : \text{magnetic}\,. \tag{42}$$

Concerning the open membrane metric, it was been pointed out that, in the decoupling limit, the open membrane equals the Boillat metric (11) upto a conformal factor z [35]:

$$\hat{G}^{\hat{a}\hat{b}}_{\mathrm{OM}} = z\,\hat{C}^{\hat{a}\hat{b}}\,. \tag{43}$$

Recently, the explicit expression for the conformal factor has been calculated for the case that $\mathcal{F}$ is a matrix of rank 2. The result is given by [36]

$$z^{-3} = K - \sqrt{K^2 - 1}\,, \qquad K \equiv \sqrt{1 + \frac{1}{24}\hat{\mathcal{H}}^2}\,. \tag{44}$$

The calculation of [36] was based upon the assumption that the open membrane metric, upon reduction, should give rise to (i) the open string metric G_{os} given in (3) *and* (ii) the open string coupling constant λ_{os} given in (4). This was combined with the fact that, on the other hand, the effective open string coupling constant λ_s has already been calculated in string theory. Comparing the string theory expression with what follows from dimensional reduction fixes the conformal factor z. The same expression for the conformal factor can be obtained, from a rather different point of view, by deformation techniques [37].

We should note that the analysis of [36] only works if the electric reduction is taken in the 1,2 direction. This assumes that we use the parametrization (14) and that the spatial extensions of the membrane are in the 1,2 directions. It does not work if we reduce in the 3,4,5 directions. Indeed, only in the case that we reduce in the 1,2 directions, the dimensional reduction is a *double* dimensional reduction with the membrane reducing to a string. The rank 2 case leads to the additional simplifications that $\mathcal{D} = \mathcal{D}_0$ and the fact that the OM–metric is *diagonal*:

- Electric rank 2 reduction : 1,2

$\mathcal{D}_0 = \mathcal{D}$, diagonal OM-metric

In a recent work we considered the case of a general rank 4 reduction [38]. The extra parameter with respect to the rank 2 reduction corresponds to the angle that describes the components of the reduction direction in the 1,2 and the 3,4,5 directions:

- general rank 4 reduction: 1,2 + 3,4,5

We found that the open membrane metric leads to the open string metric provided the conformal factor is given by

$$z^{-3} = \frac{\frac{1}{2}(\mathcal{D}_0 + 1) - \sqrt{\frac{1}{4}(\mathcal{D}_0 - 1)^2 + (\mathcal{D}_0 - \mathcal{D})}}{\sqrt{\mathcal{D}}}\,. \tag{45}$$

Note that for $\mathcal{D} = \mathcal{D}_0$ this expression reduces to the rank 2 expression given in (44). Once the conformal factor is fixed the open membrane metric is determined and its dimensional reduction leads to a prediction for the open string coupling constant λ_{os}. This prediction should be compared with the string theory calculation which in the rank 4 case reads [20]:

$$\lambda_{os} = \sqrt{\mathcal{D}}\,. \tag{46}$$

Instead, we find

$$\lambda_{os}^4 = (\sqrt{\mathcal{D}})^2 \times \left(\frac{1}{2}(\mathcal{D}_0 + 1) - \sqrt{\frac{1}{4}(\mathcal{D}_0 - 1)^2 + (\mathcal{D}_0 - \mathcal{D})}\right)\,. \tag{47}$$

The two expressions obviously do not coincide. In hindsight, the reason that the analysis of [36] only works for the rank 2 case is clear. Only in that case can we reduce along one of the spatial directions of the membrane yielding an open string. In a general rank 4 reduction one of the 3,4,5 transverse directions of the open membrane is involved in the reduction. For a mixed 1,2 (worldvolume) and 3,4,5 (tranverse) reduction we expect to obtain a bound state of an open string with an open D2–brane. Such a system is not described by the open string coupling constant λ_{os}.

The noncommutativity of an open string NCOS theory is described by a 2–form noncommutativity parameter θ^{ab}. Such a 2–form parameter naturally follows from the reduction of a 3–form open membrane parameter which we denote as the OM theta parameter. We expect that this 3–form parameter, if it exists, can teach us something about the geometry of the open membrane in the same way as the theta prameter of NCOS tells us about the noncommutative geometry of the underlying worldvolume.

One can show that the product $\hat{P}$ of the 3–form curvature $\hat{\mathcal{H}}$ and the Boillat metric $\hat{C}$ reduces, in an electric rank 2 reduction, to the NCOS theta parameter upto a conformal factor:

$$\hat{P}^{\hat{a}\hat{b}\hat{c}} = \hat{\mathcal{H}}^{\hat{a}\hat{b}\hat{d}}\hat{C}_{\hat{d}}{}^{\hat{c}} \qquad \Rightarrow$$

$$\theta^{ab} \overset{\text{el. rank 2}}{=} -z^3\,\hat{P}^{yab}\,. \tag{48}$$

The conformal factor z is defined in (44). Due to the presence of this conformal factor, this reduction does not reproduce the correct answer for a general rank 4 reduction for the same reasons as we explained in the OM metric case. However, it turns out that, in the rank 2 case, the choice of an OM theta parameter is not unique. There is another choice, called $\hat{W}$, which exactly, i.e. without a conformal factor, reduces to the NCOS theta parameter [37]. This choice is given by the 3–form curvature $\hat{\mathcal{H}}$ and *two* Boillat metrics:

$$\hat{W}^{\hat{a}\hat{b}\hat{c}} = \hat{\mathcal{H}}^{\hat{a}\hat{d}\hat{e}}\hat{C}_{\hat{d}}{}^{\hat{b}}\hat{C}_{\hat{e}}{}^{\hat{c}} \qquad \Rightarrow$$

$$\theta^{ab} \overset{\text{rank 2}}{=} \hat{W}^{yab}\,. \tag{49}$$

It has been shown that, unlike the $\hat{P}$–parameter, the $\hat{W}$ parameter has the correct scaling behaviour in the OM-theory limit [37]. It therefore seems that only $\hat{W}$ can be identified as the correct OM theta parameter.

In the same way that the 2–form theta parameter θ of NCOS describes a non-commutative geometry the existence of a natural 3–form OM theta parameter $\hat{W}$ suggests a relation with a *non–associative* geometry. In the same way that θ occurs in the string theory expression of the two–point functions the 3–form $\hat{W}$ naturally occurs in the expression of the OM three-point functions. Unfortunately, at the moment we are not able to calculate 3–point functions within OM theory.

4. Conclusions

In this lecture I have described three recent issues in M5–brane theory. Some recent progress is that the correct expressions for the OM–metric and the OM–theta parameter have been identified [36, 37, 38].

Much is still unknown about the M5–brane and the string boundary/OM theory living on its worldvolume. In this lecture I have described three open problems which, if we believe that M-theory exists, should have a natural resolution. The first problem deals with the formulation of a non-Abelian gauge theory of 2–forms on the M5–brane worldvolume. The second problem concerns which kind of non-commutatve geometry describes the worldvolume of N coinciding M5–branes. The third and last problem deals with the formulation of an open membrane theory on the M5–brane worldvolume.

What I have tried to indicate in this lecture is that, although we do not know the solutions to these three problems, it is very suggestive that the resolution requires the use of new (already existing as well as yet to be discovered) mathematics. It has been suggested in the recent literature that notions like “noncommutative loop space” [29], “non-associative geometry” [39] and “(non–Abelian) gerbes” [40] are expected to enter a proper mathematical formulation of the M5–brane.

I have given here a physicists point of view of how new mathematical notions are expected to enter a proper mathematical formulation of the M5–brane. Much further work needs to be done to arrive at a proper mathematical formulation. My hope is that the mathematicians at this workshop may get inspired by these problems and that they may motivate them to further investigate the mathematical aspects of these isssues.

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