THE BRAID STRUCTURE OF MAPPING CLASS GROUPS

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0. Introduction

It was shown by Stasheff([13]) and MacLane([7]) that monoidal categories give rise to loop spaces. A recognition principle specifies an internal structure such that a space X has such a structure if and only if X is of the weak homotopy type of n-fold loop space. It has been known for years that there is a relation between coherence problems in homotopy theory and in categories. May's recognition theorem([9]) states that for little n-cube operad $\mathcal{C}_n$, $n \geq 2$, every n-fold loop space is a $\mathcal{C}_n$-space and every connected $\mathcal{C}_n$-space has the weak homotopy type of an n-fold loop space.

E. Miller([9]) observed that there is an action of the little square operad on the disjoint union of $BDiff^+(S_{g,1})$'s extending the F-product which is induced by a kind of connected sum of surfaces. We hence have that the group completion of $\amalg_{g\geq 0} BDiff^+(S_{g,1})$ is a double loop space up to homotopy. Miller applied this result to the calculation of the homology groups of mapping class groups. However his description of the action of the little square operad is somewhat obscure. On the other hand the first author proved([4]) that the group completion of the nerve of a braided monoidal category is the homotopy type of a double loop space. This result implies that there exists a strong connection between braided monoidal category and the mapping class groups $\Gamma_{g,1}$ in view of Miller's result.

We, in this paper, show that the disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category with the product induced by the connected sum. Hence the group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is the homotopy type of a double loop space. We explicitly describe the braid structure of $\amalg_{g\geq 0}\Gamma_{g,1}$, regarding $\Gamma_{g,1}$ as the subgroup of the automorphism group of $\pi_1 S_{g,1}$ that consists of the automorphisms fixing the fundamental relator. We provide the formula for the braiding (Lemma 2.1) which is useful in dealing with the related problems. Using this braiding formula (2.2), we can make a correction on Cohen's diagram. We also show that the double loop space structure of the disjoint union of classifying spaces of mapping class groups cannot be extended to the triple loop space structure (Theorem 2.5). It seems important to note the relation between the braid structure and the double loop space structure in an explicit way.

Turaev and Reshetikhin introduced an invariant of ribbon graphs which is derived from the theory of quantum groups and is a generalization of Jones polynomial. This invariant was extended to those of 3-manifolds and of mapping class groups(cf.[11],[12],[6]). The definitions are abstract and a little complicated since they are defined through quantum groups. G.

The second author was partially supported by GARC-KOSEF.

Wright([16]) computed the Reshetikhin-Turaev invariant of mapping class group explicitly in the case $r = 4$, that is, at the sixteenth root of unity. For each $h \in \Gamma_{g,0}$ we can find the corresponding (colored) ribbon graph, whose Reshetikhin-Turaev invariant turns out to be an automorphism of the 1-dimensional summand of $V^{k_1} \otimes V^{k_1 *} \otimes \cdots \otimes V^{k_g} \otimes V^{k_g *}$ which we denote by $V_{r,g}$. We get this ribbon graph using the Heegaard splitting and the surgery theory of 3-manifolds. Wright showed as a result of her calculation that the restriction of this invariant to the Torelli subgroup of $\Gamma_{g,0}$ is equal to the sum of the Birman-Craggs homomorphisms. $\dim(V_{4,g}) = 2^{g-1}(2^g + 1)$, so the Reshetikhin-Turaev invariant of $h \in \Gamma_{g,0}$, when $r = 4$, is a $2^{g-1}(2^g+1) \times 2^{g-1}(2^g+1)$ matrix with entries of complex numbers. Wright proved a very interesting lemma that there is a natural one-to-one correspondence between the basis vectors of $V_{4,g}$ and the $\mathbb{Z}/2$-quadratic forms of Arf invariant zero. It would be interesting to check if the Reshetikhin-Turaev representation preserves the braid structure.

1. Mapping class groups and monoidal structure

Let $S_{g,k}$ be an orientable surface of genus g obtained from a closed surface by removing k open disks. The mapping class group $\Gamma_{g,k}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms of $S_{g,k}$ fixing the boundary of $S_{g,k}$ that consists of k disjoint circles. Let $Diff^+(S_{g,k})$ be the group of orientation preserving self-diffeomorphisms of $S_{g,k}$. We also have the following alternative definition :

$$\Gamma_{g,k} = \pi_o Diff^+(S_{g,k})$$

We will mainly deal with the case $k = 1$ and $k = 0$. $\Gamma_{g,1}$ and $\Gamma_{g,0}$ are generated by $2g+1$ Dehn twists(cf.[14]). There is a surjective map $\Gamma_{g,1} \to \Gamma_{g,0}$.

Figure 1. Dehn twists

Many topologists are interested in the homology of mapping class groups. An interesting observation is that there is a product on the disjoint union of $Diff^+(S_{g,1})$'s. It is known by Stasheff([13]) and MacLane([7]) that if a category $\mathcal{C}$ has a monoidal structure then its classifying space gives rise to a space which has the homotopy type of a loop space. Fiedorowicz showed([4]) that a braid structure gives rise to a double loop space structure. We now recall the definition of (strict) braided monoidal category.

Definition 1.1. A *(strict) monoidal (or tensor) category* $(\mathcal{C}, \otimes, E)$ is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *product*) and an object E (called the *unit object*) satisfying

(a) $\otimes$ is strictly associative
(b) E is a strict 2-sided unit for $\otimes$

Definition 1.2. A monoidal category $(\mathcal{C}, \otimes, E)$ is called a *(strict) braided monoidal category* if there exists a natural commutativity isomorphism $C_{A,B}$: $A\otimes B \to B\otimes A$ satisfying

(c) $C_{A,E} = C_{E,A} = 1_A$
(d) The following diagrams commute:

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "C_{A\otimes B,C}"] \arrow[dr, "1_A\otimes C_{B,C}"'] & & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "C_{A,C}\otimes 1_B"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "C_{A,B\otimes C}"] \arrow[dr, "C_{A,B}\otimes 1_C"'] & & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "1_B\otimes C_{A,C}"'] &
\end{tikzcd}$$

The first author recently gave a proof of the following lemma([4]).

Lemma 1.3. *The group completion of the nerve of a braided monoidal category is the homotopy type of a double loop space. The converse is true.*

Miller claimed in [10] that there is an action of the little square operad of disjoint squares in D^2 on the disjoint union of the $B\Gamma_{g,1}$'s extending the F-product that is induced by the connected sum. Here the F-product $\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$ is obtained by attaching a pair of *pants* (a surfaces obtained from a sphere by removing three open disks) to the surfaces $S_{g,1}$ and $S_{h,1}$ along the fixed boundary circles and extending the identity map on the boundary to the whole pants. Hence, according to May's recognition theorem on the loop spaces([9]), the group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is homotopy equivalent to a double loop space. Miller's proposition seems correct, although the details are not so transparent. In view of lemma 1.3, the disjoint union of $\Gamma_{g,1}$'s should be related to a braided monoidal category. Here we regard $\amalg_{g\geq 0}\Gamma_{g,1}$ as a category whose objects are $[g]$, $g \in \mathbb{Z}_+$ and morphisms satisfy

$$Hom([g],[h]) = \begin{cases} \Gamma_{g,1} & \text{if } g = h \\ \emptyset & \text{if } g \neq h \end{cases}$$

Without speaking of the action of little square operad, we are going to show that the group completion of $\amalg_{g\geq 0}B\Gamma_{g,1}$ is homotopy equivalent to a double loop space by showing that the disjoint union of $\Gamma_{g,1}$'s is a braiding monoidal category.

Lemma 1.4. *The disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category with the product induced by the F-product.*

Proof. Let $x_1, y_1, \ldots, x_g, y_g$ be generators of the fundamental group of $S_{g,1}$ which are induced by the Dehn twists $a_1, b_1, \cdots, a_g, b_g$, respectively. The mapping class group $\Gamma_{g,1}$ can be identified with the subgroup of the automorphism group of the free group on $x_1, y_1, \cdots, x_g, y_g$ that consists of the automorphisms fixing the fundamental relator $R = [x_1, y_1][x_2, y_2]\cdots[x_g, y_g]$. The binary operation on $\amalg_{g\geq 1}\Gamma_{g,1}$ induced by the F-product can be identified with the operation taking the free product of the automorphisms. The (r, s)-braiding on the free group on $x_1, y_1, \cdots, x_g, y_g$ can be expressed by:

$$\begin{aligned}
x_1 &\longmapsto x_{s+1}\\
y_1 &\longmapsto y_{s+1}\\
&\vdots\\
x_r &\longmapsto x_{s+r}\\
y_r &\longmapsto y_{s+r}\\
x_{r+1} &\longmapsto S^{-1}x_1S\\
y_{r+1} &\longmapsto S^{-1}y_1S\\
&\vdots\\
x_{r+s} &\longmapsto S^{-1}x_sS\\
y_{r+s} &\longmapsto S^{-1}y_sS
\end{aligned}$$

where $S = [x_{s+1}, y_{s+1}][x_{s+2}, y_{s+2}]\cdots[x_{s+r}, y_{s+r}]$.
It is easy to see that the (r, s)-braiding fixes the fundamental relator R
Moreover, the (r, s)-braiding makes the diagrams in (d) of Definition 1.2 commute. □

Lemma 1.4 explains the pseudo double loop space structure on the union of the classifying spaces of the mapping class groups observed by E. Miller. Lemma 1.3 and Lemma 1.4 imply the following:

Theorem 1.5. *The group completion of $\amalg_{g\geq 0}B\Gamma_{g,1}$ is the homotopy type of a double loop space.*

2. Braid structure

Let B_n denote Artin's braid group. B_n has $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ and is specified by the following presentation:

$$\begin{aligned}
\sigma_i\sigma_j &= \sigma_j\sigma_j && \text{if } |i-j|\geq 2\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1, \cdots, n-2
\end{aligned}$$

It has been observed for many years that there are certain connections between the braid groups and the mapping class groups. In this section we introduce a new kind of braid structure in the mapping class groups $\Gamma_{g,1}$'s in an explicit form. This explicit expression enables us to deal with a kind of Dyer-Lashof operation (or Browder operation) in an explicit form. It seems possible for us to get further applications of the formula of the braid structure in the future. First let us express explicitly the (1,1)-braiding on genus 2 surface. $\Gamma_{2,1}$ is generated by the Dehn twists $a_1, b_1, a_2, b_2, \omega_1$. Let x_1, y_1, x_2, y_2 be generators of $\pi_1 S_{g,1}$ which are induced by a_1, b_1, a_2, b_2, respectively. Regard $a_1, b_1, a_2, b_2, \omega_1$ as automorphisms on $F_{\{x_1,y_1,x_2,y_2\}}$. Then we have

$$\begin{aligned}
a_1 &: y_1 \longmapsto y_1 x_1^{-1} \\
b_1 &: x_1 \longmapsto x_1 y_1 \\
a_2 &: y_2 \longmapsto y_2 x_2^{-1} \\
b_2 &: x_2 \longmapsto x_2 y_2 \\
\omega_1 &: x_1 \longmapsto x_1 [x_2, y_2] x_2^{-1} x_1 x_2 [y_2, x_2] x_1^{-1} \\
&\quad\; y_1 \longmapsto x_1 [x_2, y_2] x_2^{-1} x_1^{-1} x_2 [y_2, x_2] y_1 x_2 [y_2, x_2] x_1^{-1} \\
&\quad\; y_2 \longmapsto x_2^{-1} x_1 x_2 y_2 x_2^{-1}
\end{aligned}$$

These automorphisms fix the generators that do not appear in the above list.

The (1,1)-braiding in genus 2 should be expressed in terms of the elements $a_1, b_1, a_2, b_2, \omega_1$ and should be specified on the generators of $\pi_1 S_{g,1}$ by the formulas:

$$\begin{aligned}
x_1 &\longmapsto x_2 \\
y_1 &\longmapsto y_2 \\
x_2 &\longmapsto [y_2, x_2] x_1 [x_2, y_2] \\
y_2 &\longmapsto [y_2, x_2] y_1 [x_2, y_2]
\end{aligned}$$

We need a hard calculation to get such a braiding. By using a computer program, we could get the following explicit formula for the braid structure.

Lemma 2.1. *The (1,1)-braiding for the monoidal structure in genus 2 is given by*

$$\beta_1 = (b_1 a_1 a_1 b_1 a_1 \omega_1 (a_1 b_1 a_1)^{-1} b_2 a_2)^{-3} (a_1 b_1 a_1)^4 \tag{2.2}$$

The braid group of all braidings in the mapping class group of genus g is generated by

$$\beta_i = (b_i a_i a_i b_i a_i \omega_i (a_i b_i a_i)^{-1} b_{i+1} a_{i+1})^{-3} (a_i b_i a_i)^4 \tag{2.3}$$

for $i = 1, 2, \cdots, g-1$. We can obtain the following formula for the (r,s)-braiding in terms of the braiding generators:

$$(\beta_r \beta_{r+1} \cdots \beta_{r+s-1})(\beta_{r-1} \beta_r \cdots \beta_{r+s-2}) \cdots (\beta_1 \beta_2 \cdots \beta_s)$$

or alternatively as

$$(\beta_r\beta_{r-1}\cdots\beta_1)(\beta_{r+1}\beta_r\cdots\beta_2)\cdots(\beta_{r+s-1}\beta_{r+s-2}\cdots\beta_s)$$

Remark 2.4 The braid structure gives rise to the double loop space structure, so it is supposed to be related to the Dyer-Lashof operation. Let $D : B_{2g} \longrightarrow \Gamma_{g,1}$ be the obvious map given by

$$D(\sigma_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ \omega_{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases}$$

F. Cohen in [3] dealt with this map D. He said that the homology homomorphism D_* induced by D is trivial, because D preserves the Dyer-Lashof operation. Precisely speaking, he made a commutative diagram

$$\begin{array}{ccc} B_p \int B_{2g} & \xrightarrow{\theta'} & B_{2pg} \\ {\scriptstyle B_p \int D}\Big\downarrow & & \Big\downarrow{\scriptstyle D} \\ B_p \int \Gamma_{g,1} & \xrightarrow{\theta} & \Gamma_{pg,1} \end{array}$$

where θ is the analogue of the Dyer-Lashof operation (it should be rather Browder operation). According to his definition, $(\sigma; 1, 1) \in B_2 \int \Gamma_{1,1}$ is mapped by θ to $\omega_1 b_2 b_1 \omega_1$. His definition of θ, however, is not well-defined. This can be detected by mapping $\Gamma_{2,1}$ to $Sp(4;\mathbb{Z})$. Here $Sp(4;\mathbb{Z})$ is the automorphism group of $H_1(S_{g,1};\mathbb{Z})$. The map $\phi : \Gamma_{2,1} \to Sp(4;\mathbb{Z})$ is described as follows:

$$a_1 \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b_1 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\omega_1 \to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The map ϕ sends $\omega_1 b_2 b_1 \omega_1$ to $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

We have

$$(\sigma_1; 1, 1)^{-1}(1; a_1, 1)(\sigma_1; 1, 1) = (1; 1, a_1) = (\sigma_1; 1, 1)(1; a_1, 1)(\sigma_1; 1, 1)^{-1}$$

This element must commute with $(1; a_1, 1)$. $(\sigma_1; 1, 1)^{-1}(1; a_1, 1)(\sigma_1; 1, 1)$ is mapped to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $(\sigma_1; 1, 1)(1; a_1, 1)(\sigma_1; 1, 1)^{-1}$ is mapped to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. However neither of these two matrices commutes with $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which corresponds to a_1.

The braiding structure (2.2) plays a key role in the correct formula for θ which should be the following:

$$(\sigma_1; 1, 1) \overset{\theta}{\longmapsto} (b_1 a_1 a_1 b_1 a_1 \omega_1 (a_1 b_1 a_1)^{-1} b_2 a_2)^{-3} (a_1 b_1 a_1)^4$$

Let $\mathcal{C}_n$ be the little n-cube operad. Let Y be an n-fold loop space. Then Y is a $\mathcal{C}_n$-space, so there is a map

$$\mathcal{C}_n(2) \times Y^2 \longrightarrow Y$$

It is known that $\mathcal{C}_n(2)$ has the same homotopy type as S^{n-1}. Hence the above map induces a homology operation

$$H_i(Y) \otimes H_j(Y) \longrightarrow H_{i+j+n-1}(Y)$$

which is called the Browder operation. It is easy to see that if Y is a $\mathcal{C}_{n+1}$-space, then the Browder operation equals zero.

Let X be the group completion of $\amalg_{g \geq 0} B\Gamma_{g,1}$. Since X is homotopy equivalent to a Ω^2-space, it is, up to homotopy, a $\mathcal{C}_2$-space. It is natural to raise the question whether X is a $\mathcal{C}_3$-space, or not. The answer is negative. In the proof of the following theorem the braid formula (2.2) again plays a key role.

Theorem 2.5. *Let X be the group completion of $\amalg_{g \geq 0} B\Gamma_{g,1}$. The double loop space structure cannot be extended to the triple loop space structure.*

Proof. We show that the Browder operation

$$\theta_* : H_i(X) \otimes H_j(X) \longrightarrow H_{i+j+1}(X)$$

is nonzero for X. We have the map

$$\phi : \mathcal{C}_2(2) \times X^2 \longrightarrow X$$

Note that $\mathcal{C}_2(2)$ has the same homotopy type as S^1. By restricting the map ϕ to each connected component we get

$$S^1 \times B\Gamma_{g,1} \times B\Gamma_{g,1} \longrightarrow B\Gamma_{2g,1}$$

This map is, in the group level, denoted by the map

$$\theta : B_2 \int \Gamma_{g,1} \longrightarrow \Gamma_{2g,1}$$

which is same as described in Remark 2.4. In order to show that θ_* is nonzero it suffices to show that

$$\tilde{\theta_*} : H_0(B\Gamma_{1,1}) \otimes H_0(B\Gamma_{1,1}) \longrightarrow H_1(B\Gamma_{2,1})$$

is nonzero. The image of the map $\tilde{\theta_*}$ equals the image of the homology homomorphism $\alpha : H_1(S^1) \to H_1(B\Gamma_{2,1})$ induced by the map $S^1 \to B\Gamma_{2,1}$ which is the restriction of the map $S^1 \times B\Gamma_{1,1} \times B\Gamma_{1,1} \to B\Gamma_{2,1}$. The map α sends the generator of $H_1(S^1)$ to the abelianization class of

$$(b_1 a_1 a_1 b_1 a_1 \omega_1 (a_1 b_1 a_1)^{-1} b_2 a_2)^{-3} (a_1 b_1 a_1)^4$$

which is nonzero, since the isomorphism $H_1(\ ) \cong (\ )_{ab}$ is natural. □

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