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SUBLINEAR AND CONTINUOUS ORDER-PRESERVING FUNCTIONS FOR NONCOMPLETE PREORDERS

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ABSTRACT. We characterize the existence of a nonnegative, sublinear and continuous order-preserving function for a not necessarily complete preorder on a real convex cone in an arbitrary topological real vector space. As a corollary of the main result, we present necessary and sufficient conditions for the existence of such an order-preserving function for a complete preorder.

1. Introduction

Necessary and sufficient conditions for the existence of a continuous linear order-preserving function for a complete preorder on a topological real vector space are already found in the literature (see e.g. Candeal and Induráin [10] and Neuefeind and Trockel [17]). It is well known that there are important applications of such results in expected utility theory and collective decision making. A characterization of the existence of a continuous linear utility function for a complete preorder on a convex set in a normed real vector space was presented by Bültel [7]. Further, some authors were concerned with the existence of a homogeneous of degree one and continuous order-preserving function for a complete preorder on a real cone in a topological real vector space (see e.g. Bosi [2], Bosi, Candeal and Induráin [3], and Dow and Werlang [13]). More recently, Bosi and Zuanon [5] presented a characterization of the existence of a nonnegative, homogeneous of degree one and continuous order-preserving function for a noncomplete preorder on a real cone in a topological real vector space. In a different context, other authors were concerned with the existence of an additive order-preserving function on a completely preordered semigroup (see e.g. Allevi and Zuanon [1], and Candeal, de Miguel and Induráin [9]). Bosi and Zuanon [6] presented a characterization of the existence of a Choquet integral representation for

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a complete preorder on the space of all continuous real-valued functions on a compact topological space.

In this paper we provide an axiomatization of the existence of a nonnegative, sublinear and continuous order-preserving function for a not necessarily complete preorder on a real convex cone in a topological real vector space. All the results we present are based upon the notion of a *decreasing scale* (or *linear separable system*) which was introduced by Herden [14, 15] in order to characterize the existence of a continuous order-preserving function for a preorder on a topological space (see also Burgess and Fitzpatrick [8], and Mehta [16]).

It should be noted that such a topic can be of some interest in the applications to economics. Consider the following example concerning decision theory under uncertainty. Let $\mathbf{M} = \{\mu_n : n \in \{1, \ldots, n^*\}\}$ be a finite family of *concave capacities* on a measurable space $(\Omega, \mathcal{A})$, with Ω the *state space*, and $\mathcal{A}$ a σ-*algebra* of subsets of Ω. We recall that a *capacity* μ on $\mathcal{A}$ (i.e., a function from $\mathcal{A}$ into $[0,1]$ such that $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\mu(A) \leq \mu(B)$ for all $A \subseteq B$, $A, B \in \mathcal{A}$) is said to be concave if for all sets $A, B \in \mathcal{A}$,

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

(see e.g. Chateauneuf [11]). Let X be a *real convex cone* of nonnegative *real random variables* (i.e., measurable real functions) on $(\Omega, \mathcal{A})$, and assume that X is contained in $L^1(\Omega, \mathcal{A}, \mu_n)$ for every $n \in \{1, \ldots, n^*\}$, where $L^1(\Omega, \mathcal{A}, \mu)$ stands for the normed space of all the real random variables x such that the *Choquet integral*

$$\int_\Omega x d\mu = \int_0^\infty \mu(\{x \geq t\}) dt + \int_{-\infty}^0 (\mu(\{x \geq t\}) - 1) dt$$

is finite (see e.g. Denneberg [12]). Define a binary relation $\precsim$ on X as follows:

$$x \precsim y \text{ if and only if } \int_\Omega x d\mu_n \leq \int_\Omega y d\mu_n \text{ for all } n \in \{1, \ldots, n^*\}.$$

It is clear that $\precsim$ is a preorder on X, and that $\precsim$ is not complete in general. For every $n \in \{1, \ldots, n^*\}$, denote by τ_n the norm topology on X which is associated to μ_n, and let τ be any (vector) topology on X which is stronger than τ_n for all $n \in \{1, \ldots, n^*\}$ (i.e., $\tau_n \subseteq \tau$ for all $n \in \{1, \ldots, n^*\}$). Then the real-valued function u on X defined by

$$u(x) = \sum_{n=1}^{n^*} \int_\Omega x d\mu_n \quad (x \in X)$$

is a nonnegative, sublinear and τ-continuous *order-preserving function* for $\precsim$. Indeed, it is clear that $x \precsim y$ implies $u(x) \leq u(y)$ for all $x, y \in X$. If $x \prec y$, then we have $u(x) < u(y)$, since $\int_\Omega x d\mu_n \leq \int_\Omega y d\mu_n$ for all $n \in \{1, \ldots, n^*\}$, and there exists at least one index $\bar{n} \in \{1, \ldots, n^*\}$ such that $\int_\Omega x d\mu_{\bar{n}} < \int_\Omega y d\mu_{\bar{n}}$. Further, u is sublinear since the functional $x \to \int_\Omega x d\mu_n$ is sublinear for all $n \in \{1, \ldots, n^*\}$. Finally, u is τ-continuous since the functional $x \to \int_\Omega x d\mu_n$ is τ_n-continuous, and therefore τ-continuous for all $n \in \{1, \ldots, n^*\}$ (see Denneberg [12, Proposition 9.4]).

2. Notation and preliminaries

A *preorder* $\preceq$ on an arbitrary set X is a reflexive and transitive binary relation on X. The *strict part* and the *symmetric part* of a given preorder $\preceq$ will be denoted by $\prec$, and respectively $\sim$. A preorder $\preceq$ on a set X is said to be *complete* if for any two elements $x, y \in X$ either $x \preceq y$ or $y \preceq x$.

If $\preceq$ is a preorder on a set X, then the pair $(X, \preceq)$ will be referred to as a *preordered set*. Define, for every $x \in X$, $L_\prec(x) = \{z \in X : z \prec x\}$, $U_\prec(x) = \{z \in X : x \prec z\}$.

Given a preordered set $(X, \preceq)$, a real-valued function u on X is said to be

(1) *increasing* if $u(x) \leq u(y)$ for every $x, y \in X$ such that $x \preceq y$;
(2) *order-preserving* if it is increasing and $u(x) < u(y)$ for every $x, y \in X$ such that $x \prec y$.

If $(X, \preceq)$ is a preordered set, and τ is a topology on X, then the triple $(X, \tau, \preceq)$ will be referred to as a *topological preordered space*. If $(X, \tau, \preceq)$ is a topological completely preordered space, then the complete preorder $\preceq$ is said to be *continuous* if $L_\prec(x)$ and $U_\prec(x)$ are open subsets of X for every $x \in X$.

Given a preordered set $(X, \preceq)$, a subset A of X is said to be *decreasing* if $y \in A$ whenever $y \preceq x$ and $x \in A$.

In the sequel, the symbol $\mathbb{Q}^{++}$ ($\mathbb{R}^{++}$) will stand for the set of all positive rational (real) numbers. If (X, τ) is a topological space, then denote by $\overline{A}$ the topological closure of any subset A of X.

We say that a family $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ is a *countable decreasing scale* (*countable linear separable system*) in a topological preordered space $(X, \tau, \preceq)$ if

(1) G_r is an open decreasing subset of X for every $r \in \mathbb{Q}^{++}$;
(2) $\overline{G_{r_1}} \subseteq G_{r_2}$ for every $r_1, r_2 \in \mathbb{Q}^{++}$ such that $r_1 < r_2$;
(3) $\bigcup_{r \in \mathbb{Q}^{++}} G_r = X$.

If E is a real vector space, then define, for every subset A of E and any real number t, $tA = \{ta : a \in A\}$. Further, if A and B are any two subsets of a (real) vector space E, then define $A + B = \{a + b : a \in A, b \in B\}$.

A subset X of a *real vector space* E is said to be

(1) a *real cone* if $tx \in X$ for every $x \in X$ and $t \in \mathbb{R}^{++}$;
(2) a *real convex cone* if it is a real cone and $x + y \in X$ for every $x, y \in X$.

A real-valued function u on a real cone X in a real vector space E is said to be *homogeneous of degree one* if $u(tx) = tu(x)$ for every $x \in X$ and $t \in \mathbb{R}^{++}$.

A real-valued function u on a real convex cone X in a real vector space E is said to be *sublinear* if it is homogeneous of degree one and *subadditive* (i.e., $u(x+y) \leq u(x) + u(y)$ for every $x, y \in X$).

Given a *topological real vector space* E, denote by τ the vector topology for E (i.e., the topology on E which makes the vector operations continuous).

If X is any subset of a topological real vector space E, denote by τ_X the topology induced on X by the vector topology τ on E.

If $(X, \preceq)$ is a preordered real cone in a topological real vector space E, then we say that a countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ in $(X, \tau_X, \preceq)$ is *homogeneous* if $qG_r = G_{qr}$ for every $q, r \in \mathbb{Q}^{++}$.

If $(X, \preceq)$ is a preordered real convex cone in a topological real vector space E, then we say that a countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ in $(X, \tau_X, \preceq)$ is *subadditive* if $G_q + G_r \subseteq G_{q+r}$ for every $q, r \in \mathbb{Q}^{++}$.

3. Existence of a sublinear continuous order-preserving function

In the following theorem we characterize the existence of a nonnegative, sublinear and continuous order-preserving function for a not necessarily complete preorder on a real convex cone in a topological real vector space.

Theorem 3.1. *Let $\preceq$ be a preorder on a real convex cone X in a topological real vector space E. Then the following conditions are equivalent:*

(1) *There exists a nonnegative, sublinear and continuous order-preserving function u for $\preceq$.*
(2) *There exists a homogeneous and subadditive countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ in $(X, \tau_X, \preceq)$ such that for every $x, y \in X$ with $x \prec y$ there exist $r_1, r_2 \in \mathbb{Q}^{++}$ with*

$$r_1 < r_2, x \in G_{r_1} \text{ and } y \notin G_{r_2}.$$

Proof. (1) $\Rightarrow$ (2). Assume that there exists a nonnegative, sublinear and continuous order-preserving function u for $\preceq$. Define $G_r = u^{-1}([0, r[)$ for every $r \in \mathbb{Q}^{++}$. Let us show that $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ is a homogeneous and subadditive countable decreasing scale satisfying condition (2). Using the fact that u is nonnegative and order-preserving, we have that for every $x, y \in X$ such that $x \prec y$, there exist $r_1, r_2 \in \mathbb{Q}^{++}$ such that $u(x) < r_1 < r_2 < u(y)$, and therefore $x \in G_{r_1}$, $y \notin G_{r_2}$. Further, since u is homogeneous of degree one,

$$qG_r = qu^{-1}([0, r[) = u^{-1}([0, qr[) = G_{qr} \text{ for every } q, r \in \mathbb{Q}^{++}.$$

Hence, $\mathcal{G}$ is homogeneous. It only remains to show that $\mathcal{G}$ is subadditive. To this aim, consider any two rational numbers $q, r \in \mathbb{Q}^{++}$, and let $z \in G_q + G_r$. Then there exist two elements $x, y \in X$ such that $z = x + y$, $u(x) < q$, $u(y) < r$. Hence, using the fact that u is subadditive, we have $u(z) = u(x+y) \leq u(x) + u(y) < q + r$, and therefore $z \in G_{q+r}$.

(2) $\Rightarrow$ (1). Define, for every $x \in X$,

$$u(x) = \inf\{r \in \mathbb{Q}^{++} : x \in G_r\}.$$

Then u is a nonnegative continuous order-preserving function for $\preceq$. Indeed, by using the fact that G_r is a decreasing set for every $r \in \mathbb{Q}^{++}$, it is easily seen that u is increasing. Further, u is order-preserving by condition (2) above, and u is continuous since $u(x) = \inf\{r \in \mathbb{Q}^{++} : x \in \overline{G_r}\}$ (see e.g. Theorem 1 in Bosi and Mehta [4] for details). In order to show that u is homogeneous of degree one, it suffices to prove that for no $r \in \mathbb{Q}^{++}$, and $x \in X$ it is $u(rx) \neq ru(x)$. Then the thesis follows by a standard continuity argument. This part of the proof is already found in Bosi and Zuanon [5, Theorem 1]. Nevertheless, we present all the details here for reader's convenience. By contradiction, assume that there exist $r \in \mathbb{Q}^{++}$, and $x \in X$ such that $u(rx) < ru(x)$. Then, from the definition of u, there exists $r' \in \mathbb{Q}^{++}$ such that $u(rx) < r' < ru(x)$, $rx \in G_{r'}$. Since $u(x) > \frac{r'}{r}$, it follows that $x \notin G_{\frac{r'}{r}} = \frac{1}{r}G_{r'}$, and therefore we arrive at the contradiction $rx \notin G_{r'}$. Analogously it can be shown that for no $r \in \mathbb{Q}^{++}$, and $x \in X$ it is $ru(x) < u(rx)$. It remains to prove that u is subadditive. By contradiction, assume that there exist two elements $x, y \in X$ such that $u(x) + u(y) < u(x + y)$. Then, from the definition of u, there exist two rational numbers $q, r \in \mathbb{Q}^{++}$ such that $u(x) + u(y) < q + r < u(x + y)$, $x \in G_q$, $y \in G_r$. Since the countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ is subadditive, we have $x + y \in G_{q+r}$, and therefore $q + r < u(x + y)$ is contradictory from the definition of u. This consideration completes the proof. □

If $\preceq$ is a *homothetic* complete preorder on a real cone X in a real vector space E (i.e., $x \preceq y$ entails $tx \preceq ty$ for every $x, y \in X$, and $t \in \mathbb{R}^{++}$), then consider the following subcones of X:

$$X_0 = \{x \in X : x \sim tx \text{ for some positive real number } t \neq 1\};$$

$$X_+ = \{x \in X : x \prec tx \text{ for some real number } t > 1\};$$

$$X_- = \{x \in X : tx \prec x \text{ for some real number } t > 1\}.$$

In the following corollary, we present a characterization of the existence of a nonnegative, sublinear and continuous order-preserving function for a complete preorder on a real convex cone in a topological real vector space. We recall that, given a preordered set $(X, \preceq)$, a subset A of X is said to be an *order-dense subset of* $(X, \preceq)$ if for every $x, y \in X$ such that $x \prec y$ there exists $a \in A$ such that $x \prec a \prec y$.

Corollary 3.2. *Let $\preceq$ be a complete preorder on a real convex cone X in a topological real vector space E, and assume that X_0 and X_+ are both nonempty, while X_- is empty. Then the following conditions are equivalent:*

(1) *There exists a nonnegative, sublinear and continuous order-preserving function u for $\preceq$.*
(2) *The following conditions are satisfied:*

(a) $\preceq$ *is homothetic;*
(b) $\preceq$ *is continuous;*
(c) *The set* $\{qx_+ : q \in \mathbb{Q}^{++}\}$ *is an order-dense subset of* $(X_+, \preceq)$ *for every element* $x_+ \in X_+$*;*
(d) $x \sim y$ *for every* $x, y \in X_0$*;*
(e) $x \prec x_+$ *for every* $x \in X_0$, $x_+ \in X_+$*;*
(f) $x + y \prec (q + r)x_+$ *for every* $x, y \in X$, $x_+ \in X_+$, $q, r \in \mathbb{Q}^{++}$ *such that* $x \prec qx_+$, $y \prec rx_+$.

Proof. (1) $\Rightarrow$ (2). Assume that there exists a nonnegative, sublinear and continuous order-preserving function u for $\preceq$. Then it is clear that $\preceq$ is homothetic and continuous. If we consider any element $x_+ \in X$ such that $x_+ \prec tx_+$ for some real number $t > 1$, then it is necessarily $u(x_+) > 0$, and therefore, using the fact that u is homogeneous of degree one, condition (2c) easily follows. Finally, it is easily seen that conditions (2d), (2e) and (2f) are verified.

(2) $\Rightarrow$ (1). Consider any element $x_+ \in X$ such that $x_+ \prec tx_+$ for some real number $t > 1$, and let $G_r = L_{\prec}(rx_+)$ for every $r \in \mathbb{Q}^{++}$. Then it is easy to check that the family $\mathcal{G} = \{G_r : r \in \mathbb{Q}^{++}\}$ is a countable decreasing scale satisfying condition (2) of Theorem 3.1. Indeed, by homotheticity of $\preceq$, $x \prec rx_+$ is equivalent to $qx \prec qrx_+$ ($q \in \mathbb{Q}^{++}$), and therefore $\mathcal{G}$ is homogeneous (see Bosi and Zuanon [5, Corollary 2]). Further, $\mathcal{G}$ is subadditive by condition (2f) since, for every $q, r \in \mathbb{Q}^{++}$, and $x, y \in X$, $x \in G_q$ and $y \in G_r$ is equivalent to $x \prec qx_+$ and $y \prec rx_+$, which implies $x + y \prec (q + r)x_+$ or equivalently $x + y \in G_{q+r}$. Finally, by condition (2c) above, for every $x, y \in X$ such that $x \prec y$ there exist $r_1, r_2 \in \mathbb{Q}^{++}$ such that

$$r_1 < r_2, \quad x \prec r_1x_+ \prec r_2x_+ \prec y,$$

or equivalently

$$x \in L_{\prec}(r_1x_+), \quad y \notin L_{\prec}(r_2x_+).$$

So the proof is complete. □

Denote by $\bar{0}$ the *zero vector* in a real vector space E. In the following corollary we are concerned with a sublinear representation of a complete preorder on a real convex cone containing the zero vector.

Corollary 3.3. *Let* $\preceq$ *be a complete preorder on a real convex cone* X *in a topological real vector space* E*, and assume that* X_+ *is nonempty, while* X_- *is empty. If in addition* $\bar{0}$ *belongs to* X*, then there exists a nonnegative, sublinear and continuous order-preserving function* u *for* $\preceq$ *if and only if* $\preceq$ *is homothetic and continuous, and it satisfies condition (2f) of Corollary 3.2.*

Proof. From the corollary in Bosi, Candeal and Induráin [3], there exists a nonnegative, homogeneous of degree one and continuous order-preserving function u for $\preceq$. Indeed, the complete preorder $\preceq$ on the real (convex) cone X is homothetic and continuous, and we have in addition $\bar{0} \in X$. Let us show that u must be subadditive as a consequence of condition (2f) of Corollary 3.2. Assume by contraposition that there exist $x, y \in X$ such that $u(x) + u(y) < u(x + y)$, and consider any element $x_+ \in X_+$. Then it must be $u(x_+) > 0$ since u is a homogeneous of degree one utility function for $\preceq$, and there exist two positive rational numbers q and r such that

$$u(x) + u(y) < (q + r)u(x_+) < u(x + y), \quad x \prec qx_+, \quad y \prec rx_+.$$

But here we have a contradiction since it should be $u(x + y) < (q + r)u(x_+)$ by condition (2f) of Corollary 3.2. So the proof is complete. $\square$

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