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ON TYCHONOFF-TYPE HYPERTOPOLOGIES

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Abstract. In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space, similar to the *Tychonoff topology* but weaker than it. In 1998, G. Dimov and D. Vakarelov [7] used a generalized version of this new topology, calling it *Tychonoff-type topology.* The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family $\mathcal{M}$ of subsets of a set X. When $\mathcal{M}$ contains all singletons, a description of all Tychonoff-type topologies $\mathcal{O}$ on $\mathcal{M}$ is given. The continuous maps of a special form between spaces of the type $(\mathcal{M}, \mathcal{O})$ are described in an isomorphism theorem. The problem of *commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology* is investigated as well. Some topological properties of the hyperspaces $(\mathcal{M}, \mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$ are briefly discussed.

1. Introduction

In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space for obtaining a generalization of the famous Kolmogoroff Theorem on operations on sets. This new topology is similar to the *Tychonoff topology* (known also as *upper Vietoris topology*, or *upper semi-finite topology* ([13]), or *kappa-topology*) but is weaker than it. In 1998, G. Dimov and D. Vakarelov [7] used a generalized version of this new topology for proving an isomorphism theorem for the category of all Tarski consequence systems. This generalized version was called *Tychonoff-type topology.*

The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family $\mathcal{M}$ of subsets of a set X. When $\mathcal{M}$ is a *natural*

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family, i.e. it contains all singletons, a description of all Tychonoff-type topologies $\mathcal{O}$ on $\mathcal{M}$ is given (see Proposition 2.32). For doing this, the notion of *T-space* is introduced. The natural morphisms for T-spaces are not enough to describe all continuous maps between spaces of the type $(\mathcal{M}, \mathcal{O})$, where $\mathcal{M}$ is a natural family and $\mathcal{O}$ is a Tychonoff-type topology on it; we obtain a characterization of those continuous maps which correspond to the morphisms between T-spaces. This is done by defining suitable categories and by proving that these categories are isomorphic (see Theorem 2.37). In such a way we extend to any natural family $\mathcal{M}$ on X the corresponding result obtained in [7] for the family $\mathcal{F}in(X)$ of all finite subsets of X. We investigate also the problem of *commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology*, i.e. when the hyperspace of any subspace A of a topological space Y is canonically representable as a subspace of the hyperspace of Y. Such investigations were done previously by H.-J. Schmidt [14] for the lower Vietoris topology, by G. Dimov [6, 8] for the Tychonoff topology and for the Vietoris topology, and by B. Karaivanov [12] for other hypertopologies. We study also such a problem for a fixed subspace A of Y. Some results of [6, 8, 15] are generalized. Finally, we study briefly some topological properties (separation axioms, compactness, weight, density, isolated points, P_∞) of the hyperspaces $(\mathcal{M}, \mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$. Some results of [10, 7] are generalized.

Let us fix the notations.

Notations 1.1. We denote by ω the set of all *positive* natural numbers, by $\mathbb{R}$ — the real line, and by $\mathbb{Z}$ — the set of all integers. We put $\mathbb{N} = \omega \cup \{0\}$.

Let X be a set. We denote by $\mathcal{P}(X)$ the set of all subsets of X. Let $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. We will use the following notations:

- $A^+_{\mathcal{M}} := \{M \in \mathcal{M} : M \subseteq A\}$;
- $\mathcal{A}^+_{\mathcal{M}} := \{A^+_{\mathcal{M}} : A \in \mathcal{A}\}$;
- $\mathcal{F}in(X) := \{M \subseteq X : |M| < \aleph_0\}$;
- $\mathcal{F}in_n(X) := \{M \subseteq X : |M| \leq n\}$, where $n \in \omega$.

We will denote by $\mathcal{A}^\cap$ (respectively by $\mathcal{A}^\cup$) the closure under finite intersections (unions) of the family $\mathcal{A}$. In other words,

- $\mathcal{A}^\cap := \{\bigcap_{i=1}^k A_i : k \in \omega, A_i \in \mathcal{A}\}$ and
- $\mathcal{A}^\cup := \{\bigcup_{i=1}^k A_i : k \in \omega, A_i \in \mathcal{A}\}$.

Let $(X, \mathcal{T})$ be a topological space. We put

- $\mathcal{C}L(X) := \{M \subseteq X : M \text{ is closed in } X,\ M \neq \emptyset\}$ and
- $\mathcal{C}omp(X) := \{M \subseteq X : M \text{ is compact}\}$.

The closure of a subset A of X in $(X, \mathcal{T})$ will be denoted by $cl_X A$ or $\overline{A}^X$; as usual, for $U \subseteq A \subseteq X$, we put

- $Ex_{A,X} U := X \setminus cl_X(A \setminus U)$.

By a *base* of $(X, \mathcal{T})$ we will always mean an open base. The weight (resp., the density) of $(X, \mathcal{T})$ will be denoted by $w(X, \mathcal{T})$ (resp., $d(X, \mathcal{T})$).

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of the category $\mathcal{C}$.

For all undefined here notions and notations, see [9] and [11].

2. Hypertopologies of Tychonoff-type

Fact 2.1. Let X be a set and $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$. Then:

(a) $\bigcap \mathcal{A}^+_{\mathcal{M}} = (\bigcap \mathcal{A})^+_{\mathcal{M}}$;
(b) $A \subseteq B$ implies that $A^+_{\mathcal{M}} \subseteq B^+_{\mathcal{M}}$ for all A, $B \subseteq X$.

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{M} \subseteq \mathcal{P}(X)$. The topology $\mathcal{O}_{\mathcal{T}}$ on $\mathcal{M}$, having as a base the family $\mathcal{T}^+_{\mathcal{M}}$, will be called *Tychonoff topology on $\mathcal{M}$ generated by* $(X, \mathcal{T})$. When $\mathcal{M} = \mathcal{C}L(X)$, then $\mathcal{O}_{\mathcal{T}}$ is just the classical Tychonoff topology on $\mathcal{C}L(X)$.

Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on $\mathcal{M}$ is called a *Tychonoff topology on* $\mathcal{M}$ if there exists a topology $\mathcal{T}$ on X such that $\mathcal{T}^+_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Definition 2.3. Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on the set $\mathcal{M}$ is called a *topology of Tychonoff-type on* $\mathcal{M}$ if the family $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ is a base for $\mathcal{O}$.

Clearly, a Tychonoff topology on $\mathcal{M}$ is always a topology of Tychonoff-type on $\mathcal{M}$, but not viceversa (see Example 2.42).

Fact 2.4. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then the family $\mathcal{B}_{\mathcal{O}} := \{A \subseteq X : A^+_{\mathcal{M}} \in \mathcal{O}\}$ is closed under finite intersections, $X \in \mathcal{B}_{\mathcal{O}}$, and, hence, $\mathcal{B}_{\mathcal{O}}$ is a base for a topology $\mathcal{T}_{\mathcal{O}}$ on X. The family $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Definition 2.5. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. We will say that the topology $\mathcal{T}_{\mathcal{O}}$ on X, introduced in Fact 2.4, is *induced by the topological space* $(\mathcal{M}, \mathcal{O})$.

Proposition 2.6. *Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on $\mathcal{M}$ is a topology of Tychonoff-type if and only if there exists a topology $\mathcal{T}$ on X and a base $\mathcal{B}$ for $\mathcal{T}$ (which contains X and is closed under finite intersections) such that $\mathcal{B}^+_{\mathcal{M}}$ is a base for $\mathcal{O}$.*

Proof. Suppose $\mathcal{O}$ is a topology of Tychonoff-type on $\mathcal{M}$. Then the topology $\mathcal{T}_{\mathcal{O}}$ induced by the topological space $(\mathcal{M}, \mathcal{O})$ (see Fact 2.4 and Definition 2.5) and the base $\mathcal{B}_{\mathcal{O}}$ have the required property.

Conversely, suppose $\mathcal{T}$ and $\mathcal{B}$ are given as in the statement. Then $\mathcal{B}^+_{\mathcal{M}}$ is a base for $\mathcal{O}$, and therefore also $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ is a base for $\mathcal{O}$. □

Definition 2.7. Let X be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. When $\mathcal{B}^+_{\mathcal{M}}$ is a base for a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$, we will say that $\mathcal{B}$ *generates a topology* on $\mathcal{M}$. (Obviously, the topology $\mathcal{O}_{\mathcal{B}}$ is of Tychonoff-type.

Proposition 2.8. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. The family $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$ if and only if the family $\mathcal{B}$ satisfies the following conditions:*

(MB1) *For any $M \in \mathcal{M}$ there exists a $U \in \mathcal{B}$ such that $M \subseteq U$;*

(MB2) *For any U_1, $U_2 \in \mathcal{B}$ and any $M \in \mathcal{M}$ with $M \subseteq U_1 \cap U_2$ there exists a $U_3 \in \mathcal{B}$ such that $M \subseteq U_3 \subseteq U_1 \cap U_2$.*

Proof. It follows from Proposition 1.2.1 [9]. □

Corollary 2.9. *Let X be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{B} = \mathcal{B}^{\cap}$ and $X \in \mathcal{B}$, then $\mathcal{B}$ generates a Tychonoff-type topology on $\mathcal{M}$.*

Definition 2.10. Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a *natural family in X* if $\{x\} \in \mathcal{M}$ for all $x \in X$.

Corollary 2.11. *Let X be a set and $\mathcal{M}$ be a natural family in X. If $\mathcal{B} \subseteq \mathcal{P}(X)$ generates a topology on $\mathcal{M}$ (see Definition 2.7), then $\mathcal{B}$ is a base for a topology on X.*

Proof. By Proposition 2.8, $\mathcal{B}$ satisfies the conditions (MB1) and (MB2). Since $\mathcal{M}$ is natural, this clearly implies that $\mathcal{B}$ satisfies the hypotesis of Proposition 1.2.1 [9]. So $\mathcal{B}$ is a base for a topology on X. □

Remark 2.12. Trivial examples show that there exist sets X and (non-natural) families $\mathcal{M}$,$\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\mathcal{B}^{+}_{\mathcal{M}}$ is a base for a topology on $\mathcal{M}$ but

(a) $\bigcup \mathcal{B} \neq X$, so that $\mathcal{B}$ cannot serve even as subbase of a topology on X (take $X = \{0, 1\}$, $\mathcal{M} = \mathcal{B} = \{\{0\}\}$);

(b) $\mathcal{B}$ is not a base of a topology of X, although $\bigcup \mathcal{B} = X$ (take $X = \{0, 1, 2\}$, $\mathcal{M} = \mathcal{B} = \{\{0, 1\}, \{0, 2\}\}$).

The example of (b) shows also that if we substitute in 2.11 naturality of $\mathcal{M}$ with the condition "$\bigcup \mathcal{M} = X$" then we cannot prove that $\mathcal{B}$ is a base of a topology on X; however, it is easy to show that the condition "$\bigcup \mathcal{M} = X$" implies that $\bigcup \mathcal{B} = X$, i.e. $\mathcal{B}$ can serve as a subbase of a topology on X.

Of course, as it follows from Fact 2.1, if $\mathcal{B}^{+}_{\mathcal{M}}$ is a base of a topology $\mathcal{O}$ on $\mathcal{M}$, then $\tilde{\mathcal{B}} = \mathcal{B}^{\cap} \cup \{X\}$ is a base for a topology on X and $\tilde{\mathcal{B}}^{+}_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Corollary 2.13. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$ be a base of $(X, \mathcal{T})$, closed under finite unions. Then $\mathcal{B}$ generates a topology of Tychonoff-type on $\mathcal{F}in(X)$ and $\mathcal{C}omp(X)$.*

Proof. It follows easily from Proposition 2.8. □

Proposition 2.14. *Let X be a set, $\mathcal{M}$,$\mathcal{B}_1$,$\mathcal{B}_2 \subseteq \mathcal{P}(X)$, and suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ generate, respectively, some topologies (of Tychonoff-type) $\mathcal{O}_{\mathcal{B}_1}$ and $\mathcal{O}_{\mathcal{B}_2}$ on $\mathcal{M}$. Then $\mathcal{O}_{\mathcal{B}_1} = \mathcal{O}_{\mathcal{B}_2}$ if and only if the following conditions are satisfied:*

(CO1) *For any* $M \in \mathcal{M}$ *and any* $U_1 \in \mathcal{B}_1$ *such that* $M \subseteq U_1$ *there exists* $U_2 \in \mathcal{B}_2$ *with* $M \subseteq U_2 \subseteq U_1$*;*
(CO2) *For any* $M \in \mathcal{M}$ *and any* $U_2 \in \mathcal{B}_2$ *such that* $M \subseteq U_2$ *there exists* $U_1 \in \mathcal{B}_1$ *with* $M \subseteq U_1 \subseteq U_2$*.*

Proof. It follows from 1.2.B [9]. □

Corollary 2.15. *Let* $(X, \mathcal{T})$ *be a topological space and* $\mathcal{B}_1$*,* $\mathcal{B}_2$ *be bases of* $(X, \mathcal{T})$*, closed under finite unions. Then they generate (see Corollary 2.13) equal topologies on* $\mathcal{F}in(X)$ *and* $\mathcal{C}omp(X)$*. In particular, every topology of Tychonoff-type on* $\mathcal{F}in(X)$ *or on* $\mathcal{C}omp(X)$*, generated by a base of* $(X, \mathcal{T})$ *which is closed under finite unions, coincides with the Tychonoff topology generated by* $(X, \mathcal{T})$ *on the corresponding set.*

Proof. Check that conditions (CO1) and (CO2) of Proposition 2.14 are satisfied. □

Corollary 2.16. *Let* $(X, \mathcal{T})$ *be a topological space,* $\mathcal{M} \subseteq \mathcal{P}(X)$*,* $\mathcal{B} \subseteq \mathcal{T}$*, and suppose that* $\mathcal{B}$ *generates a topology of Tychonoff-type* $\mathcal{O}$ *on* $\mathcal{M}$*. Then* $\mathcal{O}$ *is the Tychonoff topology on* $\mathcal{M}$ *generated by* $(X, \mathcal{T})$ *if and only if for all* $M \in \mathcal{M}$ *and for all* $V \in \mathcal{T}$ *such that* $M \subseteq V$*, there exists* $U \in \mathcal{B}$ *with* $M \subseteq U \subseteq V$*. In this case we will say that* $\mathcal{B}$ *is an* $\mathcal{M}$-base *for* $(X, \mathcal{T})$*. Clearly, if* $\mathcal{M}$ *is a natural family, then every* $\mathcal{M}$*-base of* $(X, \mathcal{T})$ *is also a base of* $(X, \mathcal{T})$*.*

Proof. Put $\mathcal{B}_1 := \mathcal{T}$ and $\mathcal{B}_2 := \mathcal{B}$. Then condition (CO2) of Proposition 2.14 is trivially satisfied. The condition required in the statement is exactly condition (CO1). □

Definition 2.17. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. A family $\mathcal{U} \subseteq \mathcal{P}(X)$ will be called an $\mathcal{M}$*-cover* of A if $A = \bigcup \mathcal{U}$ and for all $M \in \mathcal{M}$ with $M \subseteq A$ there exists some $U \in \mathcal{U}$ such that $M \subseteq U$.

Proposition 2.18. *Let* X *be a set and* $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$*. Then the following conditions are equivalent:*
(U1) *For all* $U \in \mathcal{A}$ *and for all* $x \in U$*, there exists an* $M \in \mathcal{M}$ *with* $x \in M \subseteq U$*.*
(U2) *For any* $U \in \mathcal{A} \cup \mathcal{M}$ *and for any subfamily* $\{U_\delta : \delta \in \Delta\}$ *of* $\mathcal{A} \cup \mathcal{M}$*, the equality* $U^+_{\mathcal{M}} = \bigcup_{\delta \in \Delta} (U_\delta)^+_{\mathcal{M}}$ *holds if and only if the family* $\{U_\delta\}_{\delta \in \Delta}$ *is an* $\mathcal{M}$*-cover of* U*.*

Proof. Observe that, trivially, in condition (U1) we can replace the requirement 'for all $U \in \mathcal{A}$" with "for all $U \in \mathcal{A} \cup \mathcal{M}$".

(U1)⇒(U2). Let $U^+_{\mathcal{M}} = \bigcup_{\delta \in \Delta} (U_\delta)^+_{\mathcal{M}}$, with $U, U_\delta \in \mathcal{A} \cup \mathcal{M}$ for all $\delta \in \Delta$. We will prove first that $\bigcup_{\delta \in \Delta} U_\delta = U$.

Let $x \in \bigcup_{\delta \in \Delta} U_\delta$. Then there exists a $\delta \in \Delta$ such that $x \in U_\delta$. By assumption, there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U_\delta$. Hence $M \in (U_\delta)^+_{\mathcal{M}}$. Since $U^+_{\mathcal{M}} = \bigcup_{\delta \in \Delta} (U_\delta)^+_{\mathcal{M}}$, we obtain that $M \subseteq U$. Thus $x \in U$. Therefore, $\bigcup_{\delta \in \Delta} U_\delta \subseteq U$.

Conversely, let $x \in U$. By assumption, there exists an $M \in \mathcal{M}$ such that $x \in M \subseteq U$. Hence $M \in U^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$. Therefore there exists a $\delta \in \Delta$ such that $M \in (U_\delta)^+_{\mathcal{M}}$, i.e. $M \subseteq U_\delta$ and $x \in U_\delta \subseteq \bigcup_{\delta\in\Delta} U_\delta$.

We have verified that $\bigcup_{\delta\in\Delta} U_\delta = U$.

Suppose $M \in \mathcal{M}$ and $M \subseteq U$. Then $M \in U^+_{\mathcal{M}}$ and therefore there exists some $\gamma \in \Delta$ with $M \in (U_\gamma)^+_{\mathcal{M}}$. Hence $M \subseteq U_\gamma$.

This shows that the family $\{U_\delta\}_{\delta\in\Delta}$ is an $\mathcal{M}$-cover of U.

The other implication can be easily proved. (Let's remark that condition (U1) is not used in the proof of this last implication.)

(U2)⇒(U1). Suppose $U \in \mathcal{A}$ and $x \in U$. Clearly, we have

$$U^+_{\mathcal{M}} = \bigcup\{M^+_{\mathcal{M}} : M \in \mathcal{M},\ M \subseteq U\}.$$

Then, by assumption, the family $\{M \in \mathcal{M} : M \subseteq U\}$ is an $\mathcal{M}$-cover of U. Therefore $U = \bigcup\{M : M \in \mathcal{M},\ M \subseteq U\}$. Hence there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U$. □

Proposition 2.19. *Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. Then the following conditions are equivalent:*

(a) *$\mathcal{M}$ is a natural family;*
(b) *For any $U \subseteq X$ and for any subfamily $\{U_\delta : \delta \in \Delta\}$ of $\mathcal{P}(X)$, the equality $U^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$ holds if and only if the family $\{U_\delta\}_{\delta\in\Delta}$ is an $\mathcal{M}$-cover of U.*

Proof. Put $\mathcal{A} = \mathcal{P}(X)$ in Proposition 2.18. □

Proposition 2.20. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{O}$ be a Tychonoff topology on $\mathcal{M}$ generated by a topology $\mathcal{T}$ on X and $\mathcal{M}$ be a network in the sense of Arhangel′skiĭ for $\mathcal{T}_{\mathcal{O}}$. Then $\mathcal{T} = \mathcal{B}_{\mathcal{O}}$ and $\mathcal{O}$ is generated by a unique topology on X, namely by $\mathcal{T}_{\mathcal{O}}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. We only need to show that $\mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$. Assume $A \in \mathcal{B}_{O}$. Then $A^+_{\mathcal{M}} \in \mathcal{O}$. Since $\mathcal{T}$ generates $\mathcal{O}$, we have $A^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$, where $U_\delta \in \mathcal{T}$ for all $\delta \in \Delta$. Clearly $U_\delta \in \mathcal{B}_{O}$ for all $\delta \in \Delta$. By Proposition 2.18, we obtain that $A = \bigcup_{\delta\in\Delta} U_\delta$ and therefore $A \in \mathcal{T}$. □

Remark 2.21. Trivial examples show that there exist sets X, families $\mathcal{M} \subseteq \mathcal{P}(X)$ and Tychonoff topologies on $\mathcal{M}$ which are generated by more than one topology on X.

Corollary 2.22. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$ and $\mathcal{M}$ be a network in the sense of Arhangel′skiĭ for $\mathcal{T}_{\mathcal{O}}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_{\mathcal{O}} = \mathcal{T}_{\mathcal{O}}$ (see Fact 2.4 for the notations $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. Suppose $\mathcal{B}_{\mathcal{O}} = \mathcal{T}_{\mathcal{O}}$. Then the topology $\mathcal{O}$ is generated by the topology $\mathcal{T}_{\mathcal{O}}$ on X and hence, by definition, $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$.

Suppose $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$. Then $\mathcal{O}$ is generated by some topology $\mathcal{T}$ on X. By Proposition 2.20, we get $\mathcal{T} = \mathcal{B}_{\mathcal{O}}$. Hence $\mathcal{T}_{\mathcal{O}} = \mathcal{B}_{\mathcal{O}}$. □

Corollary 2.23. *Let X be a set, $\mathcal{M}$ be a natural family in X and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_{\mathcal{O}} = \mathcal{T}_{\mathcal{O}}$.*

Proof. A natural family $\mathcal{M}$ satisfies the hypothesis of Corollary 2.22. □

Proposition 2.24. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X, $\mathcal{B} \subseteq \mathcal{T}$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$. Then*

$$\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \{A \subseteq X : A \ \text{ is } \mathcal{M}\text{-covered by a subfamily of } \mathcal{B}\},$$

$\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \subseteq \mathcal{T}$ and $\mathcal{B}^{\cap} \subseteq \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$).

Proof. It follows from Proposition 2.19 and Fact 2.4. □

Proposition 2.25. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X, $\mathcal{B}$ be a base for $(X, \mathcal{T})$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$. Let $\mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$ be the topology on X induced by $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$. Then $\mathcal{T}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{T}$.*

Proof. We have, by Proposition 2.24, that $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \subseteq \mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$. Thus $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$. As it is shown in Proposition 2.24, $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \subseteq \mathcal{T}$ and hence $\mathcal{T}_{\mathcal{O}_{\mathcal{B}}} \subseteq \mathcal{T}$. So, $\mathcal{T} = \mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$. □

Example 2.26. Let us show that in Proposition 2.25 the requirement "$\mathcal{M}$ is a natural family" is essential.

Let $X = (0, 1) \subset \mathbb{R}$ be the open unit interval with the usual topology, $\mathcal{M} = \{[a, b] : 0 < a < b < 1\}$, $\mathcal{B} = \{(a, b) : 0 < a < b < 1\}$. Then the family $\mathcal{B}$ satisfies conditions (MB1) and (MB2). Consider the set $A = (\frac{1}{2}, \frac{3}{4}) \cup \{\frac{1}{4}\}$. We have $A^+_{\mathcal{M}} = (\frac{1}{2}, \frac{3}{4})^+_{\mathcal{M}} \in \mathcal{B}^+_{\mathcal{M}}$ and therefore $A \in \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$ even though A is not open in X.

Definition 2.27. Let X be a set and $\mathcal{M}, \mathcal{U} \subseteq \mathcal{P}(X)$. We will say that $\mathcal{U}$ is an *$\mathcal{M}$-closed family* if for all $A \subseteq X$ such that A is $\mathcal{M}$-covered by some subfamily of $\mathcal{U}$, we have that $A \in \mathcal{U}$.

Proposition 2.28. *Let X be a set and $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{U} \subseteq \mathcal{P}(X)$, $\mathcal{M} \subseteq \mathcal{M}'$. Suppose that $\mathcal{U}$ is an $\mathcal{M}$-closed family. Then $\mathcal{U}$ is an $\mathcal{M}'$-closed family too.*

Proof. Suppose $A \subseteq X$ is $\mathcal{M}'$-covered by some subfamily $\mathcal{U}'$ of $\mathcal{U}$. Since $\mathcal{M} \subseteq \mathcal{M}'$, the set A is also $\mathcal{M}$-covered by $\mathcal{U}'$. By the hypothesis, $\mathcal{U}$ is an $\mathcal{M}$-closed family. Hence $A \in \mathcal{U}$. □

Proposition 2.29. *Let X be a set, $\mathcal{M}$ be a natural family in X and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\mathcal{B}_{\mathcal{O}}$ is an $\mathcal{M}$-closed family (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$).*

Proof. It follows from Proposition 2.19. □

Proposition 2.30. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X and $\mathcal{B} \subseteq \mathcal{T}$ be an $\mathcal{M}$-closed base of $(X, \mathcal{T})$. Suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$. Then $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}$, $X \in \mathcal{B}$ and $\mathcal{B}^{\cap} = \mathcal{B}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$).*

Proof. Obviously, $\mathcal{B}_{\mathcal{O}_\mathcal{B}} \supseteq \mathcal{B}$. Let us show that $\mathcal{B}_{\mathcal{O}_\mathcal{B}} \subseteq \mathcal{B}$.

Let $A \in \mathcal{B}_{\mathcal{O}_\mathcal{B}}$. Then, by Proposition 2.24, A is $\mathcal{M}$-covered by some subfamily of $\mathcal{B}$ and, since $\mathcal{B}$ is $\mathcal{M}$-closed, we conclude that $A \in \mathcal{B}$. So, $\mathcal{B}_{\mathcal{O}_\mathcal{B}} = \mathcal{B}$. Now Fact 2.4 implies that $X \in \mathcal{B}$ and $\mathcal{B}^\cap = \mathcal{B}$. □

Definition 2.31. Let X be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. The ordered triple $(X, \mathcal{B}, \mathcal{M})$ will be called a *T-space* if $\mathcal{B}$ is an $\mathcal{M}$-closed family, $X \in \mathcal{B}$ and $\mathcal{B}^\cap = \mathcal{B}$.

Note that if $(X, \mathcal{B}, \mathcal{M})$ is a T-space, then $\mathcal{B}$ is a base for a topology on X.

Proposition 2.32. *Let X be a set and $\mathcal{M}$ be a natural family in X. Let $TTT(X, \mathcal{M})$ be the set of all topologies of Tychonoff-type on $\mathcal{M}$. Denote by $T\text{-}Sp(X, \mathcal{M})$ the set of all T-spaces of the form $(X, \mathcal{B}, \mathcal{M})$. Then there is a bijective correspondence between the sets $TTT(X, \mathcal{M})$ and $T\text{-}Sp(X, \mathcal{M})$. Namely, consider the function $\alpha : TTT(X, \mathcal{M}) \to T\text{-}Sp(X, \mathcal{M})$, defined by $\alpha(\mathcal{O}) = (X, \mathcal{B}_\mathcal{O}, \mathcal{M})$ (see Fact 2.4 for the notation $\mathcal{B}_\mathcal{O}$), and the function $\beta : T\text{-}Sp(X, \mathcal{M}) \to TTT(X, \mathcal{M})$, defined by $\beta((X, \mathcal{B}, \mathcal{M})) = \mathcal{O}_\mathcal{B}$ (see Corollary 2.9 and Definition 2.7); then α and β are bijections and each one is the inverse of the other one.*

Proof. Let us show that the function α is well-defined. Let $\mathcal{O} \in TTT(X, \mathcal{M})$. By Fact 2.4, the family $\mathcal{B}_\mathcal{O}$ is closed under finite intersections and $X \in \mathcal{B}_\mathcal{O}$. By Proposition 2.29, $\mathcal{B}_\mathcal{O}$ is $\mathcal{M}$-closed. Hence $(X, \mathcal{B}_\mathcal{O}, \mathcal{M}) \in T\text{-}Sp(X, \mathcal{M})$.

We will prove now that the function β is well defined. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be an $\mathcal{M}$-closed family, closed under finite intersections and such that $X \in \mathcal{B}$. Then, by Corollary 2.9, $\mathcal{O}_\mathcal{B}$ is a topology of Tychonoff-type on $\mathcal{M}$.

Proposition 2.30 gives that $\mathcal{B}_{\mathcal{O}_\mathcal{B}} = \mathcal{B}$, i.e. $\alpha \circ \beta = \mathrm{id}_{T\text{-}Sp(X,\mathcal{M})}$.

To show that $\beta \circ \alpha = \mathrm{id}_{TTT(X,\mathcal{M})}$, let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\beta(\alpha(\mathcal{O})) = \mathcal{O}_{\mathcal{B}_\mathcal{O}}$. Since $\mathcal{O}$ is a topology of Tychonoff-type, $(\mathcal{B}_\mathcal{O})^+_\mathcal{M}$ is a base of $\mathcal{O}$. On the other hand, $(\mathcal{B}_\mathcal{O})^+_\mathcal{M}$ is, by definition, a base of $\mathcal{O}_{\mathcal{B}_\mathcal{O}}$. Hence $\mathcal{O} = \mathcal{O}_{\mathcal{B}_\mathcal{O}}$. □

Definition 2.33. We denote by $\mathcal{HT}$ (Hypertopologies of Tychonoff-type) the category defined as follows: its objects are all ordered triples $(X, \mathcal{M}, \mathcal{O})$ where X is a set, $\mathcal{M}$ is a natural family in X and $\mathcal{O}$ is a topology of Tychonoff-type on $\mathcal{M}$. To define the morphisms of $\mathcal{HT}$, let $(X, \mathcal{M}, \mathcal{O})$, $(X', \mathcal{M}', \mathcal{O}')$ be objects of $\mathcal{HT}$ and $f : X \to X'$ be a function between the sets X and X'. We will say that f *generates a morphism f_H of $\mathcal{HT}$* between $(X, \mathcal{M}, \mathcal{O})$ and $(X', \mathcal{M}', \mathcal{O}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{M}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) := f(M)$ (where the M on the left-handside is regarded as an element of $\mathcal{M}$ and the M on the right-handside is regarded as a subset of X) is continuous. The morphisms of $\mathcal{HT}$ are defined to be all f_H generated in this way.

Remark 2.34. It is easy to see that not any continuous map between spaces of the type $(\mathcal{M}, \mathcal{O})$ appears as some f_H (see Definition 2.33 for the notations). Indeed, let $(X, \mathcal{T})$ be a discrete space having more than one point;

then the constant function $c : (\mathcal{CL}(X), \mathcal{O}_\mathcal{T}) \to (\mathcal{CL}(X), \mathcal{O}_\mathcal{T})$, defined by $c(F) = X$ for all $F \in \mathcal{CL}(X)$, is continuous but is not of the type f_H (here $\mathcal{O}_\mathcal{T}$ is the classical Tychonoff topology on $\mathcal{CL}(X)$ (see Definition 2.2)).

Definition 2.35. We denote by $\mathcal{TH}$ the category defined as follows: its objects are all T-spaces $(X, \mathcal{B}, \mathcal{M})$ (see Definition 2.31). To define the morphisms of $\mathcal{TH}$, let $(X, \mathcal{B}, \mathcal{M})$, $(X', \mathcal{B}', \mathcal{M}')$ be objects of $\mathcal{TH}$ and $f : X \to X'$ be a function between the sets X and X'. We will say that f *generates a morphism* f^T *of* $\mathcal{TH}$ between $(X, \mathcal{B}, \mathcal{M})$ and $(X', \mathcal{B}', \mathcal{M}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. The morphisms of $\mathcal{TH}$ are defined to be all f^T generated in this way.

Lemma 2.36. *Let X, X' be sets, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{M}' \subseteq \mathcal{P}(X')$. Let $f : X \to X'$ be a function such that $f(\mathcal{M}) \subseteq \mathcal{M}'$ and let $f_m : \mathcal{M} \to \mathcal{M}'$ be defined by $f_m(M) := f(M)$. Then, for all $A' \subseteq X'$, we have*

$$f_m^{-1}\left((A')^+_{\mathcal{M}'}\right) = \left(f^{-1}(A')\right)^+_{\mathcal{M}}.$$

Proof. Let $M \in f_m^{-1}\left((A')^+_{\mathcal{M}'}\right)$. Then $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $f(M) \subseteq A'$. Since $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(A')$, we obtain $M \in \left(f^{-1}(A')\right)^+_{\mathcal{M}}$.

Let $M \in \left(f^{-1}(A')\right)^+_{\mathcal{M}}$. Then $M \subseteq f^{-1}(A')$ and hence $f(M) \subseteq A'$. Therefore $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $M \in f_m^{-1}\left((A')^+_{\mathcal{M}'}\right)$. □

Theorem 2.37. *The categories $\mathcal{HT}$ and $\mathcal{TH}$ are isomorphic.*

Proof. We define a functor $F : \mathcal{HT} \to \mathcal{TH}$ as follows: for all $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$, we put $F((X, \mathcal{M}, \mathcal{O})) := (X, \mathcal{B}_\mathcal{O}, \mathcal{M})$ (see Fact 2.4 for the notation $\mathcal{B}_\mathcal{O}$); for all morphisms $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$, we put $F(f_H) := f^T$ (see Definitions 2.33 and 2.35 for the notations f_H and f^T).

We define a functor $G : \mathcal{TH} \to \mathcal{HT}$ as follows: for all $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$, we put $G((X, \mathcal{B}, \mathcal{M})) := (X, \mathcal{M}, \mathcal{O}_\mathcal{B})$ (see Definition 2.7 for the notation $\mathcal{O}_\mathcal{B}$); for all morphisms $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$, we put $G(f^T) := f_H$.

Let us check that F and G are well-defined.

Let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. By Proposition 2.32, the triple $(X, \mathcal{B}_O, \mathcal{M})$ is a T-space, so that $F(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{TH}|$.

Let $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$ be a morphism in $\mathcal{HT}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{M}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) = f(M)$, is continuous. To check that

$$f^T : F(X, \mathcal{M}, \mathcal{O}) \to F(X', \mathcal{M}', \mathcal{O}')$$

(i.e. $f^T : (X, \mathcal{B}_\mathcal{O}, \mathcal{M}) \to (X', \mathcal{B}_{\mathcal{O}'}, \mathcal{M}')$) is a morphism in $\mathcal{TH}$, we need to show that $f^{-1}(\mathcal{B}_{\mathcal{O}'}) \subseteq \mathcal{B}_\mathcal{O}$. Let $B' \in \mathcal{B}_{\mathcal{O}'}$. Then $(B')^+_{\mathcal{M}'} \in \mathcal{O}'$. By the continuity of f_m, we have that $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) \in \mathcal{O}$. By Lemma 2.36, $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) = \left(f^{-1}(B')\right)^+_{\mathcal{M}}$. Hence $\left(f^{-1}(B')\right)^+_{\mathcal{M}} \in \mathcal{O}$, i.e. $f^{-1}(B') \in \mathcal{B}_\mathcal{O}$. So, we have proved that F is well-defined. Clearly, F is a functor.

Let now $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$. Then, by Proposition 2.32, the topology $\mathcal{O}_\mathcal{B}$ is a Tychonoff-type topology on $\mathcal{M}$. Hence, $G((X, \mathcal{B}, \mathcal{M})) \in |\mathcal{HT}|$.

Let $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$ be a morphism in $\mathcal{TH}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. To check that

$$f_H : (X, \mathcal{M}, \mathcal{O}_{\mathcal{B}}) \to (X', \mathcal{M}', \mathcal{O}_{\mathcal{B}'})$$

is a morphism in $\mathcal{HT}$, we need to check that the function $f_m : (\mathcal{M}, \mathcal{O}_{\mathcal{B}}) \to (\mathcal{M}', \mathcal{O}_{\mathcal{B}'})$, defined by $f_m(M) := f(M)$, is continuous. By definition (see 2.7), $(\mathcal{B}')^+_{\mathcal{M}'}$ is a base of the topology $\mathcal{O}_{\mathcal{B}'}$. Let $B' \in \mathcal{B}'$. Then $(B')^+_{\mathcal{M}'} \in (\mathcal{B}')^+_{\mathcal{M}'}$. By assumption, $f^{-1}(B') \in \mathcal{B}$. Hence $\left(f^{-1}(B')\right)^+_{\mathcal{M}} \in \mathcal{O}_{\mathcal{B}}$. Since, by Lemma 2.36, $\left(f^{-1}(B')\right)^+_{\mathcal{M}} = f_m^{-1}\left((B')^+_{\mathcal{M}'}\right)$, we obtain that $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) \in \mathcal{O}_{\mathcal{B}}$. Therefore, the function f_m is continuous. So, G is well-defined. Obviously, G is a functor.

By Proposition 2.32, we have $F \circ G = \mathrm{id}_{\mathcal{TH}}$ and $G \circ F = \mathrm{id}_{\mathcal{HT}}$ on the objects. The equalities are clearly true for the morphisms. Hence F and G are isomorphisms. □

We recall that a topological space $(X, \mathcal{T})$ is called a P_∞*-space* (see [1, 7]) if $\mathcal{T}$ is closed under arbitrary intersections.

Lemma 2.38. *A space $(X, \mathcal{T})$ is a P_∞-space if and only it it has a base $\mathcal{B}$ closed under arbitrary intersections.*

Proof. Assume $\mathcal{T}$ has a base $\mathcal{B}$ closed under arbitrary intersections. Let $\mathcal{U} \subseteq \mathcal{T}$. Since $\emptyset$ is an open set, we can assume that $\bigcap \mathcal{U} \neq \emptyset$. Let $x \in \bigcap \mathcal{U}$. For any $U \in \mathcal{U}$, let $B_U \in \mathcal{B}$ be such that $x \in B_U \subseteq U$. Then

$$x \in \bigcap \{B_U : U \in \mathcal{U}\} \subseteq \bigcap \mathcal{U}$$

and, by assumption, $\bigcap \{B_U : U \in \mathcal{U}\} \in \mathcal{B}$. Hence, $\bigcap \mathcal{U} \in \mathcal{T}$. □

Proposition 2.39. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$. The topological space $(\mathcal{M}, \mathcal{O})$ is a P_∞-space if and only if the family $\mathcal{B}_{\mathcal{O}}$ is closed under arbitrary intersections. If $(\mathcal{M}, \mathcal{O})$ is a P_∞-space then $(X, \mathcal{T}_{\mathcal{O}})$ is a P_∞-space. If $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$, then $(\mathcal{M}, \mathcal{O})$ is a P_∞-space if and only if $(X, \mathcal{T}_{\mathcal{O}})$ is a P_∞-space (see Fact 2.4 for the notations $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. The first two assertions follow from Lemma 2.38, Fact 2.1(a) and the definitions of $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$. The last assertion follows now from Corollary 2.22. □

Corollary 2.40. *Let $\mathcal{HT}_\infty$ be the full subcategory of $\mathcal{HT}$ having as objects all triples $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ for which the space $(\mathcal{M}, \mathcal{O})$ is a P_∞-space. Let $\mathcal{TH}_\infty$ be the full subcategory of $\mathcal{TH}$ whose objects are all $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$ such that the family $\mathcal{B}$ is closed under arbitrary intersections. Then $\mathcal{HT}_\infty$ and $\mathcal{TH}_\infty$ are isomorphic.*

Proof. It follows from (the proof of) Theorem 2.37 and Proposition 2.39. □

Example 2.41. We will show that there exists $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ such that $(X, \mathcal{T}_{\mathcal{O}})$ is a P_∞-space but $(\mathcal{M}, \mathcal{O})$ is not a P_∞-space.

Let $X = \omega$,

$$\mathcal{B} = \{\{n\} : n \in \omega\} \cup \{A \subseteq \omega : |\omega \setminus A| < \aleph_0\} \cup \{\emptyset\},$$

and $\mathcal{P}(\omega) \supseteq \mathcal{M} \supseteq \mathcal{F}in_2(\omega) \setminus \{\emptyset\}$. The family $\mathcal{B}$ is a base of the discrete topology $\mathcal{T}$ on ω, it is closed under finite intersections (but not under infinite intersections) and $X \in \mathcal{B}$. Let us show that $\mathcal{B}$ is $\mathcal{M}$-closed.

Let $\{B_\delta\}_{\delta\in\Delta}$ be a subfamily of $\mathcal{B} \setminus \{\emptyset\}$ which is an $\mathcal{M}$-cover of a subset B of X. Without loss of generality, we can assume that there exist at least two indices δ_1 and δ_2 such that $B_{\delta_1} \neq B_{\delta_2}$. Then there is at least one $\delta \in \Delta$ such that $|\omega \setminus B_\delta| < \aleph_0$; otherwise we would have $B_{\delta_1} = \{n_{\delta_1}\}$, $B_{\delta_2} = \{n_{\delta_2}\}$ and the set $F = \{n_{\delta_1}, n_{d_2}\}$, which belongs to $\mathcal{M}$, would be contained in B without being contained in B_δ for any $\delta \in \Delta$. Therefore $|\omega \setminus B| < \aleph_0$ and, hence, $B \in \mathcal{M}$.

Put $\mathcal{O} := \mathcal{O}_\mathcal{B}$. Then, by Proposition 2.30, $\mathcal{B} = \mathcal{B}_\mathcal{O}$ and hence $\mathcal{T} = \mathcal{T}_\mathcal{O}$. Since $\mathcal{B}$ is not closed under arbitrary intersections, we obtain, by Proposition 2.39, that $(\mathcal{M}, \mathcal{O})$ is not a P_∞-space. Clearly, the space $(X, \mathcal{T})$ is a P_∞-space because it is discrete. Observe that $\mathcal{O}$ is not a Tychonoff topology since $\mathcal{B}_\mathcal{O} \neq \mathcal{T}_\mathcal{O}$.

Example 2.42. Two more examples of Tychonoff-type, non Tychonoff topologies on some families $\mathcal{M} \subseteq \mathcal{P}(X)$.

Let X be a set with more than two elements. Let $\mathcal{M} = \mathcal{F}in(X) \setminus \{\emptyset\}$ and let $\mathcal{O} = \{\{\{x\} : x \in A\} : A \subseteq X\} \cup \{\mathcal{M}\} \cup \{\emptyset\}$. Then $\mathcal{O}$ is a topology on $\mathcal{M}$.

$\mathcal{O}$ is a topology of Tychonoff-type since

$$\mathcal{O} \cap \mathcal{P}(X)^+_\mathcal{M} = \{\{x\} : x \in X\} \cup \{X^+_\mathcal{M}\} \cup \{\emptyset\}$$

is a base for $\mathcal{O}$. Clearly $(\mathcal{M}, \mathcal{O})$ is a P_∞-space.

We have $\mathcal{B}_\mathcal{O} = \{\{x\} : x \in X\} \cup \{X\} \cup \{\emptyset\}$, and therefore $\mathcal{T}_\mathcal{O}$ is the discrete topology. By Proposition 2.22, since $\mathcal{B}_\mathcal{O} \neq \mathcal{T}_\mathcal{O}$, $\mathcal{O}$ is not a Tychonoff topology on $\mathcal{M}$.

Observe that $(\mathcal{M}, \mathcal{O})$ is not a T_0-space. In fact, the only neighbourhood of any $F \in \mathcal{M}$ such that $|F| \geq 2$ is $\mathcal{M}$, and we are assuming $|X| > 2$.

We consider now the natural family $\mathcal{M}' = \mathcal{P}(X) \setminus \{\emptyset\}$ and we define $\mathcal{O}$ as above. $\mathcal{O}$ is a Tychonoff-type topology on $\mathcal{M}'$ but it is not a Tychonoff topology. Again $\mathcal{T}_\mathcal{O}$ is the discrete topology on X. Hence $\mathcal{M}' = \mathcal{CL}((X, \mathcal{T}_\mathcal{O}))$. We observe as before that $(\mathcal{M}', \mathcal{O})$ is not a T_0-space. Note also that the family $\mathcal{B}_\mathcal{O}$ is $\mathcal{M}''$-closed for every natural family $\mathcal{M}''$ in X which contains all two-points subsets of X and $\emptyset \notin \mathcal{M}''$.

We will briefly discuss now some topological properties of the hyperspaces $(\mathcal{M}, \mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$.

Fact 2.43. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$ generated by a subfamily $\mathcal{B}$ of $\mathcal{P}(X)$. Then:

(a) the topological space $(\mathcal{M}, \mathcal{O})$ is a T_0-space (resp., T_1-space) if and only if for any $F, G \in \mathcal{M}$ with $F \neq G$, there exists a $B \in \mathcal{B}$ such that either $F \subseteq B$ and $G \not\subseteq B$, or $G \subseteq B$ and $F \not\subseteq B$ (resp., $F \subseteq B$ and $G \not\subseteq B$).

(b) if for any $x \in X$ and for any $F \in \mathcal{M}$ with $x \notin F$, there exists a $B \in \mathcal{B}$ such that $F \subseteq B$ and $x \notin B$, then $(\mathcal{M}, \mathcal{O})$ is a T_0-space.

Remark 2.44. Let us note that Fact 2.43(b) implies the following assertion, which was mentioned in [5], section 2 (after Lemma 3) (the requirement that $X \in \Omega$ has to be added there): *if $(X, \mathcal{T})$ is a regular T_1-space, $\mathcal{M}$ is a family consisting of closed subsets of $(X, \mathcal{T})$ and $\mathcal{B}$ is a base of $(X, \mathcal{T})$ such that $\mathcal{B} = \mathcal{B}^{\cap}$ and $U \in \mathcal{B}$ implies that $X \setminus \overline{U} \in \mathcal{B}$, then $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ is a T_0-space.*

Fact 2.45. Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$. Then the correspondence $(X, \mathcal{T}_{\mathcal{O}}) \to (\mathcal{M}, \mathcal{O})$, $x \mapsto \{x\}$, is a homeomorphic embedding. Hence, we have, in particular, that:

(a) $w(X, \mathcal{T}_{\mathcal{O}}) \leq w(\mathcal{M}, \mathcal{O})$;

(b) if $(\mathcal{M}, \mathcal{O})$ is a T_0-space then $(X, \mathcal{T}_{\mathcal{O}})$ is a T_0-space.

Fact 2.46.

(a) Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ be a family such that there exist $F, G \in \mathcal{M}$ with $F \subset G$ and $F \neq G$, and let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is not a T_1-space.

(b) If $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ then $(\mathcal{M}, \mathcal{O})$ is a T_1-space if and only if $(X, \mathcal{T}_{\mathcal{O}})$ is a T_1-space and $\mathcal{M} = \{\{x\} : x \in X\}$.

Fact 2.47. Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$. Then $(\mathcal{M}, \mathcal{O})$ is a compact space if and only if any $\mathcal{M}$-cover of X, consisting of elements of $\mathcal{B}_{\mathcal{O}}$, has a finite $\mathcal{M}$-subcover.

Proof. It follows from Proposition 2.19. □

Examples 2.48. There are many examples of "very nice" spaces X with non-T_0-hyperspaces $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ (see Examples 2.42 and 3.19). As an example of a non-T_0-space $(X, \mathcal{T})$ with a T_0-hyperspace $(\mathcal{M}, \mathcal{O}_{\mathcal{T}})$, consider the two-points space $X = \{0, 1\}$, with $\mathcal{T} = \mathcal{M} = \{\emptyset, X\}$.

There exist non-compact spaces X such that $(\mathcal{CL}(X), \mathcal{O}_{\mathcal{B}})$ is a compact non-T_0-space (e.g., the space $(\mathcal{CL}(\mathbb{R}), \mathcal{O}_{\mathcal{B}})$, described in Example 3.19).

To get an example of a non-compact space X and a natural family $\mathcal{M}$ such that $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ is a compact T_0-space, consider $X := \mathbb{R}$ with its natural topology, $\mathcal{M} := \mathcal{F}in_2(\mathbb{R}) \cup \{\mathbb{R}\}$ and take $\mathcal{B}$ as in Example 3.19.

As an example of a compact space $(X, \mathcal{T})$ with a non-compact hyperspace $(\mathcal{M}, \mathcal{O}_{\mathcal{T}})$, consider the unit interval $X = [0, 1]$ with its natural topology and put $\mathcal{M} = \{\{x\} : x \in (0, 1]\}$.

The next three propositions are generalizations of, respectively, Propositions 1, 2 and 3 of [10], and have proofs similar to those given in [10]. (Let us note that in Proposition 2 of [10] the requirement "$\emptyset \notin \mathcal{C}$" has to be added.)

Proposition 2.49. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$, $w(\mathcal{M}, \mathcal{O}) = \aleph_0$, $(X, \mathcal{T}_{\mathcal{O}})$ be a T_1-space, $\mathcal{B}_{\mathcal{O}}$ be closed under countable unions and $\mathcal{M}$ contain all infinite countable closed subsets of $(X, \mathcal{T}_{\mathcal{O}})$. Then $(X, \mathcal{T}_{\mathcal{O}})$ is a compact space.*

Proposition 2.50. *Let* $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ *and* $\emptyset \notin \mathcal{M}$. *Then* $d(\mathcal{M}, \mathcal{O}) = d(X, \mathcal{T}_{\mathcal{O}})$.

Proposition 2.51. *Let* $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ *and* $\emptyset \notin \mathcal{M}$. *Then* $(\mathcal{M}, \mathcal{O})$ *has isolated points if and only if* $(X, \mathcal{T}_{\mathcal{O}})$ *has isolated points.*

3. On $\mathcal{O}$-commutative spaces

3.1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Recall that A is said to be *2-combinatorially embedded in* X (see [4]) if the closures in X of any two disjoint closed in A subsets of A are disjoint.

Definition 3.2. Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. We will say that A is $2_{\mathcal{B}}$*-combinatorially embedded in* X if for any $F \in \mathcal{CL}(A)$ and for any $U \in \mathcal{B}$ with $F \subseteq U$, there exists a $V \in \mathcal{B}$ such that $\overline{F}^X \subseteq V$ and $V \cap A \subseteq U$.

Proposition 3.3. *Let* $(X, \mathcal{T})$ *be a topological space and* $A \subseteq X$. *Then* A *is 2-combinatorially embedded in* X *if and only if* A *is* $2_{\mathcal{T}}$*-combinatorially embedded in* X.

Proof. ($\Rightarrow$) Let $H \in \mathcal{CL}(A)$, $V \in \mathcal{T}$ and $H \subseteq V$. We put $U = V \cap A$ and $F = A \setminus U$. Then F and H are two disjoint closed subsets of A. Hence, by assumption, they have disjoint closures in X, i.e. $\overline{F}^X \cap \overline{H}^X = \emptyset$. Let $W = X \setminus \overline{F}^X$. Then W is open in X, $W \cap A = U = V \cap A$ and $\overline{H}^X \subseteq W$.

($\Leftarrow$) Let F and G be two disjoint closed subsets of A. Put $V = X \setminus \overline{G}^X$. Then V is open in X and $F \subseteq V$. Hence, by assumption, there exists an open set W such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Let $U = A \setminus G$. Then $Ex_{A,X}U = V$. Hence $V \cap A = U$ and $W \subseteq V$. We conclude that $\overline{F}^X \subseteq W \subseteq V = X \setminus \overline{G}^X$, i.e. $\overline{F}^X \cap \overline{G}^X = \emptyset$. □

Remark 3.4. In Example 3.22 below we will show that there exist spaces $(X, \mathcal{T})$, subspaces A of X and bases $\mathcal{B}$ of $\mathcal{T}$ such that A is $2_{\mathcal{B}}$-combinatorially embedded in X but A is not 2-combinatorially embedded in X.

Proposition 3.5. *Let* $(X, \mathcal{T})$ *be a* T_1*-space,* $\mathcal{O}$ *be a topology of Tychonoff-type on* $\mathcal{CL}(X)$ *and* $A \subseteq X$. *Put* $\mathcal{B}_A = \{U \cap A : U \in \mathcal{B}_{\mathcal{O}}\}$ *(see Fact 2.4 for the notation* $\mathcal{B}_{\mathcal{O}}$*). The family* $\mathcal{B}_A$ *generates a topology of Tychonoff-type* $\mathcal{O}_A$ *on* $\mathcal{CL}(A)$. *The function* $i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O})$*, defined by* $i_{A,X}(F) := \overline{F}^X$*, is inversely continuous (i.e. it is injective and its inverse, defined on* $i_{A,X}(\mathcal{CL}(A))$*, is a continuous function) if and only if the set* A *is* $2_{\mathcal{B}_{\mathcal{O}}}$*-combinatorially embedded in* X.

Proof. The family $\mathcal{B} := \mathcal{B}_{\mathcal{O}}$ is closed under finite intersections and $X \in \mathcal{B}$ (see Fact 2.4). Hence the family $\mathcal{B}_A$ is closed under finite intersections and $A \in \mathcal{B}_A$. Therefore, by Corollary 2.9, $\mathcal{B}_A$ generates a topology of Tychonoff-type $\mathcal{O}_A$ on $\mathcal{CL}(A)$.

The function $i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O})$ is clearly injective. Denote by g its inverse defined on $i_{A,X}(\mathcal{CL}(A))$, i.e. $g : i_{A,X}(\mathcal{CL}(A)) \to \mathcal{CL}(A)$.

($\Rightarrow$) Let $H \in \mathcal{CL}(A)$, $U \in \mathcal{B}$ and $H \subseteq U$. Then $H \in (U \cap A)^+_{\mathcal{CL}(A)} \in \mathcal{O}_A$. Since $g(\overline{H}^X) = H$, the continuity of g implies that there exists a $V \in \mathcal{B}$ such that

$$\overline{H}^X \subseteq V \text{ and } g(V^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A))) \subseteq (U \cap A)^+_{\mathcal{CL}(A)}.$$

Then $V \cap A \subseteq U \cap A$. Indeed, let $x \in V \cap A$. Since X is a T_1-space, we obtain that

$$\{x\} \in V^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A)) \text{ and } g(\{x\}) = \{x\}.$$

Hence $x \in U \cap A$. So, A is $2_{\mathcal{B}}$-combinatorially embedded in X.

($\Leftarrow$) Let $F \in i_{A,X}(\mathcal{CL}(A))$ and $g(F) = H$. Then $F = \overline{H}^X$ and $H \in \mathcal{CL}(A)$. Let $U \in \mathcal{B}_A$ be such that $H \subseteq U$. Then there exists a $V \in \mathcal{B}$ with $V \cap A = U$. Hence $H \subseteq V$. Since A is $2_{\mathcal{B}}$-combinatorially embedded in X, there exists a $W \in \mathcal{B}$ such that $F = \overline{H}^X \subseteq W$ and $W \cap A \subseteq V \cap A = U$. Then $F \in W^+_{\mathcal{CL}(X)} \in \mathcal{O}$. We will show that

$$g(W^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A))) \subseteq U^+_{\mathcal{CL}(A)}.$$

Indeed, let $K \in \mathcal{CL}(A)$, $G = \overline{K}^X$ and $G \subseteq W$. Then $g(G) = K$ and

$$K = G \cap A \subseteq W \cap A \subseteq V \cap A = U,$$

i.e. $K \in U^+_{\mathcal{CL}(A)}$, as we have to show. Hence, g is a continuous function. $\square$

Corollary 3.6 ([8], Theorem 2.1)**.** *If in Proposition 3.5 we take $\mathcal{O}$ to be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$ then the function $i_{A,X}$ is inversely continuous if and only if A is 2-combinatorially embedded in X.*

Proof. It follows from Propositions 3.5, 2.20 and 3.3. $\square$

Corollary 3.7. *Let $(X, \mathcal{T})$ be a T_2-space, $A \subseteq X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$. Let $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Assume that the following condition is satisfied:*

(*) *For any $U \in \mathcal{T}$ and for all countable $F \in \mathcal{CL}(A)$ such that $|A \setminus F| \geq \aleph_0$ and $F \subseteq U$, there exists a $V \in \mathcal{B}_{\mathcal{O}}$ with $F \subseteq V \subseteq U$.*

Then the set A is sequentially closed.

Proof. Put $\mathcal{B} := \mathcal{B}_{\mathcal{O}}$. Then, by Proposition 2.24, $\mathcal{B} \subseteq \mathcal{T}$. Assume that the set A is not sequentially closed. Then there exists a sequence $(x_n)_{n\in\omega}$ in A and a point $x \in X \setminus A$ such that $\lim_{n\to\infty} x_n = x$. Without loss of generality we can assume $x_n \neq x_m$ for all $n \neq m$.

Let us consider the sets $F = \{x_{2n} : n \in \omega\}$ and $G = \{x_{2n-1} : n \in \omega\}$. Put $U = X \setminus \overline{G}^X$. Then F is a countable closed subset of A, $|A \setminus F| \geq \aleph_0$, $F \subseteq U$ and $U \in \mathcal{T}$. By (*), there exists a $V \in \mathcal{B}$ such that $F \subseteq V \subseteq U$. Since we are assuming that the function $i_{A,X}$ is inversely continuous, we obtain, by Proposition 3.5, that the set A is $2_{\mathcal{B}}$-combinatorially embedded in X. Hence there exists a $W \in \mathcal{B}$ such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Then $x \in W$, because $x \in \overline{F}^X$. Since $W \in \mathcal{T}$ and x is a limit point of G, we have $G \cap W \neq \emptyset$. However this is a contradiction because

$$W \cap A \subseteq V \cap A \subseteq U = X \setminus \overline{G}^X,$$

and hence $G \cap W = \emptyset$. Therefore, A is sequentially closed. □

Remark 3.8. In Example 3.22 below we will show that condition (*) of Corollary 3.7 is essential, i.e., if we omit it, then the set A could fail to be sequentially closed.

Corollary 3.9 ([8], Corollary 2.3)**.** *Let $(X, \mathcal{T})$ be a T_2-space, $A \subseteq X$, $\mathcal{O}$ be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$ and $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Then the set A is sequentially closed.*

Proof. We have, by Proposition 2.20, that $\mathcal{B}_{\mathcal{O}} = \mathcal{T}$. Then condition (*) of Corollary 3.7 is trivially satisfied. Hence, by Corollary 3.7, A is sequentially closed. □

Corollary 3.10. *Let $(X, \mathcal{T})$ be a sequential T_2-space, $A \subseteq X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$. Assume that condition (*) of Corollary 3.7 is satisfied. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):*

(a) *$i_{A,X}$ is a homeomorphic embedding;*
(b) *$i_{A,X}$ is inversely continuous;*
(c) *A is closed in X.*

Proof. It is clear that (a) implies (b). The implication (c)⇒(a) is true for any X, because if A is a closed subset of X then $i_{A,X}$ is the inclusion map. Let us show that (b) implies (c). By Corollary 3.7, A is sequentially closed. Since X is a sequential space, we obtain that the set A is closed. □

Corollary 3.11 ([8], Corollary 2.4)**.** *Let $(X, \mathcal{T})$ be a sequential T_2-space, $A \subseteq X$ and $\mathcal{O}$ be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):*

(a) *$i_{A,X}$ is a homeomorphic embedding;*
(b) *$i_{A,X}$ is inversely continuous;*
(c) *A is closed in X.*

Definition 3.12. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. The space $(X, \mathcal{T})$ is called *$\mathcal{O}$-commutative* if for any $A \subseteq X$ the function $i_{A,X}$, defined in Proposition 3.5, is a homeomorphic embedding.

When $\mathcal{O}$ is the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$, the notion of "$\mathcal{O}$-commutative space" coincides with the notion of "*commutative space*", introduced in [6, 8].

Corollary 3.13. *Let $(X, \mathcal{T})$ be a sequential T_2-space, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$ and let condition (*) of Corollary 3.7 be satisfied for every subspace A of X. Then X is $\mathcal{O}$-commutative if and only if X is discrete.*

Proof. It follows from Corollary 3.10. □

Corollary 3.14 ([8], Corollary 2.5)**.** *If X is a sequential T_2-space then X is commutative if and only if X is discrete.*

Example 3.15. Let us show that there exist spaces X and topologies $\mathcal{O}$ of Tychonoff-type on $\mathcal{CL}(X)$ that are not Tychonoff topologies and that satisfy all hypothesis of Corollary 3.13.

Let $X = D(\aleph_1)$ be the discrete space of cardinality $\aleph_1$. Let

$$\mathcal{B} = \{A \subset X : |A| \leq \aleph_0\} \cup \{X\}$$

and $\mathcal{M} = \mathcal{CL}(X)$. Then $\mathcal{M}$ is a natural family on X, $\mathcal{B}$ is $\mathcal{M}$-closed, $\mathcal{B}^{\cap} = \mathcal{B}$, $X \in \mathcal{B}$ and $\mathcal{B}$ is a base for the discrete topology on X. Let $\mathcal{O}_{\mathcal{B}}$ be the topology on $\mathcal{M}$ generated by $\mathcal{B}$. Then $\mathcal{O}_{\mathcal{B}}$ is a topology of Tychonoff-type on $\mathcal{M}$, however it is not a Tychonoff topology. In fact, by Proposition 2.30, $\mathcal{B} = \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$. Hence $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \neq \mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$ and, by Corollary 2.23, $\mathcal{O}$ cannot be a Tychonoff topology. Obviously, $\mathcal{B} = \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$ satisfies condition (*) of Corollary 3.7 for any subspace A of X.

Definition 3.16. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. The space $(X, \mathcal{T})$ is called *$\mathcal{O}$-HS-space* if, for any $A \subseteq X$, the function $i_{A,X}$, defined in Proposition 3.5, is continuous.

When $\mathcal{O}$ is the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$, the notion of "$\mathcal{O}$-HS-space" coincides with the notion of "*HS-space*", introduced in [2, 3].

Corollary 3.17. *Let $(X, \mathcal{T})$ be a T_1-space and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. Then X is an $\mathcal{O}$-commutative space if and only if X is an $\mathcal{O}$-HS-space and every subset A of X is $2_{\mathcal{B}_{\mathcal{O}}}$-combinatorially embedded in X.*

Proof. It follows from Proposition 3.5. □

Corollary 3.18 ([8],Corollary 2.2)**.** *A T_1-space X is commutative if and only if X is an HS-space and every subspace of X is 2-combinatorially embedded in X.*

Proof. It follows from Corollary 3.17 and Proposition 3.3. □

Example 3.19. We will describe two Tychonoff-type, non Tychonoff topologies on two different subfamilies of $\mathcal{P}(\mathbb{R})$ generated by the family $\mathcal{B}$ of all open intervals of $\mathbb{R}$. One of the resulting spaces will be T_0 and the other one will not.

Let $\mathcal{T}$ be the natural topology on $X := \mathbb{R}$. Then the family $\mathcal{B}$ of all open intervals in X is a base for $\mathcal{T}$, it is closed under finite intersections and $X \in \mathcal{B}$. Put $\mathcal{M} := \mathcal{CL}(X, \mathcal{T})$ and $\mathcal{M}' := \mathcal{F}in_2(X)$. They are natural families. The family $\mathcal{B}$ is both $\mathcal{M}'$-closed and $\mathcal{M}$-closed. Indeed, let $U \subseteq X$ be $\mathcal{M}'$-covered by a subfamily $\mathcal{B}_U$ of $\mathcal{B}$. Then $U \in \mathcal{T}$ and for every $x, y \in U$ there exists an open interval $(\alpha, \beta) \in \mathcal{B}_U$ containing the points x and y. Hence U is a connected open set in $\mathbb{R}$, i.e. $U \in \mathcal{B}$. Therefore, $\mathcal{B}$ is an $\mathcal{M}'$-closed family. Since $\mathcal{M}' \subset \mathcal{M}$, we obtain, by Proposition 2.28, that $\mathcal{B}$ is an $\mathcal{M}$-closed family as well.

By Corollary 2.9, $\mathcal{B}$ generates Tychonoff-type topologies $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$ and $\mathcal{O}'_{\mathcal{B}}$ on $\mathcal{M}'$. As it follows from Proposition 2.30, $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}_{\mathcal{O}'_{\mathcal{B}}} = \mathcal{B} \neq \mathcal{T}$. Hence, by Corollary 2.23, $\mathcal{O}_{\mathcal{B}}$ and $\mathcal{O}'_{\mathcal{B}}$ are not Tychonoff topologies on $\mathcal{M}$, respectively $\mathcal{M}'$.

It is easy to see that $(\mathcal{M}', \mathcal{O}'_{\mathcal{B}})$ is a T_0-space. Indeed, let $\{x, y\}$ and $\{u, v\}$ be two distinct elements in $\mathcal{M}'$. We can assume $x < x + \varepsilon < u \leq v$ for some $\varepsilon > 0$. Consider the interval $B = (x + \varepsilon, +\infty)$. Then $\{u, v\} \in B^+_{\mathcal{M}'}$ but $\{x, y\} \notin B^+_{\mathcal{M}'}$.

Let's prove that $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ is not a T_0-space. Put $F = \{2k : k \in \mathbb{Z}\}$ and $G = \{2k + 1 : k \in \mathbb{Z}\}$. Then $F, G \in \mathcal{M}$ and $F \neq G$ but the only neighbourhood of both F and G in $\mathcal{M}$ is $X^+_{\mathcal{M}} = \mathcal{M}$.

Example 3.20. In the notations of Example 3.19, we will show that $(\mathbb{R}, \mathcal{T})$ *is an $\mathcal{O}_{\mathcal{B}}$-HS-space.*

We are working now with the space $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ from Example 3.19. We will write simply $\mathcal{O}$ instead of $\mathcal{O}_{\mathcal{B}}$.

Let $A \subseteq X$. We have to show that the function

$$i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O}),$$

where the topology $\mathcal{O}_A$ on $\mathcal{CL}(A)$ is generated by the family

$$\mathcal{B}_A = \{A \cap U : U \in \mathcal{B}\},$$

is continuous (see Proposition 3.5 for the notation $i_{A,X}$). Let $B \in \mathcal{B}$. We will show that $i^{-1}_{A,X}(B^+_{\mathcal{M}})$ is an open set. Take an $F \in i^{-1}_{A,X}(B^+_{\mathcal{M}})$. Then $F \in \mathcal{CL}(A)$ and $\overline{F}^X \subseteq B$. There exists an $E \in \mathcal{B}$ such that

$$\overline{F}^X \subseteq E \subseteq \overline{E}^X \subseteq B$$

(this is clear if F is bounded, since in this case $\overline{F}^X$ is compact; if F is unbounded below, but is bounded above, then $B = (-\infty, \beta)$, for some $\beta \in \mathbb{R}$, and we can pick $E = (-\infty, \gamma)$ with $\sup F < \gamma < \beta$; similarly if F is unbounded above but not below; if F is unbounded both above and below then we have $B = \mathbb{R}$ and we put $E := B$). Then

$$F \in (E \cap A)^+_{\mathcal{CL}(A)} \subseteq i^{-1}_{A,X}(B^+_{\mathcal{M}}).$$

Indeed, let $G \in (E \cap A)^+_{\mathcal{CL}(A)}$. Then

$$\overline{G}^X \subseteq \overline{E}^X \subseteq B,$$

i.e. $i_{A,X}(G) \in B^+_{\mathcal{M}}$.

Remark 3.21. Let us note that a similar proof shows that *every subspace* Y *of* $(\mathbb{R}, \mathcal{T})$ *is an* $\mathcal{O}_{\mathcal{B}_Y}$*-HS-space* (see Examples 3.19 and 3.20 for the notations).

More generally, let Y be a topological space and $\mathcal{D}$ be a base of Y. We will say that Y is $\mathcal{D}$*-normal* if for every $F \in \mathcal{C}L(Y)$ and for every $D \in \mathcal{D}$ such that $F \subseteq D$ there exists an $E \in \mathcal{D}$ with $F \subseteq E \subseteq \overline{E}^Y \subseteq D$. Now, arguing as in Example 3.20, we can prove that *if* Y *is a* $\mathcal{D}$*-normal space,* $\mathcal{D} = \mathcal{D}^{\cap}$ *and* $Y \in \mathcal{D}$*, then* Y *is an* $\mathcal{O}$*-HS-space, where* $\mathcal{O}$ *is the Tychonoff-type topology on* $\mathcal{C}L(Y)$ *generated by* $\mathcal{D}$*.* This generalizes the result of M. Sekanina [15] that any normal space is a HS-space.

Example 3.22. In the notations of Examples 3.19 and 3.20, we will show that *the function* $i_{A,\mathbb{R}}$ *is a homeomorphic embedding for any open interval* A*.*

We will argue for $A = (0, 1)$; the proof for any other open interval is similar. We know, by Example 3.20, that the function $i_{A,X}$ is continuous. Therefore we only need to prove that $i_{A,X}$ is inversely continuous. By Proposition 3.5, it is enough to show that the set A is $2_{\mathcal{B}}$-combinatorially embedded in X. So, let H be a closed subset of $(0, 1)$ and let $B = (\alpha, \beta) \in \mathcal{B}$ be such that $H \subseteq B$. We have to find a $D \in \mathcal{B}$ such that $\overline{H}^X \subset D$ and $D \cap (0, 1) \subseteq B$. Clearly, $\overline{H}^X \subseteq [0, 1]$. If $\overline{H}^X \subset (0, 1)$, we can take $D = B$. If $0 \in \overline{H}^X$ but $1 \notin \overline{H}^X$ then $\alpha \leq 0$ and we can put $D = (-1, \beta)$. If $1 \in \overline{H}^X$ but $0 \notin \overline{H}^X$ then $\beta \geq 1$ and we can put $D = (\alpha, 2)$. If $0, 1 \in \overline{H}^X$ then $\alpha \leq 0$ and $\beta \geq 1$ and we put $D = (-1, 2)$. Therefore, A *is* $2_{\mathcal{B}}$*-combinatorially embedded in* $(\mathbb{R}, \mathcal{T})$*.*

Note that A *is not 2-combinatorially embedded in* $(\mathbb{R}, \mathcal{T})$*.*

Observe that the triple $((\mathbb{R}, \mathcal{T}), A, \mathcal{O})$ satisfies all hypothesis of Corollary 3.7 except for condition (*), but A is not sequentially closed.

Example 3.23. Let $Y \subseteq \mathbb{R}$. We will say, as usual, that a point $x \in Y$ is *isolated from the right (left) (in* Y*)* if there exists an $\varepsilon > 0$ such that if we put $U = (x, x + \varepsilon)$ $(U = (x - \varepsilon, x))$ then $U \cap Y = \emptyset$. Now, in the notations of Examples 3.19 and 3.20, we have: *a subspace* Y *of* $(\mathbb{R}, \mathcal{T})$ *is* $\mathcal{O}_{\mathcal{B}_Y}$*-commutative if and only if every point of* Y *is either isolated from the right or from the left.*

We first show that a space Y that has a point y_0 which is non-isolated both from the left and from the right cannot be $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Indeed, put $A = Y \setminus \{y_0\}$. We will prove that A is not $2_{\mathcal{B}_Y}$-combinatorially embedded in Y. By Proposition 3.5, this will imply that the function $i_{A,Y}$ is not inversely continuous and hence the space Y will be not $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Let $H = \{y_n : n \in \omega\}$ be a decreasing sequence in Y converging to y_0. Then H is a closed subset of A and $H \subset (y_0, +\infty) \cap Y$. Suppose that there exists $B \in \mathcal{B}$ such that $cl_Y H \subseteq B$ and $B \cap A \subseteq (y_0, +\infty)$. Since $y_0 \in cl_Y H \subseteq B$ and y_0 is not isolated from the left, we have that $(B \cap A) \setminus (y_0, +\infty) \neq \emptyset$, which is a contradiction. Hence, A is not $2_{\mathcal{B}_Y}$-combinatorially embedded in Y.

Now we will show that a space Y having only points which are isolated either from the left or from the right is $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Let $A \subset Y$. We know, by Remark 3.21, that the function $i_{A,Y}$ is continuous. Hence it is enough to show that it is inversely continuous, i.e., according to Proposition 3.5, that A is $2_{\mathcal{B}_Y}$-combinatorially embedded in Y. So, let $H \in \mathcal{C}L(A)$ and let $H \subseteq B \cap Y$ for some $B = (\alpha, \beta)$. We have to find a $D \in \mathcal{B}$ such that $\overline{H}^Y \subset D \cap Y$ and $D \cap A \subseteq B$. We have $cl_Y H \subseteq \overline{B}^X = [\alpha, \beta]$. If $cl_Y H \subseteq B$, we can take $D = B$ and we are done. If $\alpha \in cl_Y H$ and $\beta \notin cl_Y H$ then $\alpha \in Y$ and α is not isolated from the right, being a limit point of H. Hence, by the assumption, α is isolated from the left. Thus there exists a $\gamma < \alpha$ such that $(\gamma, \alpha) \cap Y = \emptyset$. Then $D = (\gamma, \beta)$ is the required interval. The other two possible cases are treated analogously.

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