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TRANSFINITE SEQUENCES OF CONTINUOUS AND BAIRE 1 FUNCTIONS ON SEPARABLE METRIC SPACES

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Abstract. We investigate the existence of well-ordered sequences of Baire 1 functions on separable metric spaces.

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set X is partially ordered by the pointwise order, that is $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. In other words put $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and $f(x) \neq g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible length of the increasing or decreasing well-ordered sequences of functions in $\mathcal{F}$ with respect to this order.

A classical theorem of Kuratowski asserts, that if $\mathcal{F}$ is the set of continuous or Baire 1 functions defined on a Polish space X, then there exists a monotone sequence of length ξ in $\mathcal{F}$ iff $\xi < \omega_1$ (see [2, §24. III.2']). Moreover, P. Komjáth proved in [1] that the corresponding question concerning Baire α functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

In the present paper we investigate what happens if we drop the condition of completeness and replace the Polish space X by a separable metric space.

Our main results are the following. Let $d(X)$ denote the density of a space X.

Theorem. *Let (X, ϱ) be a metric space. Then there exists a well-ordered sequence of length ξ of continuous real-valued functions defined on X iff $\xi < d(X)^+$.*

Corollary. *A metric space is separable iff every well-ordered sequence of continuous functions defined on it is countable.*

Theorem. *There exists a separable metric space on which there exists a well-ordered sequence of length ω_1 of Baire 1 functions.*

Theorem. *The following statement: 'There exists a separable metric space on which there exists a well-ordered sequence of length ω_2 of Baire 1 functions' is independent of $ZFC + \neg CH$.*

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Remark. During and after the conference Kenneth Kunen answered one of my questions, and also improved some of the results and proofs. These results will appear in a forthcoming joint paper.

References

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