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FELL-CONTINUOUS SELECTIONS AND TOPOLOGICALLY WELL-ORDERABLE SPACES II

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Abstract. The present paper improves a result of [3] by showing that a space X is topologically well-orderable if and only if there exists a selection for $\mathcal{F}_2(X)$ which is continuous with respect to the Fell topology on $\mathcal{F}_2(X)$. In particular, this implies that $\mathcal{F}(X)$ has a Fell-continuous selection if and only if $\mathcal{F}_2(X)$ has a Fell-continuous selection.

1. Introduction

Let X be a topological space, and let $\mathcal{F}(X)$ be the family of all non-empty closed subsets of X. Also, let τ be a topology on $\mathcal{F}(X)$ and $\mathcal{D} \subset \mathcal{F}(X)$. A map $f : \mathcal{D} \to X$ is a *selection* for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A map $f : \mathcal{D} \to X$ is a *τ-continuous* selection for $\mathcal{D}$ if it is a selection for $\mathcal{D}$ which is continuous with respect to the relative topology τ on $\mathcal{D}$ as a subspace of $\mathcal{F}(X)$.

Two topologies on $\mathcal{F}(X)$ will play the most important role in this paper. The first one is the *Vietoris topology* τ_V which is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \cap V \neq \emptyset,\ V \in \mathcal{V},\ \text{and } S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X. The other one is the *Fell topology* τ_F which is defined by all basic Vietoris neighbourhood $\langle \mathcal{V} \rangle$ with the property that $X \setminus \bigcup \mathcal{V}$ is compact.

Finally, let us recall that a space X is *topologically well-orderable* (see Engelking, Heath and Michael [2]) if there exists a linear order "$\prec$" on X such that X is a linear ordered topological space with respect to $\prec$, and every non-empty closed subset of X has a $\prec$-minimal element.

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Recently, the topologically well-orderable spaces were characterized in [3, Theorem 1.3] by means of Fell-continuous selections for their hyperspaces of non-empty closed subsets.

Theorem 1.1 ([3])**.** *A Hausdorff space X is topologically well-orderable if and only if $\mathcal{F}(X)$ has a τ_F-continuous selection.*

In the present paper, we improve Theorem 1.1 by showing that one may use τ_F-continuous selections only for the subset $\mathcal{F}_2(X) = \{S \in \mathcal{F}(X) : |S| \leq 2\}$ of $\mathcal{F}(X)$. Namely, the following theorem will be proven.

Theorem 1.2. *A Hausdorff space X is topologically well-orderable if and only if $\mathcal{F}_2(X)$ has a τ_F-continuous selection.*

About related results for Vietoris-continuous selections, the interested reader is referred to van Mill and Wattel [6].

Theorem 1.2 is interesting also from another point of view. According to Theorem 1.1, it implies the following result which may have an independent interest.

Corollary 1.3. *If X is a Hausdorff space, then $\mathcal{F}(X)$ has a τ_F-continuous selection if and only if $\mathcal{F}_2(X)$ has a τ_F-continuous selection.*

A word should be said also about the proof of Theorem 1.2. In general, it is based on the proof of Theorem 1.1 stated in [3], and is separated in a few different steps which are natural generalizations of the corresponding ones given in [3]. In fact, the paper demonstrates that all statements of [3] remain true if $\mathcal{F}(X)$ is replaced by $\mathcal{F}_2(X)$. Related to this, the interested reader may consult an alternative proof of Theorem 1.2 given in [1] and based again on the scheme in [3].

2. A reduction to locally compact spaces

In the sequel, all spaces are assumed to be at least Hausdorff.

In this section, we prove the following generalization of [3, Theorem 2.1].

Theorem 2.1. *Let X be a space such that $\mathcal{F}_2(X)$ has a τ_F-continuous selection. Then X is locally compact.*

Proof. We follow the proof of [3, Theorem 2.1]. Namely, let f be a τ_F-continuous selection for $\mathcal{F}_2(X)$ and suppose, if possible, that X is not locally compact. Hence, there exists a point $p \in X$ such that $\overline{V}$ is not compact for every neighbourhood V of p in X. Claim that there exists a point $q \in X$ such that

(1) $\quad q \neq p$ and $f(\{p, q\}) = p$.

To this purpose, note that there exists $F \in \mathcal{F}(X)$ such that F is not compact and $p \notin F$. Then, $f^{-1}(X \setminus F)$ is a τ_F-neighbourhood of $\{p\}$ in $\mathcal{F}_2(X)$, so there exists a finite family $\mathcal{W}$ of open subsets of X such that $X \setminus \bigcup \mathcal{W}$ is compact and

$$\{p\} \in \langle \mathcal{W} \rangle \cap \mathcal{F}_2(X) \subset f^{-1}(X \setminus F).$$

Then, $F \cap W \neq \emptyset$ for some $W \in \mathcal{W}$ because F is not compact. Therefore, there exists a point $q \in F \cap (\bigcup \mathcal{W})$. This q is as required.

Let q be as in (1). Since X is Hausdorff, $f(\{q\}) \neq f(\{p, q\})$, and f is τ_F-continuous, there now exist two finite families $\mathcal{U}$ and $\mathcal{V}$ of open subsets of X such that $X \setminus \bigcup \mathcal{U}$ is compact, $\{q\} \in \langle \mathcal{U} \rangle$, $\{p, q\} \in \langle \mathcal{V} \rangle$, and $\langle \mathcal{U} \rangle \cap \langle \mathcal{V} \rangle = \emptyset$. Then,

$$p \in V_p = \bigcap \{V \in \mathcal{V} : p \in V\} \subset X \setminus \bigcup \mathcal{U}. \tag{2}$$

Indeed, suppose there is a point $\ell \in V_p \cap (\bigcup \mathcal{U})$. Then, $\{\ell, q\} \in \langle \mathcal{U} \rangle$ because $\{q\} \in \langle \mathcal{U} \rangle$. However, we also get that $\{\ell, q\} \in \langle \mathcal{V} \rangle$ because $q \notin V$ for some $V \in \mathcal{V}$ implies $p \in V$, hence $\ell \in V_p \subset V$. Thus, we finally get that $\{\ell, q\} \in \langle \mathcal{U} \rangle \cap \langle \mathcal{V} \rangle$ which is impossible. So, (2) holds as well.

To finish the proof, it remains to observe that this contradicts the choice of p. Namely V_p becomes a neighbourhood of p which, by (2), has a compact closure because $X \setminus \bigcup \mathcal{U}$ is compact. □

3. A reduction to compact spaces

For a locally compact space X we will use αX to denote the one point compactification of X. For a non-compact locally compact X let us agree to denote by α the point of the singleton $\alpha X \setminus X$.

In what follows, to every family $\mathcal{D} \subset \mathcal{F}(X)$ we associate a family $\alpha(\mathcal{D}) \subset \mathcal{F}(\alpha X)$ defined by

$$\alpha(\mathcal{D}) = \{S \in \mathcal{F}(\alpha X) : S \cap X \in \mathcal{D} \cup \{\emptyset\}\}.$$

The following extension theorem was actually proven in [3, Theorem 3.1].

Theorem 3.1. *Let X be a locally compact non-compact space X, and $\mathcal{D} \subset \mathcal{F}(X)$. Then, $\mathcal{D}$ has a τ_F-continuous selection if and only if $\alpha(\mathcal{D})$ has a τ_V-continuous selection g such that $g^{-1}(\alpha) = \{\{\alpha\}\}$.*

Proof. Just the same proof as in [3, Theorem 3.1] works. Namely, if f is a τ_F-continuous selection for $\mathcal{D}$, we may define a selection g for $\alpha(\mathcal{D})$ by $g(S) = f(S \cap X)$ if $S \cap X \neq \emptyset$ and $g(S) = \alpha$ otherwise, where $S \in \alpha(\mathcal{D})$. Clearly $g^{-1}(\alpha) = \{\{\alpha\}\}$ and, as shown in [3, Theorem 3.1], g is τ_V-continuous. If now g is a τ_V-continuous selection for $\alpha(\mathcal{D})$, with $g^{-1}(\alpha) = \{\{\alpha\}\}$, then $g(S \cup \{\alpha\}) \in S$ for every $S \in \mathcal{D}$, so we may define a selection f for $\mathcal{D}$ by $f(S) = g(S \cup \{\alpha\})$, $S \in \mathcal{D}$. The verification that f is τ_F-continuous was done in [3, Theorem 3.1]. □

4. Special selections and connected sets

In what follows, to every selection $f : \mathcal{F}_2(X) \to X$ we associate an order-like relation "$\prec_f$" on X (see Michael [5]) defined for $x \neq y$ by

$$x_1 \prec_f x_2 \text{ iff } f(\{x_1, x_2\}) = x_1.$$

Further, we will need also the following $\prec_f$-intervals:

$$(x, +\infty)_{\prec_f} = \{z \in X : x \prec_f z\}$$

and

$$[x, +\infty)_{\prec_f} = \{z \in X : x \preceq_f z\}.$$

Now, we provide the generalization of [3, Theorem 4.1] for the case of $\mathcal{F}_2(X)$.

Theorem 4.1. *Let X be a space, $a \in X$, and let $A \in \mathcal{F}(X)$ be a connected set such that $|A| > 1$ and $a \in A \cap \overline{X \setminus A}$. Also, let $f : \mathcal{F}_2(X) \to X$ be a τ_V-continuous selection for $\mathcal{F}_2(X)$. Then, $f^{-1}(a) \neq \{\{a\}\}$.*

Proof. Suppose, on the contrary, that $f^{-1}(a) = \{\{a\}\}$. By hypothesis, there exists a point $b \in A$, with $b \neq a$. Since f is τ_V-continuous, $f(\{a, b\}) = b$ and $a \in \overline{X \setminus A}$, we can find a point $c \in X \setminus A$ such that $f(\{b, c\}) = b$. Then, $B = A \cap (c, +\infty)_{\prec_f}$ is a clopen subset of A because $B = A \cap [c, +\infty)_{\prec_f}$, see [5]. However, this is impossible because $b \in A \setminus B$, while $a \in B$. □

5. A further result about special selections

Following [3], we shall say that a point $a \in X$ is a *partition* of X if there are open subset $L, R \subset X \setminus \{a\}$ such that $\overline{L} \cap \overline{R} = \{a\}$ and $L \cap R = \emptyset$.

We finalize the preparation for the proof of Theorem 1.2 with the following result about special Vietoris continuous selections and partitions which generalizes [3, Theorem 5.1].

Theorem 5.1. *Let X be a compact space, f a τ_V-continuous selection for $\mathcal{F}_2(X)$, and let $a \in X$ be a partition of X such that $f^{-1}(a) = \{\{a\}\}$. Then, X is first countable at a.*

Proof. By definition, there are open sets $L, R \subset X \setminus \{a\}$ such that $\overline{L} \cap \overline{R} = \{a\}$ and $L \cap R = \emptyset$. Hence, both L and R are non-empty. Take a point $\ell_0 \in L$. Then, by hypothesis, $f(\{\ell_0, a\}) = \ell_0$. Since f is τ_V-continuous, this implies the existence of a neighbourhood $L_0 \subset L$ of ℓ_0 and a neighbourhood V_0 of a such that

$$L_0 \cap V_0 = \emptyset \text{ and } f(\langle\{L_0, V_0\}\rangle \cap \mathcal{F}_2(X)) \subset L_0.$$

Since $a \in \overline{R}$, there exists a point $r_0 \in V_0 \cap R$. Observe that $f(\{a, r_0\}) = r_0 \in V_0$. Hence, just like before, we may find a neighbourhood $R_0 \subset R \cap V_0$ of r_0 and a neighbourhood $W_0 \subset V_0$ of a such that

$$R_0 \cap W_0 = \emptyset \text{ and } f(\langle\{R_0, W_0\}\rangle \cap \mathcal{F}_2(X)) \subset R_0.$$

Thus, by induction, we may construct a sequence $\{\ell_n : n < \omega\}$ of points of L, a sequence $\{r_n : n < \omega\}$ of points of R, and open sets $L_n, V_n, R_n, W_n \subset X$ such that

$$(3)\qquad \begin{array}{l} \ell_n \in L_n, \\ a \in V_n, \\ L_n \cap V_n = \emptyset \text{ and} \\ f(\langle\{L_n, V_n\}\rangle \cap \mathcal{F}_2(X)) \subset L_n, \end{array}$$

$$(4)\qquad \begin{array}{l} r_n \in R_n, \\ a \in W_n, \\ R_n \cap W_n = \emptyset \text{ and} \\ f(\langle\{R_n, W_n\}\rangle \cap \mathcal{F}_2(X)) \subset R_n, \end{array}$$

and

$$(5)\qquad \begin{array}{l} V_{n+1} \subset W_n \subset V_n, \\ L_{n+1} \subset L \cap W_n \text{ and} \\ R_n \subset R \cap V_n. \end{array}$$

Since X is compact, $\{\ell_n : n < \omega\}$ has a cluster point ℓ, and $\{r_n : n < \omega\}$ has a cluster point r. We claim that $\ell = r$. Indeed, suppose for instance that $\ell \prec_f r$ (the case $r \prec_f \ell$ is symmetric). Then, there are disjoint open sets U_ℓ and U_r such that $\ell \in U_\ell$, $r \in U_r$, and $x \prec_f y$ for every $x \in U_\ell$ and $y \in U_r$, see [4]. Next, take $\ell_n \in U_\ell$ and $r_m \in U_r$ such that $n > m$. Then, we have $\ell_n \prec_f r_m$. However, by (3), (4) and (5), we get that $\{r_m, \ell_n\} \in \langle\{R_m, W_m\}\rangle \cap \mathcal{F}_2(X)$, and therefore $f(\{r_m, \ell_n\}) = r_m$. This is clearly impossible, so $\ell = r$.

Having already established this, let us observe that $b = \ell = r$ implies $b \in \overline{L} \cap \overline{R}$ because $\ell \in \overline{L}$ and $r \in \overline{R}$. However, $\overline{L} \cap \overline{R} = \{a\}$ which finally implies that $b = a$.

We are now ready to prove that, for instance, $\{W_n : n < \omega\}$ is a local base at a. To this end, suppose if possible that this fails. Hence, there exists an open neighbourhood U of a such that $W_n \setminus U \neq \emptyset$ for every $n < \omega$. Next, whenever $n < \omega$, take a point $t_n \in W_n \setminus U$. Since X is compact, $\{t_n : n < \omega\}$ has a cluster point $t \notin U$. Then, $t \prec_f a$ and, as before, we may find disjoint open sets U_t and U_a such that $t \in U_t$, $a \in U_a$, and $x \prec_f y$ for every $x \in U_t$ and $y \in U_a$. Next, take $t_n \in U_t$ and $r_m \in U_a$ such that $n > m$. Then, $t_n \prec_f r_m$, while, by (4) and (5), $r_m \prec_f t_n$ because $\{r_m, t_n\} \in \langle\{R_m, W_m\}\rangle \cap \mathcal{F}_2(X)$. The contradiction so obtained completes the proof. □

6. Proof of Theorem 1.2

In case X is a topologically well-orderable space, we may use Theorem 1.1.

Suppose that $\mathcal{F}_2(X)$ has a τ_F-continuous selection. If X is compact, then Theorem 1.2 is, in fact, a result of van Mill and Wattel [6]. Let X be non-compact. By Theorem 2.1, X is locally compact. Then, by Theorem 3.1,

$\mathcal{F}_2(\alpha X)$ has a τ_V-continuous selection f such that $f^{-1}(\alpha) = \{\{\alpha\}\}$. Relying once again on the result of [6], αX is a linear ordered topological space with respect to some linear order "$<$" on αX. It now suffices to show that there exists a compatible (with the topology of αX) linear order "$\prec$" on αX such that α is either the first or the last element of αX, see [2, Lemma 4.1]. We show this following precisely the proof of Theorem 1.1 in [3]. Namely, let

$$L = \{x \in \alpha X : x < \alpha\} \text{ and } R = \{x \in \alpha X : \alpha < x\}.$$

Note that $L, R \subset \alpha X \setminus \{\alpha\} = X$ are open subsets of αX. In case one of these sets is also closed, the desired linear order "$\prec$" on αX can be defined by exchanging the places of L and R. Namely, by letting for $x, y \in \alpha X$ that $x \prec y$ if and only if

$$\begin{aligned} &x, y \in \overline{L} \text{ and } x < y, \text{ or} \\ &x, y \in \overline{R} \text{ and } x < y, \text{ or} \\ &x \in \overline{R} \text{ and } y \in \overline{L}. \end{aligned}$$

Finally, let us consider the case $\overline{L} \cap \overline{R} = \{\alpha\}$. Then, α is a partition of αX. Hence, by Theorem 5.1, αX is first countable at α. Let $\mathcal{C}[\alpha]$ be the connected component of α in αX. Since $f^{-1}(\alpha) = \{\{\alpha\}\}$, it now follows from Theorem 4.1 that $\mathcal{C}[\alpha] = \{\alpha\}$. Indeed, $\mathcal{C}' = \mathcal{C}[\alpha] \cap \{x \in \alpha X : x \leq \alpha\}$ and $\mathcal{C}'' = \mathcal{C}[\alpha] \cap \{x \in \alpha X : x \geq \alpha\}$ are both connected subsets of X with $\alpha \in \mathcal{C}' \cap \overline{(X \setminus \mathcal{C}')}$ and $\alpha \in \mathcal{C}'' \cap \overline{(X \setminus \mathcal{C}'')}$ (consider that $X \setminus \mathcal{C}' \supset R$ and $X \setminus \mathcal{C}'' \supset L$), so that $\mathcal{C}' = \{\alpha\}$ and $\mathcal{C}'' = \{\alpha\}$, whence also $\mathcal{C}[\alpha] = \{\alpha\}$. Then, αX has a clopen base at α. Indeed, let $\ell \in L$ and $r \in R$. Since $\mathcal{C}[\alpha]$ is also the quasi-component of the point α, there are clopen neighbourhoods U_ℓ, U_r of α such that $\ell \notin U_\ell$ and $r \notin U_r$. Then,

$$U = \{x \in U_\ell \cap U_r : \ell < x < r\} = \{x \in U_\ell \cap U_r : \ell \leq x \leq r\}$$

is a clopen neighbourhood of α with $U \subset \{x \in X : \ell < x < r\}$.

That is, αX has a clopen base at α and it is first countable at this point. Then, let $\{U_n : n < \omega\}$ be a decreasing clopen base at α, with $U_0 = \alpha X$. Next, for every point $x \in X$, let $n(x) = \max\{n : x \in U_n\}$ and, for convenience, $n(\alpha) = \omega$. Finally, we may define a linear order "$\prec$" on αX by putting $x \prec y$ if and only if

$$\text{either} \quad n(x) < n(y) \quad \text{or} \quad n(x) = n(y) \text{ and } x < y.$$

Since $\{U_n : n < \omega\}$ is a decreasing clopen base at α, the order "$\prec$" is compatible with the topology of αX. It is clear that, with respect to "$\prec$", α is the last element of X. This completes the proof.

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