Convergence of Banach valued stochastic processes of Pettis and McShane integrable functions *†

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Abstract

It is shown that if $(X_n)_n$ is a Bochner integrable stochastic process taking values in a Banach lattice E, the convergence of $f(X_n)$ to $f(X)$ where f is in a total subset of E^* implies the a.s. convergence. For any Banach space E-valued stochastic process of Pettis integrable strongly measurable functions $(X_n)_n$, the convergence of $f(X_n)$ to $f(X)$ for each f in a total subset of E^* implies the convergence in the Pettis norm. Also convergence theorems of Mc-Shane integrable martingales are given.

1. Introduction

In [4] and [7] it is proved that if $(X_n)_n$ is a stochastic process of Bochner integrable functions taking values in a Banach space E, the convergence of $f(X_n)$ to $f(X)$ where f is in a total subset of E^*, implies the scalar convergence of X_n to X. The same result is extended to stochastic processes taking values in a Banach lattice E.

It is known that the weak Radon-Nikodym property is equivalent to the convergence in Pettis norm of a uniformly integrable martingale (see [10]). If this property does not hold, we ask for which class T of functionals f the convergence of the real valued stochastic process $f(X_n)$ to $f(X)$ implies the convergence of X_n to X in Pettis norm. In section 4 we prove that for Pettis-integrable strongly measurable martingales, T can be a total subset of E^* (Theorem 3).

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In the last section we shall deal with martingales of McShane integrable functions and the analogous of Theorem 3 is proved (see Theorem 8).

2. Preliminaries

Let E be a Banach space with norm $\|\cdot\|$, $B(E)$ its unit ball and E^* its dual. A subset T of E^* is called a *total set* over E if $f(x) = 0$ for each $f \in T$ implies $x = 0$.

Throughout $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_{n\in I\!N}$ a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$ if $m < n$. Moreover, without loss of generality, we will assume that $\mathcal{F}$ is the completion of $\sigma(\cup_n \mathcal{F}_n)$.

Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$, then a function $X : \Omega \to E$ is called *weakly* $\mathcal{F}_0$ *-measurable* if the function $f(X)$ is $\mathcal{F}_0$-measurable for every $f \in E^*$. A weakly $\mathcal{F}$-measurable function is called *weakly measurable.* A function $X : \Omega \to E$ is said to be *Pettis integrable* if $f(X)$ is Lebesgue integrable on Ω for each $f \in E^*$ and there exists a set function $\nu : \mathcal{F} \to E$ such that

$$f\nu(A) = \int_A fX$$

for all $f \in E^*$ and $A \in \mathcal{F}$. In this case we write $\nu(A) = P\int_A X$ and we call $\nu(\Omega)$ the *Pettis integral* of X *over* Ω and ν is the *indefinite Pettis integral of* X. The space of all E-valued Pettis integrable functions is denoted by $\mathcal{P}(E)$. The Pettis norm of a Pettis integrable functions is:

$$|X|_P = \sup\left\{\int_\Omega |f(X)| : f \in B(E^*)\right\}.$$

The pair $(X_n, \mathcal{F}_n)$ is called a *stochastic process of Pettis integrable functions* if, for each $n \in I\!N$, $X_n : \Omega \to E$ is Pettis integrable, X_n is weakly $\mathcal{F}_n$-measurable and the *Pettis conditional expectation* $E(X_n|\mathcal{F}_m)$ of X_n exists for all $n \geq m$. It should be noted that, in general, if X is only Pettis integrable, even it is strongly measurable, there is no *Pettis conditional expectation* of X with respect to a sub-σ-algebra of $\mathcal{F}$. The stochastic process $(X_n, \mathcal{F}_n)$ is called a *martingale* if $E(X_n|\mathcal{F}_m) = X_m$ for $n \geq m$.

A martingale $(X_n, \mathcal{F}_n)$ is

(*i*) *convergent* in $\mathcal{P}(E)$ if there exists a function $X \in \mathcal{P}(E)$ such that

$$\lim_{n\to\infty} |X_n - X|_P = 0;$$

(*ii*) *variationally bounded* if $\sup_n |\nu_n|(\Omega) < \infty$ where $\nu_n(A) = P\int_A X_n$ and $|\nu_n|$ denotes the variation of ν_n;

(*iii*) *uniformly continuous* if $\lim_{P(A)\to 0} P\int_A X_n = 0$ uniformly with respect to n;

(*iv*) *uniformly integrable* if it is variationally bounded and uniformly continuous.

3. Banach lattice valued stochastic processes

In this section we consider stochastic processes consisting of strongly measurable Bochner integrable random variables taking values in a Banach lattice (see [5], Chapter VIII). For an element $x \in E$ we denote by x^+ the least upper bound between x and 0. The Banach lattice E is said *to have the order continuous norm* or, briefly, to be *order continuous*, if for every downward directed set $\{x_\alpha\}_\alpha$ in E with $\wedge_\alpha x_\alpha = 0$, then $\lim_\alpha \|x_\alpha\| = 0$. The norm on a Banach space has the *Kadec-Klee property with respect to a set* $D \subset E^*$ if whenever $\lim_n f(x_n) = f(x)$ for every $f \in D$ and $\lim_n \|x_n\| = \|x\|$, then $\lim_n x_n = x$ strongly. If $D = E^*$ we say that the norm has the *Kadec-Klee property.* It was proved in [3] the following renorming Theorem for Banach lattices.

Theorem 1 *A Banach lattice E is order continuous if and only if there is an equivalent lattice norm on E with the Kadec-Klee property.*

It is obvious that if E is separable, the equivalent norm has the Kadec-Klee property with respect to a countable set of functionals.
A *stopping time* is a map $\tau : \Omega \to I\!N \cup \{\infty\}$ such that, for each $n \in I\!N$, $\{\tau \leq n\} = \{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathcal{F}_n$. We denote by Γ be the collection of all simple stopping times (i.e. taking finitely many values and not taking the value ∞), then Γ is a set filtering to the right. A stochastic process $(X_n, \mathcal{F}_n)$ is called a *subpramart* if for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all τ and σ in Γ, $\tau, \sigma \geq \tau_0$ then

$$P(\{\|(X_\sigma - E(X_\tau|\mathcal{F}_\sigma))^+\| > \varepsilon\}) \leq \varepsilon.$$

If $(X_n, \mathcal{F}_n)$ is a positive supramart, then for each $f \in (E^*)^+$, where $(E^*)^+$ denotes the nonnegative cone in E^*, $(f(X_n), \mathcal{F}_n)$ and $(\|X_n\|, \mathcal{F}_n)$ are real valued positive subpramarts ([5], Lemma viii.1.12)
If E has the Radon-Nikodym property each L^1-bounded subpramart converges strongly a.s.. Without assuming this property we ask which class of functionals has the property that the scalar convergence of $f(X_n)$ to $f(X)$ for each f in the class implies the strong convergence. We are able to prove the following theorem.

Theorem 2 *([8], Theorem 3.8) Let E be an order continuous Banach lattice, which is weakly sequentially complete and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a positive subpramart with an L^1-bounded subsequence and let X be a strongly measurable random variable. Assume that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

Proof. Since (X_n) and X are strongly measurable it is possible to assume that E is separable. By a decomposition theorem ([5], Lemma viii.1.17) and the fact that a subsequence of $(X_n)_n$, still denoted by $(X_n)_n$, is L^1-bounded we can also assume that

$$X_{n_k} = Y_{n_k} + Z_{n_k}$$

where Y_{n_k} and Z_{n_k} are $\mathcal{F}_{n_k}$-measurable, $(Y_{n_k})_k$ is uniformly integrable and $\lim_k Z_{n_k} = 0$, a.s.. For each $f \in (E^*)^+$, $f(X_n)_n$ is a real valued subpramart with an L^1-bounded subsequence, then it converges a.s. to a real random variable X_f. Also $f(Y_{n_k})$ converges to X_f a.s. and in L^1. In particular for each $f \in T$, $\lim_k f(Y_{n_k}) = f(X)$. So for $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\lim_k \int_A f(Y_{n_k})$$

exists in $I\!R$. Hence $(\int_A Y_{n_k})_k$ is weakly Chauchy. Since the Banach lattice E is weakly sequentially complete, let for every $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\mu(A) = w - \lim_k \int_A Y_{n_k}.$$

Then μ is a measure of bounded variation and it is absolutely continuous with respect to P. For each $f \in T$ we have

$$f(\mu(A)) = \lim_k \int_A f(Y_{n_k}) = \int_A f(X).$$

Let $A_n = \{\|X\| \leq n\}$, then XI_{A_n} is Bochner integrable and

$$f(\mu(A_n)) = \int_{A_n} f(X) = f \int_{A_n} X.$$

Since T is a total set it follows that

$$\mu(A_n) = \int_{A_n} X.$$

Moreover the uniform integrability of $(Y_{n_k})_k$ implies that

$$\int_{A_n} \|X\| = \|\mu\|(A_n) \leq \sup_k \int_\Omega Y_{n_k}, \tag{1}$$

and since X is strongly measurable, $P(\cup_n(\|X\| \leq n)) = 1$. Letting $n \to \infty$ in (1), we get that X is Bochner integrable and for each $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\mu(A) = \int_A X.$$

It follows that

$$\int_A f(X) = f(\mu(A)) = \lim_k \int_A f(Y_{n_k}) = \int_A X_f,$$

for each $f \in (E^*)^+$ and $A \in \cup_n \mathcal{F}_n$. Hence $f(X) = X_f$ a.s. and for each $f \in (E^*)^+$, $f(X_n)$ converges to $f(X)$ a.s.. Let $\|\!|\cdot|\!\|$ denote the Kadec-Klee norm equivalent to $\|\cdot\|$, as in Theorem 1, and let $D \in (E^*)^+$ be a countable norming subset. Applying ([5], Lemma viii.1.15) to the sequence $\{(f(X_n), \mathcal{F}_n), n \in I\!N, f \in D\}$ it follows that $\lim_n \|\!|X_n|\!\| = \|\!|X|\!\|$, a.s.. Now invoking again Theorem 1 we get the strong convergence of X_n to X and the assert follows. □

Considering that if a Banach space E does not contain c_0, it is order continuous and weakly sequentially complete, the following corollary holds.

Corollary 1 *Let E be a Banach lattice not containing c_0 as an isomorphic copy and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a positive subpramart with an L^1-bounded subsequence and let X be a strongly measurable random variable. Assume that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

4. Convergence of Pettis integrable stochastic processes

In this section we consider Pettis integrable stochastic processes.

Theorem 3 *Let $(X_n, \mathcal{F}_n)$ be an uniformly integrable martingale of Pettis integrable strongly measurable functions, X a weakly measurable function. Let T be a total subset of X^*, and assume that $f(X_n)$ converges to $f(X)$ a.s. for each $f \in T$ (the null set depends on f). Then $X \in \mathcal{P}(E)$ and X_n converges to X in the Pettis norm.*

Proof. By Pettis measurability Theorem we can assume that E is separable, then since T is closed and *weak**-dense, the assert follows from [9] Theorem 1. □

Remark 1 *Since in Theorem 3 we can suppose E separable, the weak measurability of X can be replaced by the measurability of the functions $f(X)$ for all $f \in T$ (see [2]).*

We will extend Theorem 3 to more general stochastic processes $(X_n, \mathcal{F}_n)$.

Definition 1 *A stochastic process $(X_n, \mathcal{F}_n)$ of Bochner integrable functions is said to be L^1-bounded if $\sup_n \int_\Omega \|X_n\| < \infty$.*

Definition 2 *A stochastic process $(X_n, \mathcal{F}_n)$ of strongly measurable functions is said to be a* game which becomes fairer with time *(briefly a P-*martingale*), if for each $\varepsilon > 0$*

$$\lim_n \sup_{m \geq n} P(\|E(X_m|\mathcal{F}_n) - X_n\| > \varepsilon) = 0.$$

If for each $\varepsilon > 0$

$$\lim_n \sup_{m \geq n} P(\sup_{n \leq q \leq m} \|E(X_m|\mathcal{F}_q) - X_q\| > \varepsilon) = 0$$

the sequence $(X_n, \mathcal{F}_n)$ *is called a* mil.

Definition 3 *A stochastic process* $(X_n, \mathcal{F}_n)$ *of Pettis integrable functions is* σ-bounded *if there exists an increasing sequence* $(B_n)_n$*,* $B_n \in \mathcal{F}_n$*, such that* $\lim_n P(B_n) = 1$ *and the sequence* (X_n) *restricted to each* B_m*,* $m = 1, 2, \dots,$ *is* L^1*-bounded.*

For more details and the proofs of the following Theorems see [9].

Theorem 4 *Let* $(X_n, \mathcal{F}_n)$ *be a* σ*-bounded* P*-martingale of Pettis integrable functions and* X *a weakly measurable function. Let* T *be a total subset of* E^**, and assume that* $f(X_n)$ *converges to* $f(X)$ *a.s. for each* $f \in T$ *(the null set depends on* f*). Then* X_n *converges to* X *in probability (i.e. for every* $\varepsilon > 0$ *we have* $\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$*).*

Theorem 5 *Let* $(X_n, \mathcal{F}_n)$ *be a* σ*-bounded mil of Pettis integrable strongly measurable functions and* X *a weakly measurable function. Moreover let* T *be a total subset of* E^**, and assume that* $f(X_n)$ *converges to* $f(X)$ *a.s. for each* $f \in T$ *(the null set depends on* f*). Then* X_n *converges to* X *a.s. in the strong topology.*

As we noted in Remark 1 the hypothesis of weak measurability of X in Theorems 4 and 5 can be substituted by the measurability of the functions $f(X)$ for all $f \in T$.
Assuming a weaker strong measurability condition on the martingale $(X_n, \mathcal{F}_n)$, in Theorem 3 we obtain:

Theorem 6 *Let* $(X_n, \mathcal{F}_n)$ *be an uniformly integrable martingale of Pettis integrable functions such that the indefinite integrals of all* X_n *have norm relatively compact range and let* X *be a weakly measurable function. Assume that there exists an increasing sequence of measurable sets* $(B_m)_m$*,* $B_m \in \mathcal{F}_m$*, such that* $\lim_m P(B_m) = 1$ *and that the function* X_n *restricted to each* B_m *is strongly measurable,* $n = 1, 2, \dots$*. Assume, moreover, that for each* $f \in T$*, where* T *is a total set,* $f(X_n)$ *converges to* $f(X)$ *a.s. (the null set depends on* f*). Then* $X \in \mathcal{P}(E)$ *and* X_n *converges to* X *in the Pettis norm.*

Theorem 3 and Theorem 6 hold also for amarts, changing the proofs as in [13] Theorem 2.

5. Martingale of McShane integrable functions

In this section we consider stochastic processes of McShane integrable functions.
Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a probability space which is a quasi-Radon, outer regular and compact probability space. A *Mc-Shane partition* of Ω is a set $\{(S_i, \omega_i), i = 1, \ldots, p\}$ where $(S_i)_i$ is a disjoint family of measurable sets of finite measure, $P(\Omega \setminus \cup_{i=1}^{p} S_i) = 0$ and $\omega_i \in \Omega$ for each $i = 1, \ldots, p$. A *gauge* on Ω is a function $\Delta : \Omega \to \mathcal{A}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. A Mc-Shane partition $\{(S_i, \omega_i), i = 1, \ldots, p\}$ is *subordinate to a gauge* Δ if $S_i \subset \Delta(\omega_i)$ for $i = 1, \ldots, p$.
A function $f : \Omega \to E$ is *McShane integrable* (briefly M-integrable), with *McShane integral* $z \in E$ if for each $\varepsilon > 0$ there exists a gauge $\Delta : \Omega \to \mathcal{A}$, such that

$$\left\| \sum_{i=1}^{p} P(S_i) f(\omega_i) - z \right\| < \varepsilon$$

for each McShane partition $\{(S_i, \omega_i) : i = 1, \ldots, p\}$ subordinate to Δ.
It is known that if $f : \Omega \to E$ is M-integrable, then $\nu_f(\Omega) = \left\{(M)\int_A f : A \in \mathcal{F}\right\}$ is totally bounded (see [1], Theorem B and [6], Corollary 3E), hence it is norm relatively compact. Denote by $M(E)$ the set of all M-integrable functions endowed with the seminorm

$$|X|_M = \sup\left\{ \int_\Omega |f(X)| : f \in B(E^*) \right\},$$

which is equivalent to the seminorm ([11])

$$\sup\left\{ \left\| M\int_A X \right\| : A \in \mathcal{F} \right\}.$$

If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, X is McShane integrable and Y is McShane integrable on $(\Omega, \mathcal{A}, \mathcal{G}, P)$, then Y is called the *McShane conditional expectation* of X with respect to $\mathcal{G}$ if

- (i) Y is weakly $\mathcal{G}$-measurable;
- (ii) for every $A \in \mathcal{G}$, $M\int_A Y = M\int_A X$.

The symbol $Y = E_M(X|\mathcal{G})$ will denote the McShane conditional expectation of X with respect to $\mathcal{G}$.
We say that $(X_n, \mathcal{F}_n)$ is a *stochastic process of M-integrable functions*, if for each $n \in I\!N$, X_n is M-integrable, X_n is weakly measurable with respect to $\mathcal{F}_n$ and the McShane conditional expectation $E_M(X_n|\mathcal{F}_m)$ of X_n exists for all $n \geq m$. Also we observe that the conditional expectation of a M-integrable function does not always exist, indeed the same is true for strongly measurable Pettis integrable functions and a strongly measurable Pettis integrable function is McShane integrable.

As in case of a stochastic process of Pettis integrable functions, we say that $(X_n, \mathcal{F}_n)$ is a martingale if X_n is a M-integrable function for each n, and if for all $n \geq m$ $E_M(X_n|\mathcal{F}_m) = X_m$ or equivalently for all $A \in \mathcal{F}_m$

$$M\int_A X_m = M\int_A X_n.$$

If X is M-integrable and $E_M(X|\mathcal{F}_n)$ exists for all n, then $X_n = E_M(X|\mathcal{F}_n)$ is called a closed martingale. Since a M-integrable function is Pettis integrable and $\nu_f(\Omega) = \{(M)\int_A f : A \in \mathcal{F}\}$ is norm relatively compact, there exists a sequence of simple functions $f_n : \Omega \to E$, converging to f in $|\cdot|_M$, i.e. $\lim |f_n - f|_M = 0$. The following proposition is an extension of Lemma 1.4 of [12] to a martingale of McShane integrable functions. The proof follows with suitable changes.

Proposition 1 *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then the following are equivalent:*

- (i) *there exists a M-integrable function X such that X_n is $|\cdot|_M$ convergent to X;*
- (ii) *there exists a M-integrable function X such that $E_M(X|\mathcal{F}_n) = X_n$ for each $n \in I\!N$;*
- (iii) *there exists a M-integrable function X such that for each $A \in \cup_n \mathcal{F}_n$*

$$\lim_n M\int_A X_n = M\int_A X.$$

The condition $(ii) \Rightarrow (i)$ in the previous Proposition says that a closed martingale is $|\cdot|_M$ convergent. We have the following:

Proposition 2 *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then, for all $A \in \cup_n \mathcal{F}_n$, the set function $\mu(A) = \lim_n M\int_A X_n$ is absolutely continuous and has norm relatively compact range if and only if the martingale $(X_n, \mathcal{F}_n)$ is $|\cdot|_M$ Chauchy.*

Proof. First we prove the necessary part.
Since μ has norm relatively compact range, by Hoffman-Jorgensen Theorem for each $\varepsilon > 0$ there exists a function $H_\varepsilon : \Omega \to E$ such that $H_\varepsilon = \sum_{i=1}^k x_i I_{A_i}$, with $A_i \in \cup_n \mathcal{F}_n$ and $x_i \in E$, so that

$$\sup\left\{\left\|\mu(A) - \int_A H_\varepsilon\right\| : A \in \cup_n \mathcal{F}_n\right\} < \varepsilon.$$

Take $\varepsilon > 0$ and let $H = H_{\varepsilon/4}$, there exists m_0 for which $A_i \in \mathcal{F}_{m_0}$, for $i = 1, \ldots, k$. Since $\mu(A) = \lim_n M\int_A X_n$ there is m_0 such that $\|\mu(A) - M\int_A X_n\| < \frac{\varepsilon}{4}$ for $n > m_0$. Let $n, m \geq m_0$.

We have

$$
\begin{aligned}
& \sup\left\{\left\|M\textstyle\int_A(X_n - X_m)\right\| : A \in \cup_n\mathcal{F}_n\right\} \\
\leq\ & \sup\left\{\left\|M\textstyle\int_A(X_n - H)\right\| : A \in \cup_n\mathcal{F}_n\right\} + \sup\left\{\left\|M\textstyle\int_A(H - X_m)\right\| : A \in \cup_n\mathcal{F}_n\right\} \\
\leq\ & \sup\left\{\left\|M\textstyle\int_A X_n - \mu(A)\right\| : A \in \cup_n\mathcal{F}_n\right\} + \sup\left\{\left\|\mu(A) - M\textstyle\int_A H\right\| : A \in \cup_n\mathcal{F}_n\right\} \\
+\ & \sup\left\{\left\|M\textstyle\int_A X_m - \mu(A)\right\| : A \in \cup_n\mathcal{F}_n\right\} + \sup\left\{\left\|\mu(A) - M\textstyle\int_A H\right\| : A \in \cup_n\mathcal{F}_n\right\} \\
<\ & \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

Then $|X_n - X_m|_M < \varepsilon$ for $n, m \geq m_0$.
Conversely choose $\varepsilon > 0$ and find m_0 such that if $n, m \geq m_0$ then $|X_n - X_m|_M < \varepsilon$. If $\mu_n(A) = M\int_A X_n$ for $A \in \cup\mathcal{F}_n$ then

$$\|\mu_n(A) - \mu_m(A)\|_M \leq |X_n - X_m|_M < \varepsilon.$$

So the sequence of measures μ_n is Chauchy, therefore $\lim_n \mu_n(A) = \mu(A)$ exists. The functions X_n are M-integrable, then μ_n has a norm relatively compact range and since the convergence is uniform in $A \in \cup_n\mathcal{F}_n$, it follows that μ is absolutely continuous and has a norm relatively compact range. □

Proposition 1 and Proposition 2 hold also for M-integrable martingales indexed by a directed set.
We will prove now two convergence theorems for a M-integrable martingale.

Theorem 7 *Let $(X_n, \mathcal{F}_n)$ be an uniformly integrable martingale of M-integrable functions and suppose that there exists a weakly measurable function $X : \Omega \to E$ such that $f(X_n)$ converges to $f(X)$ a.s.. Then X_n is $|\cdot|_M$ convergent to X.*

Proof. Since $(X_n)_n$ is uniformly integrable the set function $\nu : \cup_n\mathcal{F}_n \to E$ defined as

$$\nu(A) = \lim_n M\int_A X_n$$

is an absolutely continuous measure of bounded variation and it can be extended to the whole $\mathcal{F}$ to an absolutely continuous measure of bounded variation. Moreover for each $\omega \notin N$ with $P(N) = 0$, $f(X_n(\omega))$ converges to $f(X(\omega))$ for each $f \in E^*$. Hence it follows from [6] Theorem 4A that X is M-integrable and $\nu(\Omega) = M\int_\Omega X$. Then for each $A \in \cup_n\mathcal{F}_n$

$$\lim_n M\int_A X_n = M\int_A X$$

and the assert follows from Proposition 1. □

Definition 4 *A function $X : \Omega \to E$ is called* weakly asymptotically measurable *with respect to an increasing family $(\mathcal{F}_n)_n$ of sub-σ-algebras of $\mathcal{F}$ if there exists an integer N such that for all $n > N$ and for all $f \in E^*$ $f(X)$ is $\mathcal{F}_n$-measurable.*

Theorem 8 *Let $(X_n, \mathcal{F}_n)$ be an uniformly integrable martingale of M-integrable functions and let T be a weak*-sequentially dense subset of E^*. Assume that there exists a weakly measurable function $X : \Omega \to E$ such that X is weakly asymptotically measurable with respect to $(\mathcal{F}_n)$ and such that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n is $|\cdot|_M$ convergent to X.*

Proof. Since each McShane integrable function is Pettis integrable it follows by [9] Theorem 1 that X is Pettis integrable, (X_n) converges to X in the Pettis norm and

$$\mu(A) = \lim_n M\int_A X_n = P\int_A X$$

for all $A \in \cup_n \mathcal{F}_n$. We want to prove that X is M-integrable. Since X is weakly asymptotically measurable there exists $N \in I\!N$ such that X is weakly $\mathcal{F}_N$-measurable, then

$$E(X|\mathcal{F}_N) = X \tag{2}$$

and also

$$E(X|\mathcal{F}_N) = X_N. \tag{3}$$

Then (2) and (3) implies that $X = X_N$ a.s. and X is M-integrable. Therefore the assert follows from Proposition 1. □

References

[1] Zvi Artstein, Lyapounov convexity theorem and Riemann-type integrals, Indiana Univ. Math. J., Vol.25, N.8, 717-724, (1976).

[2] A. Bozorgnia and M. Bhaskara Rao, Measurability and convergence theorems for random elements in Banach spaces in terms of total subset of linear functionals, Bulletin of the Institute of Math. Academia Sinica, Vol. 14,N. 4, 383-416,(1986).

[3] W. J. Davis, N. Ghoussoub and J. Lindenstrauss, A lattice renorming theorem and applications to vector-valued processes, Trans. Amer. Math. Soc., Vol. 263, N. 2, 531-540, (1981).

[4] W. J. Davis, N. Ghoussoub, W. B. Johnson, S. Kwapien and B. Maurey, Weak convergence of vector valued martingales, Probability in Banach spaces, Vol. 6, 41-50, (1990).

[5] L. Egghe, Stopping Time Techniques for Analysts and Probabilist, London Mathematical Society, Lecture Notes, vol. 100, Cambridge University Press, (1984), Cambridge.

[6] D. H. Fremlin, The generalized McShane integral, Illinois J. Math., Vol. 39, N.1, 39-67, (1995).

[7] V. Marraffa, On almost sure convergence of amarts and martingales without the Radon-Nikodym property, J. of Theoretical Prob., Vol.1, N.3, 255-261, (1988).

[8] V. Marraffa, Convergenza di processi stocastici senza la proprietà di Radon-Nikodym, Ph.D. Univ. of Palermo, (1988).

[9] V. Marraffa, Stochastic processes of vector valued Pettis and McShane integrable functions, preprint.

[10] K. Musial, Martingale of Pettis integrable functions, Lecture Notes in Math., Springer-Verlag, Vol. 794, 324-339, (1980).

[11] C. Swartz, Beppo Levi's theorem for vector valued McShane integral and applications, Bull. Belg. Math. Soc., Vol. 4, N. 5, 589-599, (1997).

[12] J.J. Uhl, Jr., Martingales of strongly measurable Pettis integrable functions, Trans. of Amerrican Math. Soc., Vol. 167, 369-378, (1972).

[13] J.J. Uhl, Jr., Pettis mean convergence of vector-valued asymptotic martingales, Z. Wahr. verw. Geb., Vol. 37, 291-295, (1977).

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