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GEOMETRIC SYMMETRY GROUPS, CONSERVATION LAWS AND GROUP-INVARIANT SOLUTIONS OF THE WILLMORE EQUATION

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Abstract. The present paper is concerned with the geometric (point) Lie symmetry groups of the Willmore equation – the Euler-Lagrange equation associated with the Willmore functional. The ten-parameter group of special conformal transformations in three-dimensional Euclidean space, which is known to be the symmetry group of the Willmore functional, is recognized as the largest group of geometric transformations admitted by the Willmore equation in Mongé representation. The conserved currents of ten linearly independent conservation laws, which correspond to the variational symmetries of the Willmore equation and hold on its smooth solutions, are derived. All types of non-equivalent group-invariant solutions of the Willmore equation are identified, an optimal system of one-dimensional subalgebras being given together with the invariants of the corresponding one-parameter groups, up to one exception. Special attention is paid to the rotationally-invariant (axially-symmetric) solutions.

1. Introduction

The so-called Willmore functional

$$\mathcal{W} = \int_{\mathcal{S}} H^2 \, \mathrm{d}A \tag{1}$$

which assigns to each surface $\mathcal{S}$ its total squared mean curvature H (here $\mathrm{d}A$ is the area element on the surface) has drawn much attention after the appearance of Willmore's paper [18] in 1965. In this work, Willmore proposed to study the surfaces providing extremum to the functional (1), which are now referred to as Willmore