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ON UNCERTAINTY, BRAIDING AND JACOBI FIELDS IN GEOMETRIC QUANTUM MECHANICS

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Abstract. Acting within the framework of the geometric quantum mechanics, we give a Jacobi field interpretation of the quantum mechanical uncertainty. Then a link with elliptic curves via the classical integrability of Schrödinger dynamics and the cross ratio interpretation of quantum probabilities is established. Furthermore a geometrical construction of all special unitary representations of the three-strand braid group on the quantum one-qubit space is provided.

1. Integrability in Quantum Mechanics

1.1. Geometry of Quantum Mechanics

We start this paper by briefly reviewing geometric quantum mechanics and, in particular, the completely integrable structure of Schrödinger dynamics in finite dimensional quantum space (cf. [1–4, 6, 15]).

Throughout the paper we assume $\hbar = 1$. Let V be a complex Hilbert space of dimension $n+1$ with a scalar product $\langle\cdot|\cdot\rangle$ which is linear in the second variable. We shall freely use the Dirac notation. The space of pure states in quantum mechanics is the projective space associated to V, denoted by $\mathbb{P}(V)$, of complex dimension n, whose points are the rays $[v]$ (directions) pertaining to nonzero vectors $|v\rangle$. Considering the actions of the unitary group $\mathrm{U}(V)$ associated to $(V, \langle\cdot|\cdot\rangle)$ and its Lie algebra $\mathfrak{u}(V)$, consisting of all skew-hermitian endomorphisms of V (the quantum observables, and with a slight abuse of language), the projective space $\mathbb{P}(V)$ becomes a $\mathrm{U}(V)$-homogeneous Kähler manifold. Furthermore we can identify a point in $\mathbb{P}(V)$ with the projection operator

$$[v] = \frac{|v\rangle\langle v|}{\|v\|^2}$$

and then compute the fundamental vector field $A^\sharp$ associated to $A \in \mathfrak{u}(V)$ (evaluated at $[v] \in \mathbb{P}(V)$, $\|v\| = 1$) as

$$A^\sharp|_{[v]} = |v\rangle\langle Av| + |Av\rangle\langle v|.$$

The action of the complex structure J is

$$J|_{[v]}A^\sharp|_{[v]} = |v\rangle\langle \mathrm{i}Av| + |\mathrm{i}Av\rangle\langle v|$$

while the expressions for the natural Fubini–Study metric g

$$g_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = \Re\{\langle Av|Bv\rangle + \langle v|Av\rangle\langle v|Bv\rangle\}$$

and the Kähler form ω

$$\omega_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = g_{[v]}(J|_{[v]}A^\sharp|_{[v]}, B^\sharp|_{[v]}) = \frac{\mathrm{i}}{2}\langle v|[A,B]v\rangle$$

yield a Riemannian and a symplectic structure in the pure quantum state space $\mathbb{P}(V)$, respectively.

1.2. Toral Action and Integrability

Every quantum system with a non-degenerate Hamiltonian H can be viewed as a completely integrable classical system (cf. [2] and references therein). Indeed, given a quantum Hamiltonian and orthonormal basis of eigenvectors $|e_j\rangle$, $j = 0, \ldots, n$, we have the basic relations

$$|v\rangle = \sum_{j=0}^{n} \alpha_j |e_j\rangle$$

for each vector $|v\rangle \in V$ and

$$H = \sum_{j=0}^{n} \lambda_j |e_j\rangle\langle e_j| = \sum_{j=0}^{n} \lambda_j P_j$$

where P_j is the projection operators onto the line $\langle e_j\rangle$. In general, given an observable F we can define the function $f : \mathbb{P}(V) \to \mathbb{R}$

$$f([v]) := \langle v|\mathrm{i}Fv\rangle.$$

Moreover it can be shown that there is an effective group action of the n-dimensional torus $\mathbb{T}^n$ on (a dense subset of) $\mathbb{P}(V)$ induced by

$$e_j \mapsto \exp(\mathrm{i}\beta_j)e_j$$

with $\beta_j \in [0, 2\pi)$. In view of this we can state the following

Theorem 1 (cf. [2]). *The "classical" Hamiltonian system* $(\mathbb{P}(V), \omega, h)$ *is completely integrable. The Lagrangian tori are provided by the orbits of the* n*-dimensional torus action above. The* action *variables* I_j *coincide with the transition probabilities* $|\alpha_j|^2 = p_j([v])$, $j = 1, 2, \ldots, n$, *the* angle *variables are the phases* $\beta_j - \beta_0$, *say, and the orbit space can be identified with the standard open* n*-simplex in the Euclidean space* $\mathbb{R}^n$.

For a more precise statement see [2]. Note that in a two level quantum system this theorem is simply pictured by a sphere $\mathbb{S}^2 \cong \mathbb{P}(\mathbb{C}^2)$, the tori correspond to parallels (related, in turn, to the "action" variables) whereas the angles around the axis through the poles represent the "angle" variable.

1.3. Cross Ratio and Transition Probabilities

Another outstanding geometric property of the quantum transition probabilities is the fact that given two pure quantum states, $[\xi]$ and $[\eta]$ in $\mathbb{P}(V)$ and given their respective hermitian-orthogonal states $[\xi^{\perp}]$ and $[\eta^{\perp}]$ on the projective line $\overline{[\xi][\eta]}$ which they determine, their cross-ratio equals the transition probability between $[\xi]$ and $[\eta]$

$$k^2 := ([\xi], [\eta], [\eta^{\perp}], [\xi^{\perp}]) = \frac{|\langle \xi | \eta \rangle|^2}{\langle \xi | \xi \rangle \langle \eta | \eta \rangle}$$

(see e.g., [4]). Notice that if $\overline{[\xi][\eta]}$ is regarded as a sphere, then $[\xi]$ and $[\xi^{\perp}]$, and $[\eta]$, $[\eta^{\perp}]$, respectively, become antipodal points thereon.

2. Jacobi Fields and Elliptic Functions in Quantum Mechanics

2.1. Uncertainty and Jacobi Fields

Now we discuss an interpretation of quantum uncertainty in terms of Jacobi fields and a connection with the theory of elliptic curves. This can be done via classical integrability of Schrödinger dynamics and the cross-ratio interpretation of quantum transition probabilities discussed above.

We confine ourselves to the case of a two level system with the non-degenerate Hamiltonian

$$H = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$$

with $\lambda_0 < \lambda_1$. The **dispersion** of the observable $\mathcal{A} \in \mathrm{u}(V)$ in the state $[v]$ is

$$\Delta_{[v]} A = \|Av - \langle v|Av\rangle v\| = \|A^{\sharp}|_{[v]}\| := \sqrt{g_{[v]}(A^{\sharp}|_{[v]}, A^{\sharp}|_{[v]})} = \|J|_{[v]} A^{\sharp}|_{[v]}\|$$

so we can define the vector field $\mathcal{J} := J|_{[v]} H^{\sharp}|_{[v]}$ taken along a minimal geodesic curve joining two orthogonal eigenstates of H. This is just a half-meridian in

$\mathbb{S}^2 \cong \mathbb{P}(\mathbb{C}^2)$, viewed as a totally geodesic submanifold of the full projective space, and it is perpendicular to the geodesic at every point.

We need also to observe that the relationship between the canonical metrics of the two-sphere and the complex projective line is

$$g_{\mathbb{P}(\mathbb{C}^2)} = \frac{1}{4}\, g_{\mathbb{S}^2}$$

i.e., the Fubini–Study metric on the projective line is the metric on a sphere of radius $\frac{1}{2}$, whence the curvature $K_{\mathbb{P}(\mathbb{C}^2)} = 4\, K_{\mathbb{S}^2} = 4$ (see e.g., [5, 7]).

So we can state the following

Theorem 2 (cf. [3]). i) *The dispersion* $\Delta_{[v]}H$ *equals* $\delta h \cdot r_\vartheta$, *where* r_ϑ *is the radius of the parallel with colatitude* ϑ *pertaining to the sphere with radius* $\frac{1}{2}$.

ii) *The corresponding vector field* $\mathcal{J}$ *is a Jacobi vector field when restricted to a geodesic connecting two orthogonal eigenstates corresponding to different energy levels.*

Notice that in view of this theorem the Heisenberg Uncertainty Principle appears essentially as a manifestation of the *curvature* of the quantum mechanical space $\mathbb{P}(\mathbb{C}^2)$ (see also [6, 15]).

2.2. A Link with Elliptic Functions

A relation, rather speculative but interesting, between two level quantum systems and elliptic functions, will be shown here (see also [3]), but first let us review some basic definitions and properties of these objects (see e.g. [12, 14, 16, 18]).

The complete elliptic integral of the first kind, according to Legendre's classification, associated to the Jacobi modulus $k \in \mathbb{C} \setminus \{0, 1\}$ is

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}\, \mathrm{d}\phi\,.$$

An important case, especially for applications, arises when the Jacobi modulus k is real and $0 < k < 1$. By this restriction k^2 can be interpreted as a cross-ratio of the four roots of a complex polynomial appearing in a generic elliptic integral in Weierstraß form

$$\int \frac{\mathrm{d}x}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}$$

where $z^2 = 4(x - e_1)(x - e_2)(x - e_3)$ is a non singular cubic $\mathcal{C}$ in $\mathbb{P}^2$ with $e_1 + e_2 + e_3 = 0$. The elliptic integral above is explicitly inverted by the Weierstraß function $\wp = \wp(z, \tau)$, fulfilling the above equation with $x = \wp$, $z = \wp'$. It is known that this cubic is diffeomorphic to a torus $\mathcal{T} := \mathbb{C}/\Lambda$, defined by quotienting $\mathbb{C}$

via the normalized lattice $\Lambda = \mathbb{Z}(1,\tau)$, where $\tau = \mathrm{i}\frac{K'(k)}{K(k)}$. Furthermore, the j-invariant specified below parameterizes the isomorphism classes of elliptic curves

$$j = \frac{4}{27}\frac{(1-k^2k'^2)^3}{k^4k'^4}.$$

Now we come back to the two level quantum system and recall, as explained in Subsection 1.3, that

$$|\langle 1|v\rangle|^2 = ([v],[1],[0],[v^\perp]) =: k^2$$

and $|\langle 0|v\rangle|^2 =: k'^2 = 1-k^2$. So the idea is that we may regard the transition probability $k^2 = |\langle 1|v\rangle|^2$ *as the Jacobi modulus (squared) of an elliptic curve* $\mathcal{C} = \mathcal{C}_{k^2} = \mathcal{C}_j$. *The modulus* k^2 *will also be the cross-ratio of the corresponding Weierstraß roots.*

The main result is the following

Theorem 3 ([3]). i) *There exists a family of elliptic curves* $\mathcal{C}_{k^2}$ *parameterized by* k^2, *building up a (topologically trivial, i.e., having a contractible base) fibration* $\mathcal{F} \longrightarrow (0,1)$ *by abelian tori, wherein the dynamical Lagrangian tori (parallels on the unit sphere) can be embedded and made to correspond, in the normalized lattice* $\mathbb{Z}(1,\tau)$ *to the* τ*-one-cycle. The one-one-cycle can be associated to a meridian passing through the poles, and can be called* collapse cycle, *since the measurement of the Hamiltonian forces collapse onto an eigenstate, with the appropriate probability.*

ii) *The tori have varying complex structures (induced by* τ*), ultimately governed by the geometrical uncertainty, which appears directly in the expression for the* j*-invariant.*

Further properties and explicit calculations can be found in [3].

3. Braiding in One-Qubit Space

3.1. The Braid Group B_3 and $\mathrm{SL}(2,\mathbb{Z})$

In this Section we investigate a different but related property of the one-qubit space $\mathbb{P}(\mathbb{C}^2) \cong \mathbb{S}^2$, more precisely we study special unitary representations of the three-strand braid group B_3 from a geometrical viewpoint as an action on quantum states. Let us briefly review some of its basic properties (see e.g., [8,10,17]). The n-strand braid group B_n, $n \geq 3$, can be presented via its generators b_i, $i = 1,2,\ldots,n-1$ subject to the relations

$$\begin{aligned} &b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} && \text{for} \quad i = 1,2,\ldots,n-1 \\ &b_i b_j = b_j b_i && \text{for} \quad |i-j| \geq 2\,. \end{aligned}$$

Adjoining the relations ${b_i}^2 = 1$, $i = 1, 2, \dots, n-1$, we get a presentation of the symmetric group S_n. There is a natural surjection $B_n \to S_n$, and its kernel is given by the pure (or coloured) braid group P_n. In our case $n = 3$, so we have the single condition $b_1 b_2 b_1 = b_2 b_1 b_2$.
The centre Z of B_3 is generated by $(b_1 b_2)^3$ and one has $B_3/Z \cong \mathrm{PSL}(2, \mathbb{Z})$ (the latter being the modular group).

3.2. $\mathrm{SU}(2)$-representations of B_3

Now we are going to give a purely geometrical representation of the braid group in terms of the rotations in $\mathbb{R}^3$. To this aim, recall that any $\mathrm{SU}(2)$ matrix can be written in the form

$$U_{\mathbf{n}}(\varphi) = \exp(\mathrm{i}\frac{\varphi}{2}\boldsymbol{\sigma} \cdot \mathbf{n}) = \cos\frac{\varphi}{2} I_2 + \sin\frac{\varphi}{2}\,\mathrm{i}\,\boldsymbol{\sigma} \cdot \mathbf{n}$$

with $\varphi \in [0, 2\pi)$, $\mathbf{n}$ a unit vector, $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ (the Pauli matrices) which furnish a rotation in the ordinary three-space of angle φ around the axis $\mathbf{n}$. Now, let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be two unit vectors, and $\mathbf{a} \cdot \mathbf{b} =: \cos\Omega$. Then we have

$$U_{\mathbf{a}}(\alpha) \cdot U_{\mathbf{b}}(\beta) = pI_2 + q\mathrm{i}\boldsymbol{\sigma} \cdot \mathbf{a} + r\mathrm{i}\boldsymbol{\sigma} \cdot \mathbf{b} + s\mathrm{i}\boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}$$

with

$$\begin{aligned} p = {} & \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\Omega \\ q = {} & \sin\frac{\alpha}{2}\cos\frac{\beta}{2} \\ r = {} & \cos\frac{\alpha}{2}\sin\frac{\beta}{2} \\ s = {} & -\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\,. \end{aligned}$$

Applying the latter in the braid relation for $n = 3$, we arrive at the following

Theorem 4 ([3]). i) *There exists a unique family of* $\mathrm{SU}(2)$*-representation classes of the three-strand braid group* B_3*, where the rotation angle* α *of both generators and the angle* Ω *between their respective axes are related by means of the formula*

$$\sin\frac{\alpha}{2}\cos\frac{\Omega}{2} = \frac{1}{2}$$

with $\Omega \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$*. The equivalent forms are*

$$\cos\Omega = \frac{\cos\alpha}{1 - \cos\alpha} \qquad \textit{and} \qquad \cos\alpha = \frac{\cos\Omega}{1 + \cos\Omega}$$

with $\alpha \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ *(trivial representations are included).*

ii) *The above representations induce, in turn, special unitary representations of* $\mathrm{SL}(2,\mathbb{Z})$ *and of the modular group* $\mathrm{PSL}(2,\mathbb{Z})$.

(See also [9, 10]). Observe that all non trivial special unitary representations of B_3 are genuine braid group representations in the sense that they do *not* induce representations of the symmetric group S_3. Indeed, this is the case if the extra condition $b_1^2 = b_2^2 = 1$ is fulfilled, which never happens unless the representation is trivial (see also [17]). The characters of the representations read, in turn $\chi(U_{\mathbf{a}}(\alpha)) = \mathrm{Tr}\,(U_{\mathbf{a}}(\alpha)) = 2\cos\frac{\alpha}{2} = \chi(U_{\mathbf{b}}(\alpha))$.

3.3. Three-Strand Braiding in One-Qubit Space and the Weierstraß Roots

In [3] we have found the unitary representations of B_3 involving braiding of the Weierstraß roots e_i, $i = 1, 2, 3$ with $e_1 + e_2 + e_3 = 0$. The calculations show that the Weierstraß roots, when represented by points on a sphere, form an equilateral triangle inscribed in a great circle and the braid generators induce rotations of angle π, with their respective axes forming an angle of $\frac{2\pi}{3}\cdot$ One is led to the *anharmonic* ratio of the Weierstraß roots and to the square lattice case.

Theorem 5 (cf. [3]). *There exists a unique "physical"* (*i.e., with Jacobi modulus* $0 < k^2 < 1$) *unitary representation* (*class*) *of the three-strand braid group* B_3, *causing braiding of the three roots* $e_1 = \sqrt{3}$, $e_2 = 0$, $e_3 = -\sqrt{3}$ *of the natural elliptic cubic, associated to the Jacobi modulus* $k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{1}{2}\cdot$

Conclusions

In this work we have reviewed some basic geometric properties of the finite dimensional quantum mechanics, stemming from the complete integrability of the Schrödinger dynamics in the "classical" sense. We have shown a link between quantum uncertainty and Jacobi fields and have pointed out a connection with elliptic functions coming from to the cross-ratio interpretation of quantum transition probabilities. We also gave a geometrical construction, in terms of the rotations in the ordinary Euclidean space, of some special unitary representations of the three-strand braid group on the quantum one-qubit space. From this we derived an action of the braid group directly on the three Weierstraß roots, when viewed as points in the one-qubit space.

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