Research Article

δ-Small Submodules and δ-Supplemented Modules

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Received 9 January 2007; Accepted 18 June 2007

Recommended by Akbar Rhemtulla

Let $R$ be a ring and $M$ a right $R$-module. It is shown that (1) $\delta(M)$ is Noetherian if and only if $M$ satisfies ACC on $\delta$-small submodules; (2) $\delta(M)$ is Artinian if and only if $M$ satisfies DCC on $\delta$-small submodules; (3) $M$ is Artinian if and only if $M$ is an amply $\delta$-supplemented module and satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules.

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1. Introduction and preliminaries

In this note, all rings are associative with identity and all modules are unital right modules unless otherwise specified.

Let $R$ be a ring and $M$ a module. The concept of $\delta$-small submodules was introduced by Zhou in [1]. Motivated by [2–4], we study modules with ACC (resp., DCC) on $\delta$-small submodules and prove that $\delta(M)$ is Noetherian (resp., Artinian) if and only if $M$ satisfies ACC (resp., DCC) on $\delta$-small submodules in Section 2. In Section 3, we give the concepts of (amply) $\delta$-supplemented modules via $\delta$-small submodules. It is shown that $M$ is Artinian if and only if $M$ is an amply $\delta$-supplemented module and satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules. In Section 4, we introduce the concept of $\delta$-semiperfect modules and investigate the connections between $\delta$-supplemented modules and $\delta$-semiperfect modules.

Let $M$ be a module and $N \leq M$. $N$ is said to be $\delta$-small in $M$ (see [5]) if, whenever $N + X = M$ with $M/X$ singular, we have $X = M$. $\delta(M) = \text{Rej}_M(\mathcal{P}) = \cap\{N \leq M \mid M/N \in \mathcal{P}\}$, where $\mathcal{P}$ be the class of all singular simple modules. $M$ is called an amply supplemented module if for any two submodules $A$ and $B$ of $M$ with $A + B = M$, $B$ contains a supplement of $A$. $M$ is called a supplemented module if for each submodule $A$ of $M$ there exists
a submodule $B$ of $M$ such that $M = A + B$ and $A \cap B \ll B$. The notions which are not explained here will be found in [6].

**Lemma 1.1** (see [7, Proposition 5.20]). Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll \leq M_1 \oplus M_2$ if and only if $K_1 \leq \leq M_1$ and $K_2 \leq \leq M_2$.

### 2. Modules with chain conditions on $\delta$-small submodules

In this section, we study modules with chain conditions on $\delta$-small submodules and prove that $\delta(M)$ is Noetherian (resp., Artinian) if and only if $M$ satisfies ACC (resp., DCC) on $\delta$-small submodules. Let us start with the following.

**Lemma 2.1** (see [1, Lemma 1.3]). Let $M$ be a module.

(i) For submodules $N$, $K$, $L$ of $M$ with $K \leq N$,

1. $N \ll \delta M$ if and only if $K \ll \delta M$ and $N/K \ll \delta M/K$;
2. $N + L \ll \delta M$ if and only if $N \ll \delta M$ and $L \ll \delta M$.

(ii) If $K \ll \delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll \delta N$. In particular, if $K \ll \delta M \leq N$, then $K \ll \delta N$.

(iii) Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll \delta M_1 \oplus M_2$ if and only if $K_1 \ll \delta M_1$ and $K_2 \ll \delta M_2$.

**Lemma 2.2** (see [1, Lemma 1.5]). Let $M$ and $N$ be modules.

1. $\delta(M) = \sum_{i=1}^{\infty} M_i$ is a $\delta$-small submodule of $M$.
2. If $f : M \to N$ is a homomorphism, then $f(\delta(M)) \leq \delta(N)$.
3. If $M = \bigoplus_{i=1}^{\infty} M_i$, then $\delta(M) = \bigoplus_{i=1}^{\infty} \delta(M_i)$.
4. If every proper submodule of $M$ is contained in a maximal submodule of $M$, then $\delta(M)$ is the unique largest $\delta$-small submodule of $M$.

**Theorem 2.3.** Let $M$ be a module. Then $\delta(M)$ is Noetherian if and only if $M$ satisfies ACC on $\delta$-small submodules.

**Proof.** “$\Rightarrow$” It is clear by Lemma 2.2.

“$\Leftarrow$” Suppose that $\delta(M)$ is not Noetherian. Let $A_1 \leq A_2 \leq \cdots$ be an infinite ascending chain of submodules of $\delta(M)$. Let $a_1 \in A_1$ and $a_j \in A_j - A_{j-1}$ for each $j > 1$. For any $k \geq 1$, let $N_k = \sum_{j=1}^{k} a_j R$. Then $N_k$ is finitely generated and $N_k \leq \delta(M)$. Hence $N_k \ll \delta M$. It is clear that $N_1 \leq N_2 \leq \cdots$ and so $M$ fails to satisfy ACC on $\delta$-small submodules. This completes the proof.

Recall that a module $M$ has finite uniform dimension $k$, for some nonnegative $k$, if $M$ does not contain any infinite direct sum of nonzero submodules and $k$ is the maximal number of summands in a direct sum of nonzero submodules of $M$. In this case, we call $k$ the uniform dimension of $M$, and write $\udim M = k$.

**Proposition 2.4.** Let $M$ be a module. Then the following statements are equivalent.

1. $\delta(M)$ has finite uniform dimension.
2. Every $\delta$-small submodule of $M$ has finite uniform dimension and there exists a positive integer $k$ such that $\udim N \leq k$ for any $N \ll \delta M$.
3. $M$ does not contain an infinite direct sum of nonzero $\delta$-small submodules.
Proof. “(1)⇒(2)” It is obvious because udim\(N\) ≤ udim\(δ(M)\) for any \(N \ll_δ M\).

“(2)⇒(3)” Let \(N_1 \oplus N_2 \oplus \cdots\) be an infinite direct sum of nonzero \(δ\)-small submodules of \(M\). Then \(N_1 \oplus \cdots \oplus N_{k+1}\) is a \(δ\)-small submodule of \(M\) and udim\((N_1 \oplus \cdots \oplus N_{k+1})\) ≥ \(k+1\). This is a contradiction.

“(3)⇒(1)” Let \(N_1 \oplus N_2 \oplus \cdots\) be an infinite direct sum of nonzero submodules of \(δ(M)\). For every \(i \geq 1\), let \(n_i\) be a nonzero element of \(N_i\). Then \(n_1R \ll_δ M\). Thus \(n_1R + n_2R + \cdots\) is an infinite direct sum of nonzero \(δ\)-small submodules of \(M\). This is a contradiction and so \(δ(M)\) has finite uniform dimension. □

Theorem 2.5. Let \(M\) be a module. Then the following statements are equivalent.

(1) \(δ(M)\) is Artinian.
(2) Every \(δ\)-small submodule of \(M\) is Artinian.
(3) \(M\) satisfies DCC on \(δ\)-small submodules.

Proof. “(1)⇒(2)⇒(3)” They are clear.

“(3)⇒(1)” It suffices to prove that any factor module of \(δ(M)\) is finitely cogenerated. If there exists a factor module of \(δ(M)\) that is not finitely cogenerated, then the set \(Ω\) of submodules of \(δ(M)\), such that \(δ(M)/L\) is not finitely cogenerated, is nonempty. Let \(\{L_1 : λ \in Λ\}\) be any chain of submodules in \(Ω\). Let \(L = \bigcap_{λ \in Λ} L_λ\). If \(L \notin Ω\), then \(δ(M)/L\) is finitely cogenerated and hence \(L = L_λ\) for some \(λ \in Λ\). Thus \(L \in Ω\). By Zorn’s lemma, \(Ω\) has a minimal member \(A\).

Let \(N\) be a finitely generated submodule of \(δ(M)\). Then \(N\) is a \(δ\)-small submodule of \(M\) and hence Artinian by hypothesis. Thus \(δ(M)\) is locally Artinian. Now let \(x \in δ(M), x \notin Ω\). Then \(xR\) is Artinian and \((xR + A)/A = xR/(xR \cap A)\). So \((xR + A)/A\) is a nonzero Artinian module and hence \(δ(M)/A\) has essential socle. Let \(S\) denote the submodule of \(δ(M)\), containing \(A\), such that \(S/A\) is the socle of \(δ(M)/A\). Thus \(S/A\) is not finitely generated by [7, Proposition 10.7].

Next we show that \(A \ll_δ M\). If \(M = A + B\) for some \(B \leq M\) and \(M/B\) is singular, then \(S = A + (S \cap B)\). Suppose that \(A \cap B \neq Λ\). Then \(δ(M)/(A \cap B)\) is finitely cogenerated by the choice of \(A\). But \(S/A = (A + (S \cap B))/A = (S \cap B)/(A \cap B) \ll_δ Soc(δ(M)/(A \cap B))\) and hence \(S/A\) is finitely generated. This is a contradiction. Thus \(A = A \cap B \leq B\) and we have \(M = A + B = B\). So \(A \ll_δ M\).

Now suppose that \(M = S + V\) of some submodule \(V\) of \(M\) and \(M/V\) is singular. Then \(M/(A + V) = (S + V)/(A + V) \approx S/(A + (S \cap V))\). Thus \(M/(A + V)\) is semisimple. If \(M \neq A + V\), then there exists a maximal submodule \(W\) of \(M\) such that \(A + V \leq W\). But \(S \leq δ(M) \leq W\) since \(M/W\) is a singular simple module and this gives the contradiction \(M = W\). Thus \(M = A + V\), hence \(M = V\) since \(A \ll_δ M\). Thus \(S \ll_δ M\) and hence \(S\) is Artinian by hypothesis. It follows that \(S/A\) is Artinian, and, in particular, \(S/A\) is finitely generated. This is a contradiction. Thus \(δ(M)\) is Artinian.

Example 2.6. Let \(R = \mathbb{Z}\), \(p\) is a prime and \(M = \mathbb{Z}(p^∞)\), the Prüfer \(p\)-group, then every proper submodule of \(M\) is Noetherian, but \(M\) is not Noetherian. Indeed, every proper submodule of \(M\) is \(δ\)-small. Moreover, \(M = δ(M)\). Thus every \(δ\)-small submodule of \(M\) is Noetherian, but \(δ(M)\) is not Noetherian.
Corollary 2.7. Let \( R \) be a ring which satisfies DCC on \( \delta \)-small right ideals. Then \( R \) satisfies ACC on \( \delta \)-small right ideals.

Let \( N \leq M \). \( N \) is called a \( \delta \)-semimaximal submodule of \( M \) if \( N = \bigcap_{i=1}^{n} L_i \) with \( M/L_i \) singular simple for any \( i = 1, \ldots, n \).

Proposition 2.8. Let \( M \) be a module. Then the following statements are equivalent.

1. \( M \) is Artinian.
2. \( M \) satisfies DCC on \( \delta \)-small submodules and on \( \delta \)-semimaximal submodules.
3. \( M \) satisfies DCC on \( \delta \)-small submodules and \( \delta(M) \) is a \( \delta \)-semimaximal submodule.

Proof. “(1) \( \Rightarrow \) (2)” It is clear.

“(2) \( \Rightarrow \) (3)” Suppose that \( M \) satisfies DCC on \( \delta \)-semimaximal submodules. Let \( N \) be a minimal \( \delta \)-semimaximal submodule of \( M \). Clearly \( \delta(M) \leq N \). If \( M = \delta(M) \), then \( \delta(M) = N \). Suppose that \( M \neq \delta(M) \). If \( P \) is a maximal submodule of \( M \) with \( M/P \) singular, then \( N \cap P \) is a \( \delta \)-semimaximal submodule of \( M \) and hence \( N = N \cap P \), so that \( N \leq P \). It follows that \( N \leq \delta(M) \). Hence \( N = \delta(M) \). Thus \( \delta(M) \) is a \( \delta \)-semimaximal submodule of \( M \).

“(3) \( \Rightarrow \) (1)” It is clear \( \delta(M) \) is Artinian. If \( M = \delta(M) \), then \( M \) is Artinian. Suppose that \( M \neq \delta(M) \). Then \( \delta(M) = P_1 \cap P_2 \cap \cdots \cap P_n \), where \( M/P_i \) is singular simple for any \( i = 1, \ldots, n \). It follows that \( M/\delta(M) \) embeds in the finitely generated semisimple module \( M/P_1 \oplus \cdots \oplus M/P_n \). Hence \( M/\delta(M) \) is Artinian and so \( M \) is Artinian.

3. \( \delta \)-supplemented modules

Let \( M \) be a module. Let \( N \) and \( L \) be submodules of \( M \). \( N \) is called a \( \delta \)-supplement of \( L \) if \( M = N + L \) and \( N \cap L \ll \delta N \). \( N \) is called a \( \delta \)-supplement submodule if \( N \) is a \( \delta \)-supplement of some submodule of \( M \). \( M \) is called a \( \delta \)-supplemented module if every submodule of \( M \) has a \( \delta \)-supplement. On the other hand, \( M \) is called an amply \( \delta \)-supplemented module if for any submodules \( A, B \) of \( M \) with \( M = A + B \) there exists a \( \delta \)-supplement \( P \) of \( A \) such that \( P \leq B \). Clearly, supplemented modules are \( \delta \)-supplemented modules and every amply \( \delta \)-supplemented module is \( \delta \)-supplemented. But the converses are not true.

Lemma 3.1. Let \( M \) be a \( \delta \)-supplemented module. Then

1. \( M/\delta(M) \) is semisimple;
2. \( L \) a submodule of \( M \) with \( L \cap \delta(M) = 0 \), then \( L \) is semisimple.

Proof. (1) Let \( N \) be any submodule of \( M \) containing \( \delta(M) \). Then there exists a \( \delta \)-supplement \( K \) of \( N \) in \( M \), that is, \( M = N + K \) and \( N \cap K \ll \delta K \). Thus \( M/\delta(M) = N/\delta(M) \oplus (K + \delta(M))/\delta(M) \), and so every submodule of \( M/\delta(M) \) is a direct summand. Therefore \( M/\delta(M) \) is semisimple.

(2) It is clear by (1), since \( L \cong L \oplus \delta(M)/\delta(M) \leq M/\delta(M) \).

Proposition 3.2. Let \( M \) be an amply \( \delta \)-supplemented module. Then homomorphic images are amply \( \delta \)-supplemented modules.

Proof. Assume \( M \) is amply \( \delta \)-supplemented and \( f : M \to N \) is any epimorphism. We want to show that \( N \) is amply \( \delta \)-supplemented. Let \( N = A + B \). Then \( M = f^{-1}(A) + f^{-1}(B) \).
Since $M$ is amply $\delta$-supplemented, there exists a submodule $X$ of $M$ such that $M = f^{-1}(A) + X$, $f^{-1}(A) \cap X \ll X \leq f^{-1}(B)$. Now, $N = A + f(X)$ and $A \cap f(X) = f(f^{-1}(A) \cap X) \ll _\delta f(X)$. Clearly $f(X) \leq B$. 

**Proposition 3.3.** Let $M$ be a $\delta$-supplemented module. Then $M = N \oplus L$ for some semisimple module $N$ and some module $L$ with $\delta(L) \leq _\varepsilon L$.

**Proof.** For $\delta(M)$, there exists $N \leq M$ such that $N \cap \delta(M) = \{0\}$ and $N \oplus \delta(M) \leq _\varepsilon M$. Since $M$ is a $\delta$-supplemented module, there exists $L \leq M$ such that $N + L = M$ and $N \cap L \ll _\delta L$. Since $N \cap L = N \cap (N \cap L) \leq N \cap \delta(L) \leq N \cap \delta(M) = \{0\}$, $M = N \oplus L$. By Lemma 3.1, $N$ is semisimple. Thus $\delta(M) = \delta(N) \oplus \delta(L)$. Since $N \oplus \delta(L) \leq _\varepsilon M = N \oplus L$, $\delta(L) \leq _\varepsilon L$ by Lemma 1.1. This completes the proof.

**Lemma 3.4.** Let $M_1, U \leq M$ and let $M_1$ be a $\delta$-supplemented module. If $M_1 + U$ has a $\delta$-supplement in $M$, then so does $U$.

**Proof.** Since $M_1 + U$ has a $\delta$-supplement in $M$, there exists $X \leq M$ such that $X + (M_1 + U) = M$ and $X \cap (M_1 + U) \ll _\delta X$. For $(X + U) \cap M_1$, since $M_1$ is a $\delta$-supplemented module, there exists $Y \leq M_1$ such that $(X + U) \cap M_1 + Y = M_1$ and $(X + U) \cap Y \ll _\delta Y$. Thus we have $X + U + Y = M$ and $(X + U) \cap Y \ll _\delta Y$, that is, $Y$ is a $\delta$-supplement of $X + U$ in $M$. Next, we will show that $X + Y$ is a $\delta$-supplement of $U$ in $M$. It is clear that $(X + Y) + U = M$, so it suffices to show that $(X + Y) \cap U \ll _\delta X + Y$. Since $Y + U \leq M_1 + U$, $X \cap (Y + U) \leq X \cap (M_1 + U) \ll _\delta X$. Thus $(X + Y) \cap U \leq X \cap (Y + U) + Y \cap (X + U) \ll _\delta X + Y$ by Lemma 2.1, as required.

**Proposition 3.5.** Let $M_1$ and $M_2$ be $\delta$-supplemented modules. If $M = M_1 + M_2$, then $M$ is a $\delta$-supplemented module.

**Proof.** Let $U$ be a submodule of $M$. Since $M_1 + M_2 + U = M$ trivially has a $\delta$-supplement in $M$, $M_2 + U$ has a $\delta$-supplement in $M$ by Lemma 3.4. Thus $U$ has a $\delta$-supplement in $M$ by Lemma 3.4 again. So $M$ is a $\delta$-supplemented module.

**Proposition 3.6.** If $M$ is a $\delta$-supplemented module, then every finitely $M$-generated module is a $\delta$-supplemented module.

**Proof.** From Proposition 3.5, we know that every finite sum of $\delta$-supplemented modules is a $\delta$-supplemented module. Next we will show that every factor module of a $\delta$-supplemented module is again a $\delta$-supplemented module.

Let $M$ be a $\delta$-supplemented module and $M/N$ any factor module of $M$. For any submodule $L$ of $M$ containing $N$, since $M$ is a $\delta$-supplemented module, there exists $K \leq M$ such that $L + K = M$ and $L \cap K \ll _\delta K$. Thus $M/N = L/N + (N + K)/N$ and $(L/N) \cap ((N + K)/N) = (N + (L \cap K))/N \ll _\delta (N + K)/N$, that is, $(N + K)/N$ is a $\delta$-supplement of $L/N$ in $M/N$, as required.

**Proposition 3.7.** Let $M$ be a module. If every submodule of $M$ is a $\delta$-supplemented module, then $M$ is an amply $\delta$-supplemented module.

**Proof.** Let $L, N \leq M$ and $M = N + L$. By assumption, there is $H \leq L$ such that $(L \cap N) + H = L$ and $(L \cap N) \cap H = N \cap H \ll _\delta H$. Thus $H + N \geq H + (L \cap N) = L$ and hence $H + N \geq (N + L) = M$. Therefore, $M = H + N$ as desired.
Corollary 3.8. Let $R$ be any ring. Then the following statements are equivalent.

1. Every module is an amply $\delta$-supplemented module.
2. Every module is a $\delta$-supplemented module.

A module $M$ is said to be $\pi$-projective if for every two submodules $U$, $V$ of $M$ with $U + V = M$ there exists $f \in \text{End}(M)$ with $\text{Im} f \leq U$ and $\text{Im}(1 - f) \leq V$.

Theorem 3.9. Let $M$ be a module. If $M$ is a $\pi$-projective $\delta$-supplemented module, then $M$ is an amply $\delta$-supplemented module.

Proof. Let $A$, $B$ be submodules of $M$ such that $M = A + B$. Since $M$ is $\pi$-projective, there exists an endomorphism $e$ of $M$ such that $e(M) \leq A$ and $(1 - e)(M) \leq B$. Note that $(1 - e)(A) \leq A$. Let $C$ be a $\delta$-supplement of $A$ in $M$. Then $M = e(M) + (1 - e)(M) = e(M) + (1 - e)(A + C) \leq A + (1 - e)(C) \leq M$, so that $M = A + (1 - e)(C)$. Note that $(1 - e)(C)$ is a submodule of $B$. Let $y \in A \cap (1 - e)(C)$. Then $y \in A$ and $y = (1 - e)(x) = x - e(x)$ for some $x \in C$. Next $x = y + e(x) \in A$, so that $y \in (1 - e)(A \cap C)$. But $A \cap C \leq \delta C$ gives that $A \cap (1 - e)(C) = (1 - e)(A \cap C) \leq \delta (1 - e)(C)$. Thus $(1 - e)(C)$ is a $\delta$-supplement of $A$ in $M$. It follows that $M$ is an amply $\delta$-supplemented module. $\Box$

Theorem 3.10. Let $M$ be a module. Then $M$ is Artinian if and only if $M$ is an amply $\delta$-supplemented module and satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules.

Proof. The necessity is clear. Conversely, suppose that $M$ is an amply $\delta$-supplemented module which satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules. Then $\delta(M)$ is Artinian by Theorem 2.5. Next, it suffices to show that $M/\delta(M)$ is Artinian. It is clear that $M/\delta(M)$ is semisimple by Lemma 3.1.

Now suppose that $\delta(M) \leq N_1 \leq N_2 \leq N_3 \leq \cdots$ is an ascending chain of submodules of $M$. Because $M$ is an amply $\delta$-supplemented module, there exists a descending chain of submodules $K_1 \geq K_2 \geq \cdots$ such that $K_i$ is a $\delta$-supplement of $N_i$ in $M$ for each $i \geq 1$. By hypothesis, there exists a positive integer $t$ such that $K_t = K_{t+1} = K_{t+2} = \cdots$. Because $M/\delta(M) = N_i/\delta(M) \oplus (K_i + \delta(M))/\delta(M)$ for all $i \geq t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/\delta(M)$ is Noetherian, and hence finitely generated. So $M/\delta(M)$ is Artinian, as desired. $\Box$

Example 3.11. For $\mathbb{Z}_2$, the only $\delta$-supplement submodules are 0 and $\mathbb{Z}$ and the only $\delta$-small submodule is 0, but $\mathbb{Z}_2$ is not Artinian.

Corollary 3.12. Let $M$ be a finitely generated $\delta$-supplemented module. Then $M$ is Artinian if and only if $M$ satisfies DCC on $\delta$-small submodules.

Proof. “$\Leftarrow$” Since $M/\delta(M)$ is semisimple and $M$ is finitely generated, $M/\delta(M)$ is Artinian. Now that $M$ satisfies DCC on $\delta$-small submodules, $\delta(M)$ is Artinian by Theorem 2.5. Thus $M$ is Artinian.

“$\Rightarrow$” It is clear. $\Box$

Remark 3.13. Let $R$ be a ring. If $R_R$ is an amply $\delta$-supplemented module, then $R$ is a right Artinian ring if and only if $R$ satisfies DCC on $\delta$-small right ideals. Thus a right perfect ring which satisfies DCC on $\delta$-small right ideals is a right Artinian ring.
Let us end this section with the following.

**Proposition 3.14.** If $M$ is a $\delta$-supplemented module and satisfies DCC on $\delta$-small submodules, then so does $M/A$ for any submodule $A$ of $M$.

**Proof.** Let $A$ be any submodule of $M$ and $B_i/A \leq B_j/A \leq \cdots$ where each $B_i/A \ll_{\delta} M/A$. Let $C$ be a $\delta$-supplement of $A$ in $M$. Then $M/A = (A + C)/A \simeq C/A \cap C$. Since $B_i/A$ is $\delta$-small in $M/A$, $B_i/A \approx D_i/A \cap C \ll C/A \cap C$ for some $D_i$. Next we prove that $D_1 \ll_{\delta} M$. Let $D_1 + E = M$ with $M/E$ singular. Then $(D_1 + (E + A \cap C))/A \cap C = M/A \cap C$. Hence $E + A \cap C = M$ and $E = M$. Thus we have $D_1 \leq D_2 \leq \cdots$. Since $M$ satisfies ACC on $\delta$-small submodules, there exists $n$ such that $D_k = D_{k+1}$ for all $k \geq n$. Thus $B_k/A = B_{k+1}/A$ for all $k \geq n$. Therefore $M/A$ satisfies ACC on $\delta$-small submodules, as required. □

4. $\delta$-semiperfect modules

In this section, we introduce the concept of $\delta$-semiperfect modules and investigate the interconnections between $\delta$-supplemented modules and $\delta$-semiperfect modules. Let $P$ and $M$ be modules, we call an epimorphism $f : P \rightarrow M$ a $\delta$-cover in case Ker$f \ll_{\delta} P$. A $\delta$-cover $f : P \rightarrow M$ is called a projective $\delta$-cover in case $P$ is a projective module.

**Definition 4.1.** A module $M$ is called a $\delta$-semiperfect module if any homomorphic image of $M$ has a projective $\delta$-cover.

**Proposition 4.2.** If $f : M \rightarrow N$ is an epimorphism with Ker$f \leq \delta(M)$, then $\delta(N) = f(\delta(M))$.

**Proof.** It follows from [7, Corollary 8.17]. □

**Lemma 4.3.** If both $f : P \rightarrow M$ and $g : M \rightarrow N$ are $\delta$-covers, then $g f : P \rightarrow N$ is a $\delta$-cover.

**Proof.** If both $f : P \rightarrow M$ and $g : M \rightarrow N$ are $\delta$-covers, then Ker$f \ll_{\delta} P$ and Ker$g \ll_{\delta} M$. We want to show that Ker$g f \ll_{\delta} P$. Let $P = $ Ker$g f + L$ with $P/L$ singular. Then $M = $ Ker$g f + f(L)$. Since $M/f(L)$ is singular, $M = f(L)$. This implies that $P = L$ since $P/L$ is singular and Ker$f \ll_{\delta} P$, as desired. □

**Lemma 4.4.** If each $f_i : P_i \rightarrow M_i$ $(i = 1, 2, \ldots, n)$ is a $\delta$-cover, then $\bigoplus_{i=1}^n f_i : \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$ is a $\delta$-cover.

**Proof.** It is straightforward. □

**Theorem 4.5.** Let $M$ be a module and $U \leq M$. Then the following statements are equivalent.

(1) $M/U$ has a projective $\delta$-cover.

(2) If $V \leq M$ and $M = U + V$, then $U$ has a $\delta$-supplement $U'$ \leq $V$ such that $U'$ has a projective $\delta$-cover.

(3) $U$ has a $\delta$-supplement $U'$ which has a projective $\delta$-cover.

**Proof.** “(1)⇒(2)” Let $f : P \rightarrow M/U$ be a projective $\delta$-cover. Since $M = U + V$, $g : V \rightarrow M/U$ via $v \mapsto v + U$ is an epimorphism. Since $P$ is projective, there is a homomorphism $h : P \rightarrow V$ such that $f = gh$. It is easy to see that $M = U + h(P)$, where $h(P) \leq V$. Now Ker$f \ll_{\delta} P$, so we have $U \cap h(P) = h(\text{Ker} f) \ll_{\delta} h(P)$ and $h(P)$ is a $\delta$-supplement of $U$ in $M$. Since Ker$h \subseteq \text{Ker} f \ll_{\delta} P$, $h : P \rightarrow h(P)$ is a projective $\delta$-cover.
“(2)⇒(3)” It is obvious.
“(3)⇒(1)” Let \( f : P \to U' \) be a projective \( \delta \)-cover. Since \( U' \) is a \( \delta \)-supplement of \( U \), the natural epimorphism \( g : U' \to U'/U \cap U' \simeq U + U'/U = M/U \) is a \( \delta \)-cover. Hence \( hgf : P \to M/U \) is a projective \( \delta \)-cover by Lemma 4.3, where \( h : U'/U \cap U' \simeq U + U'/U \) is an isomorphism.

**Theorem 4.6.** Let \( M \) be a module. Then the following statements are equivalent.

1. \( M \) is \( \delta \)-semiperfect.
2. \( M \) is amply \( \delta \)-supplemented by \( \delta \)-supplements which have projective \( \delta \)-covers.
3. \( M \) is \( \delta \)-supplemented by \( \delta \)-supplements which have projective \( \delta \)-covers.

**Proof.** It is clear from Theorem 4.5. \( \square \)

**Example 4.7.** A \( \delta \)-semiperfect module is not necessarily semiperfect. Let \( Q = \Pi_{i=1}^{\infty} F_i \), where each \( F_i = \mathbb{Z}_2 \). Let \( R \) be the subring of \( Q \) generated by \( \bigoplus_{i=1}^{\infty} F_i \) and \( 1_Q \). Then \( R_R \) is \( \delta \)-semiperfect but not semiperfect. It is also seen that \( R_R \) is a \( \delta \)-supplemented module but not a supplemented module (see [1, Example 4.1]).

**References**


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